Conditional Value-at-Risk (CVaR) Norm: Stochastic Case

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Abstract

The concept of Conditional Value-at-Risk (CVaR) is used in various applications in uncertain environment. This paper introduces CVaR norm for a random variable, which is by definition CVaR of absolute value of this random variable. It is proved that CVaR norm is indeed a norm in the space of random variables. CVaR norm is defined in two variations: scaled and non-scaled. L-1 and L-infinity norms are limiting cases of the CVaR norm. In continuous case, scaled CVaR norm is a conditional expectation of the random variable. A similar representation of CVaR norm is valid for discrete random variables. Several properties for scaled and non-scaled CVaR norm, as a function of confidence level, were proved. Dual norm for CVaR norm is proved to be the maximum of L-1 and scaled L-infinity norms. CVaR norm, as a Measure of Error, generates a Regular Risk Quadrangle. Negative CVaR function, which is a non convex extension for CVaR norm, is introduced analogously to function L-p for p < 1. Linear regression problems were solved by minimizing CVaR norm of regression residuals.

Keywords: CVaR norm, L_p norm, Conditional Value-at-Risk, CVaR

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1. Introduction

The concept of Conditional Value-at-Risk (CVaR) is widely used in risk management and various applications in uncertain environment. This paper introduces a concept of *CVaR norm* in the space of random variables. *CVaR norm* in $\mathbb{R}^n$ was introduced and developed in [7], [5]. This section provides a short introduction in the *CVaR norm* in $\mathbb{R}^n$ and shows the relation with the *CVaR norm* in the space of random variables. Also, we consider special cases of *CVaR norm* for random variables with discrete and continuous distributions and provide some examples. The *negative CVaR function* is defined, both in $\mathbb{R}^n$ and in the space of random variables, which is an extension of *CVaR norm* (but it is not actually a norm).

Section 2.1 gives a formal definition of *CVaR norm* in stochastic case and proves that *CVaR norm* is indeed a norm. This section also shows an equivalence of definitions for special cases given in the introductory section and the general definition. *CVaR norm* is a parametric family of norms with respect to the confidence parameter α. Section 2.2 proves properties of *CVaR norm* as a function of α. Section 2.3 defines the dual norm to the *CVaR norm* and proves several basic statements about normed space generated by the *CVaR norm* (Banach and reflexive space). Section 2.4 gives a short introduction to the concept of Risk Quadrangle (see [9]). We define the quadrangle generated by the *CVaR norm* as a measure of error and we prove that this quadrangle is regular. Section 3 defines the *negative CVaR function* and proves several basic properties. Section 4 illustrates properties of *CVaR norm* with a case study.

The concept of this paper is motivated by applications of norms in optimization. We consider norms in $\mathbb{R}^n$ and in the space of random variables. We use symbols $\mathbf{x}$ and x_i for a vector and an i-th vector component in $\mathbb{R}^n$, i.e. $\mathbf{x} = (x_1, \dots, x_n)$. We use symbol X for a random variable. l_p norms are broadly used in $\mathbb{R}^n$, and L_p norms are considered in the space of random variables. For $p \in [1, \infty]$ the norms l_p and L_p are defined as follows[1]:

$$l_p(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n}|x_i|^p\right)^{1/p}, \quad L_p(X) = (E|X|^p)^{1/p},$$

where E is the expectation sign. The most popular cases are $p = 1, 2, \infty$, i.e.,

- $l_1(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}|x_i|, \quad L_1(X) = E|X|;$
- $l_\infty(\mathbf{x}) = \max_{i=1,\dots,n}|x_i|, \quad L_\infty(X) = \sup|X|;$
- $l_2(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)^{1/2}, \quad L_2(X) = (EX^2)^{1/2};$

It is known that $l_1(\mathbf{x}) \le l_2(\mathbf{x}) \le l_\infty(\mathbf{x})$ and $L_1(X) \le L_2(X) \le L_\infty(X)$, see [2]. Also, the following inequalities holds: $l_p(\mathbf{x}) \le l_q(\mathbf{x})$ and $L_p(X) \le L_q(X)$ for $p < q$, see [2].

[1] Note that the classic definition for l_p norm is $l_p(\mathbf{x}) = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$, it does not satisfy inequality $l_p(\mathbf{x}) \le l_q(\mathbf{x})$ for $p < q$. This paper uses an equivalent scaled version of this norm $l_p(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$, which satisfies that inequality. L_p norm is commonly defined as $L_p(f) = \|f\|_p \equiv \left(\int_S |f|^p \,\mathrm{d}\mu\right)^{1/p}$, where S is a considered space. It is known (see, e.g., [2]) that for $\|\cdot\|_p$ and $\|\cdot\|_q$ norms inequality $\|f\|_p \le \mu(S)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$ holds for $1 \le p \le q \le \infty$, where S is a considered space and $\mu(S)$ is the measure of the space S. When S is a probability space, $\mu(S) = 1$ and inequality $L_p(X) \le L_q(X)$ holds for $1 \le p \le q \le \infty$, where $L_p(X) = (E|X|^p)^{1/p}$ and E is an expectation sign.

Further we will illustrate the general concept of *CVaR norm* in $\mathbb{R}^n$, as well as in spaces of discrete and continuous random variables.

CVaR norm in $\mathbb{R}^n$ is considered in [7] and [5]. According to [7], the *CVaR norm* in $\mathbf{x} \in \mathbb{R}^n$ is defined as follows. Let $|x|_{(i)}$ be ordered absolute values of components of $\mathbf{x} \in \mathbb{R}^n$, i.e.

$$\{|x_1|, \ldots, |x_n|\} = \{|x|_{(1)}, \ldots, |x|_{(n)}\},$$

and

$$|x|_{(i)} \leq |x|_{(i+1)}, \text{ for } i = 1, \ldots, n-1.$$

Then, for $j = 0, \ldots, n-1$ and $\alpha_j = j/n$, *scaled CVaR norm* (or just *CVaR norm* in this paper) is defined by

$$\langle\langle \mathbf{x} \rangle\rangle_{\alpha_j}^S = (|x|_{(j+1)} + \ldots + |x|_{(n)})/(n-j).$$

For $j = n$ we have $\alpha_j = j/n = 1$ and the norm is $\langle\langle \mathbf{x} \rangle\rangle_{\alpha_n}^S = |x|_{(i)} = \max_i |x_i|$. For $\alpha_j < \alpha < \alpha_{j+1}$, the norm $\langle\langle \mathbf{x} \rangle\rangle_{\alpha}^S$ equals to the weighted sum,

$$\langle\langle \mathbf{x} \rangle\rangle_{\alpha}^S = \mu \langle\langle \mathbf{x} \rangle\rangle_{\alpha_j}^S + (1-\mu)\langle\langle \mathbf{x} \rangle\rangle_{\alpha_{j+1}}^S,$$

where

$$\mu = \frac{(\alpha_{j+1} - \alpha)(1 - \alpha_j)}{(\alpha_{j+1} - \alpha_j)(1 - \alpha)}.$$

A similar norm, called D-norm, was introduced in [3] in a different way: for $p \in [1, n]$, $M = \{1, \ldots, n\}$, $|S|$ is cardinality of a set S, $\lfloor p \rfloor = \max\{l \in \mathbb{Z} | l \leq p\}$ (i.e., $\lfloor p \rfloor$ is a maximal integer number which is not greater than p)

$$|||\mathbf{x}|||_p = \max_{\{S \cup \{t\}: S \subset M, |S| \leq \lfloor p \rfloor, t \in M \setminus S\}} \left\{ \sum_{j \in S} |x_j| + (p - \lfloor p \rfloor)|x_t| \right\}.$$

For $\alpha \in \left[0, \frac{n-1}{n}\right]$, the CVaR norm coinsides with the D-norm $|||\mathbf{x}|||_p$ with parameter p defined by $p = n(1-\alpha)$, see [7].

The second variant of the norm, called *non-scaled CVaR norm*, is defined in [7] as follows[2]:

$$\langle\langle \mathbf{x} \rangle\rangle_{\alpha} = \begin{cases} (1-\alpha)\langle\langle \mathbf{x} \rangle\rangle_{\alpha}^S, & \text{for } 0 \leq \alpha < 1; \\ 0, & \text{for } \alpha = 1. \end{cases}$$

For example, $\langle\langle \mathbf{x} \rangle\rangle_{j/n} = (|x|_{(j+1)} + \ldots + |x|_{(n)})/n$. This paper shows that the $\langle\langle \cdot \rangle\rangle_{\alpha}$ norm in $\mathbb{R}^n$ can be considered as a special case of *CVaR norm* in the space of discrete random variables, which is defined in the following paragraph.

Now, we consider CVaR norm for discretely distributed random variables. Suppose that a random variable X takes values $\{x_i\}_{i=1}^N$ with probabilities $\{p_i\}_{i=1}^N$, and $\sum_{i=1}^N p_i = 1$, where $x_i \in \mathbb{R}$ and $N \in \mathbb{N}$ or $N = \infty$ (we use the notation $\mathbb{N}$ for the set of natural numbers). Let us denote by the sequence $\{|x|_{(i)}\}_{i=1}^N$ an ordered sequence $\{|x_i|\}_{i=1}^N$, i.e. $|x|_{(i)} \leq |x|_{(i+1)}$. Let us denote for discretely distributed X by $\{|p|_{(i)}\}_{i=1}^N$ a corresponding to the $\{|x|_{(i)}\}_{i=1}^N$ sequence of values from the $\{p_i\}_{i=1}^N$, i.e. if $|x|_{(i)}$ corresponds to $|x_j|$, then $|p|_{(i)} = p_j$. Let

[2] In [7] it is defined as $\langle\langle \mathbf{x} \rangle\rangle_{\alpha} = n(1-\alpha)\langle\langle \mathbf{x} \rangle\rangle_{\alpha}^S$, but in this paper we will stick to slightly different definition in the sake of consistency with stochastic case, where $\langle\langle X \rangle\rangle_{\alpha} = (1-\alpha)\langle\langle X \rangle\rangle_{\alpha}^S$.

us define $\alpha_j = \sum_{i=1}^{j} |p|_{(i)}$. The *non-scaled CVaR norm* with confidence level α_j for the discretely distributed random variable X is defined by the following expression:

$$\langle\!\langle X \rangle\!\rangle_{\alpha_j} = (1-\alpha_j)\mathrm{CVaR}_{\alpha_j}(|X|) = \begin{cases} \sum_{i=j+1}^{N} |x|_{(i)}|p|_{(i)}, & \text{for } j = 0,\dots,N-1; \\ 0, & \text{for } j = N, \end{cases}$$

here $CVaR_\alpha(|X|)$ denotes conditional value at risk for random variable $|X|$, see [8]. If $N = \infty$, then

$$\langle\!\langle X \rangle\!\rangle_{\alpha_j} = \sum_{i=j+1}^{\infty} |x|_{(i)}|p|_{(i)}, \quad \text{for } j \in \mathbb{N}.$$

Similarly to the definition of *non-scaled CVaR norm* in $\mathbb{R}^n$, for $\alpha_j < \alpha < \alpha_{j+1}$ the $\langle\!\langle X \rangle\!\rangle_\alpha$ equals to the weighted sum

$$\langle\!\langle X \rangle\!\rangle_\alpha = (1-\lambda)\langle\!\langle X \rangle\!\rangle_{\alpha_j} + \lambda \langle\!\langle X \rangle\!\rangle_{\alpha_{j+1}},$$

where $\lambda = (\alpha - \alpha_j)/(\alpha_{j+1} - \alpha_j)$.

If $N = n$, $p_i = 1/n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then the CVaR norm of X coincides with the deterministic CVaR norm of $\mathbf{x}$: $\langle\!\langle X \rangle\!\rangle_\alpha = \langle\!\langle \mathbf{x} \rangle\!\rangle_\alpha$.

Let us illustrate the definition of non-scaled CVaR norm in stochastic case with the following example.

Example 1.

Consider the random variable X taking the values $x_i = 1 - 2^{-i}$ with the probabilities $p_i = 2^{-i}$ for $i \in \mathbb{N}$. Since $x_i > 0$ and $x_i < x_{i+1}$, then $x_i = |x|_{(i)}$. For $\alpha = \alpha_j$,

$$\langle\!\langle X \rangle\!\rangle_{\alpha_j} = \sum_{i=j+1}^{\infty} (1-2^{-i})2^{-i} = \sum_{i=j+1}^{\infty} 2^{-i} - 4^{-i} = \frac{2^{-(j+1)}}{1-2^{-1}} - \frac{4^{-(j+1)}}{1-4^{-1}} = 2^{-j} - 4^{-j}/3.$$

For $j \to \infty$, there is convergence $\alpha_j \to 1$ and

$$\langle\!\langle X \rangle\!\rangle_{\alpha_j} \to 0 = (1-1)CVaR_1(|X|).$$

For $\alpha = 0.2$, which is between $\alpha_0 = 0$ and $\alpha_1 = 0.5$, the value $\langle\!\langle X \rangle\!\rangle_{0.2}$ is a weighted sum of $\langle\!\langle X \rangle\!\rangle_0$ and $\langle\!\langle X \rangle\!\rangle_{0.5}$ with coefficient $\lambda = (\alpha - \alpha_0)/(\alpha_1 - \alpha_0) = 0.2/0.5 = 0.4$. We have

$$\langle\!\langle X \rangle\!\rangle_0 = 1 - 1/3 = 2/3, \ \langle\!\langle X \rangle\!\rangle_{0.5} = 0.5 - 0.25/3 = 1.25/3,$$

therefore,

$$\langle\!\langle X \rangle\!\rangle_{0.2} = (1-\lambda)\langle\!\langle X \rangle\!\rangle_0 + \lambda \cdot 0.5 \cdot \langle\!\langle X \rangle\!\rangle_{0.5} = 0.6 \cdot 2/3 + 0.4 \cdot 1.25/3 = 1.7/3 \approx 0.567.$$

Figure 1 shows a plot of $\langle\!\langle X \rangle\!\rangle_\alpha$ depending upon α. Notice that $\langle\!\langle X \rangle\!\rangle_\alpha$ is a non-increasing, concave and piecewise-linear function w.r.t. α. Paper [7] showed these properties for CVaR norm in $\mathbb{R}^n$. Section 2.2 proves these properties of $\langle\!\langle X \rangle\!\rangle_\alpha$ in stochastic case.

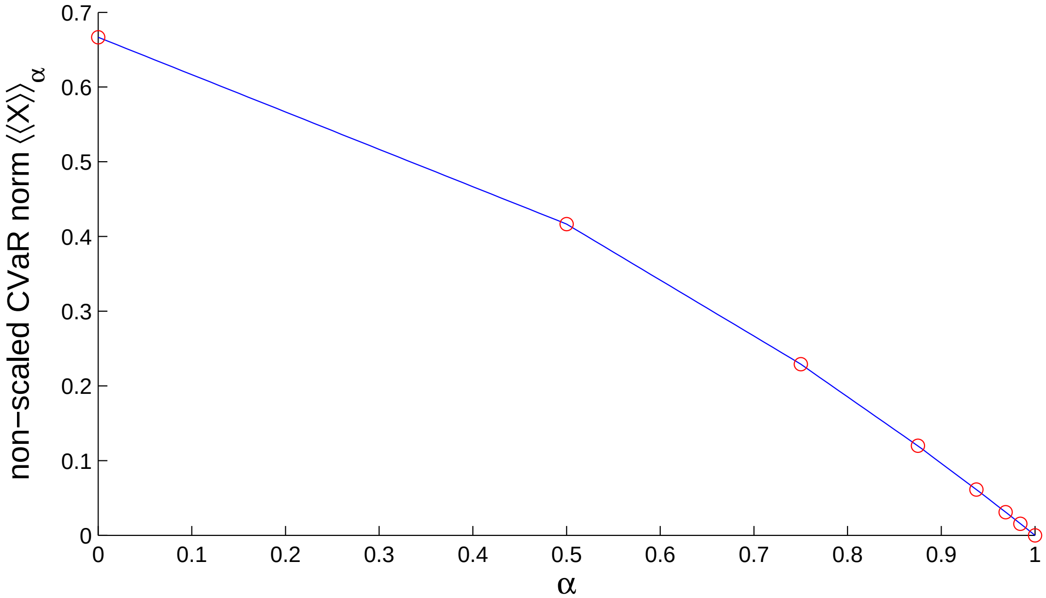

Figure 1: Non-scaled CVaR norm $\langle\!\langle X\rangle\!\rangle_\alpha$ of random variable X with atoms $x_i = 1-2^{-i}$ and probabilities $p_i = 2^{-i}$ as a function of α.

For discrete random variable X, the *scaled CVaR norm* is defined as follows:

$$\langle\!\langle X\rangle\!\rangle_\alpha^S = \begin{cases} \frac{1}{1-\alpha}\langle\!\langle X\rangle\!\rangle_\alpha, & \text{for } \alpha < 1; \\ \sup|X|, & \text{for } \alpha = 1, \end{cases}$$

where $\sup|X|$ denotes the essential supremum[3] of the random variable. If $N = n$, $p_i = 1/n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\langle\!\langle X\rangle\!\rangle_\alpha^S = \langle\!\langle \mathbf{x}\rangle\!\rangle_\alpha^S$.

Example 2.

Consider the random variable X taking the values $x_i = 1-2^{-i}$ with the probabilities $p_i = 2^{-i}$ for $i \in \mathbb{N}$. For $\alpha = \alpha_j$, values of the $\langle\!\langle X\rangle\!\rangle_\alpha^S$ norm are:

$$\langle\!\langle X\rangle\!\rangle_{\alpha_j}^S = \frac{2^{-j} - 4^{-j}/3}{1 - \sum_{i=1}^{j} 2^{-i}} = \frac{2^{-j} - 4^{-j}/3}{2^{-j}} = 1 - 2^{-j}/3.$$

For $j \to \infty$, there is convergence value $\alpha_j \to 1$ and CVaR norm $\langle\!\langle X\rangle\!\rangle_{\alpha_j}^S \to 1 = \langle\!\langle X\rangle\!\rangle_1^S = \sup|X|$.
For $\alpha = 0.2$,

$$\langle\!\langle X\rangle\!\rangle_{0.2}^S = (1.2/3 + 0.5/3)/0.8 \approx 0.708.$$

Figure 2 shows a plot of $\langle\!\langle X\rangle\!\rangle_\alpha^S$ depending upon α. Notice that the norm is an increasing continuous function w.r.t. α. Paper [7] showed these properties for CVaR norm in $\mathbb{R}^n$. Section 2.2 proves these properties of $\langle\!\langle X\rangle\!\rangle_\alpha^S$ for stochastic case.

[3]By definition, $\operatorname{ess\,sup} X = \inf\{a \in \mathbb{R} | P(X > a) = 0\}$.

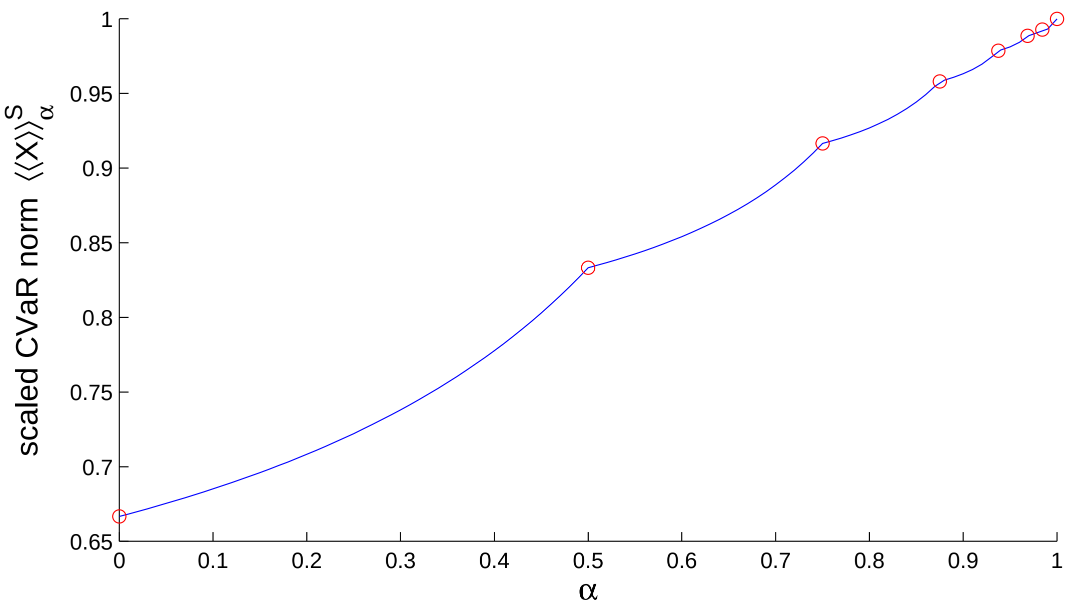

Figure 2: Scaled CVaR norm $\langle\!\langle X \rangle\!\rangle_\alpha$ of random variable X with atoms $x_i = 1 - 2^{-i}$ and probabilities $p_i = 2^{-i}$ as a function of α.

Furthermore, we consider the space of random variables with continuous distribution functions. This is an important special case.

Let $F_X(x)$ be a continuous cumulative distribution function of a random variable X, and $F_X^{-1}(\alpha)$ be an inverse function to $F_X(x)$ (i.e., $F_X^{-1}(F_X(x)) = x$), and $q_\alpha(X)$ be an α-quantile of the random variable X (i.e., $P(X \le q_\alpha(X)) = \alpha$, $q_\alpha(X) = F_X^{-1}(\alpha)$).

CVaR(X), in this case, is a conditional expectation: $CVaR_\alpha(X) = E(X|X > F_X^{-1}(\alpha))$, or, equivalently, $\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp$, see [8].

In continuous case, the CVaR norm is defined as follows: $\langle\!\langle X \rangle\!\rangle_\alpha^S = \mathrm{CVaR}_\alpha(|X|)$. We prove in Section 2.1 that CVaR norm is indeed a norm. Also, we show that $\langle\!\langle X \rangle\!\rangle_0^S = L_1(X)$ and $\langle\!\langle X \rangle\!\rangle_1^S = L_\infty(X)$.

We use the sign $\propto$ as a notation for words «distributed by». For example, $X \propto \mathcal{N}(0,1)$ means that the random variable X is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$.

Let us illustrate the definition of CVaR norm in the space of continuous random variables with the following example.

Example 3.

Consider an exponentially distributed random variable $X \propto Exp(\lambda)$ with the probability density function

$$f_X(x) = \left\{\lambda e^{-\lambda x}, x \ge 0; 0, x < 0\right\},$$

and with cumulative distribution function

$$F_X(x) = \left\{1 - e^{-\lambda x}, x \ge 0; 0, x < 0\right\}. \tag{1}$$

Since X is a non negative random variable, then $X = |X|$. The expression (1) for $F_X(x)$ implies the following equation for quantile $q_\alpha(X)$:

$$1 - e^{-\lambda q_\alpha(X)} = \alpha.$$

Consequently,

$$-\lambda q_\alpha(X) = \ln(1-\alpha),$$

$$q_\alpha(X) = -\frac{1}{\lambda}\ln(1-\alpha).$$

For $\alpha = 1$, the quantile $q_1(X) = \infty$. Then, for $\alpha \in [0,1]$

$$\begin{aligned}\langle\!\langle X\rangle\!\rangle_\alpha^S &= \frac{1}{1-\alpha}\left[\int_{q_\alpha(X)}^{\infty} x\lambda e^{-\lambda x}dx\right] = \frac{1}{1-\alpha}\left[\left(-xe^{-\lambda x}\right)\Big|_{q_\alpha(X)}^{\infty} + \int_{q_\alpha(X)}^{\infty}\lambda e^{-\lambda x}dx\right] = \\ &= \frac{1}{1-\alpha}\left[q_\alpha(X)(1-\alpha) + \left(-\frac{1}{\lambda}e^{-\lambda x}\right)\Big|_{q_\alpha(X)}^{\infty}\right] = \frac{1}{1-\alpha}\left[q_\alpha(X)(1-\alpha) + \frac{1}{\lambda}(1-\alpha)\right] = \\ &= \frac{1}{\lambda}[1-\ln(1-\alpha)].\end{aligned}$$

For $\alpha = 0$,

$$\langle\!\langle X\rangle\!\rangle_0^S = \frac{1}{\lambda} = EX = E|X| = L_1(X).$$

For $\alpha = 1$,

$$\langle\!\langle X\rangle\!\rangle_1^S = \infty = \sup X = \sup|X| = L_\infty(X).$$

Figure 3 shows a plot of the probability density function $f_X(x)$ of X and Figure 4 shows a plot of the function $\langle\!\langle X\rangle\!\rangle_\alpha^S$ as a function of α.

Similar to the discrete distribution case, we consider the *non-scaled CVaR norm*:

$$\langle\!\langle X\rangle\!\rangle_\alpha = (1-\alpha)\langle\!\langle X\rangle\!\rangle_\alpha^S.$$

$\mathrm{CVaR}_\alpha(|X|)$). It is easy to see that $\langle\!\langle X\rangle\!\rangle_\alpha = E(I(|X| > F_{|X|}^{-1}(\alpha))|X|)$, where $I(A)$ is an indicator function:

$$I(A) = \begin{cases} 1, & \text{if } A \text{ is } true; \\ 0, & \text{if } A \text{ is } false. \end{cases}$$

Section 2.2 shows that the non-scaled CVaR norm is a decreasing concave function of α and changes from $L_1(X)$ to 0 when α changes from 0 to 1.

Let us illustrate the definition of non-scaled CVaR norm $\langle\!\langle X\rangle\!\rangle_\alpha$ in the space of continuous random variables with the following example.

Example 4.

Consider an exponentially distributed random variable $X \propto Exp(\lambda)$. For $\alpha \in [0,1)$, the non-scaled CVaR norm equals

$$\langle\!\langle X\rangle\!\rangle_\alpha = \frac{1-\alpha}{\lambda}[1-\ln(1-\alpha)].$$

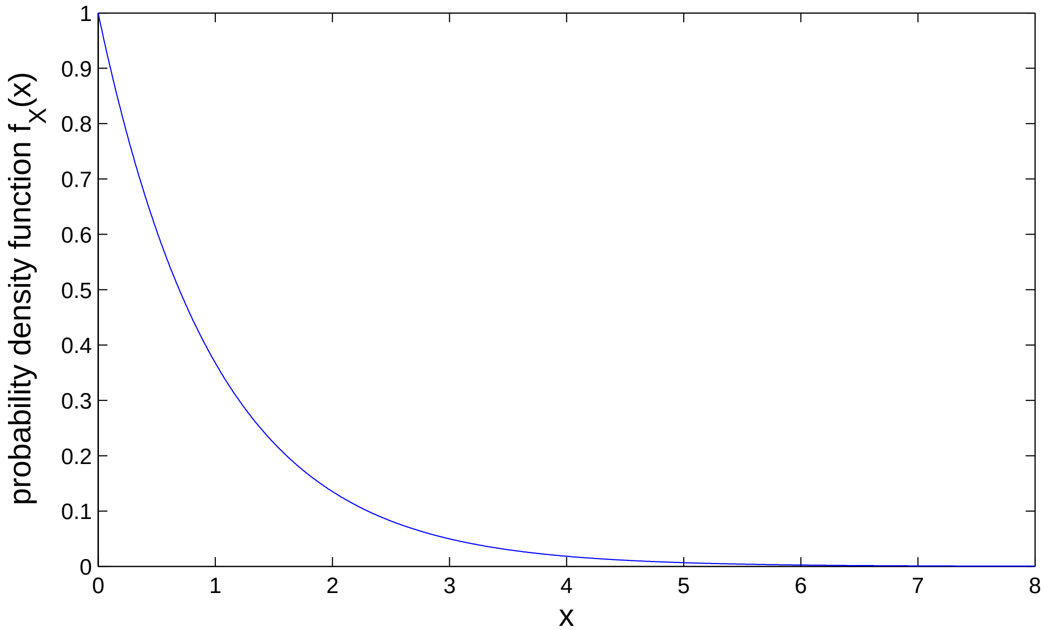

Figure 3: Probability density function $f_X(x)$ for $X \propto Exp(1)$.

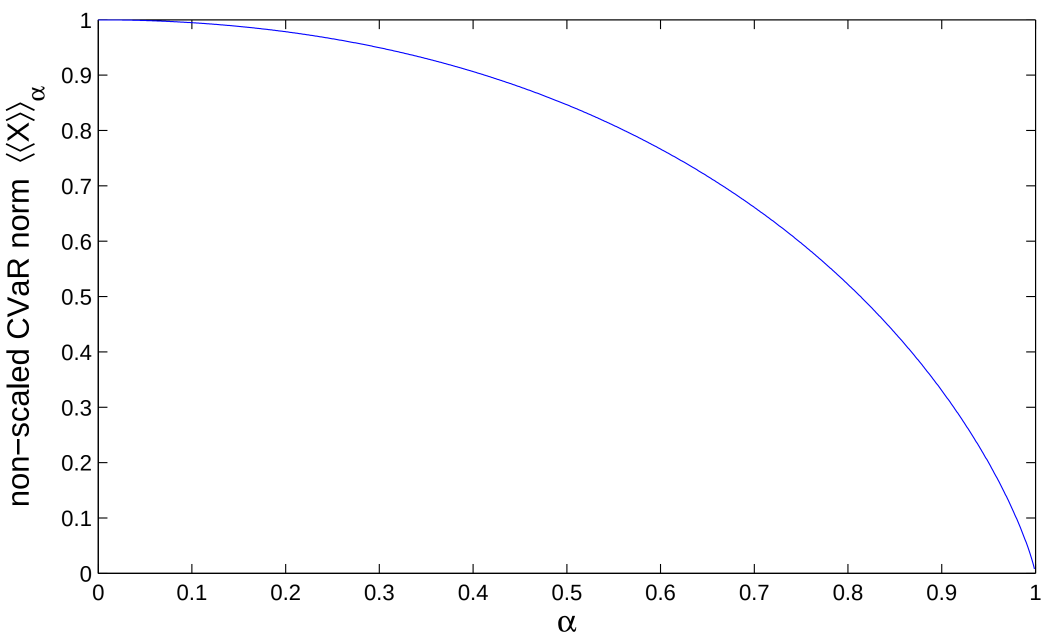

Figure 4: Non-scaled CVaR norm $\langle\langle X \rangle\rangle_\alpha$ for $X \propto Exp(1)$, as a function of α.

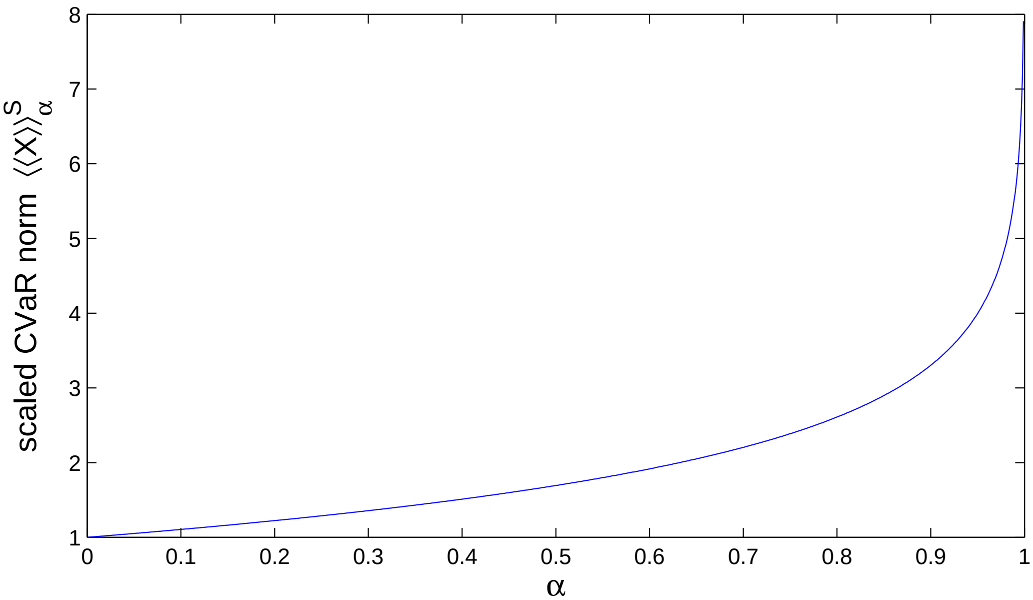

Figure 5: Scaled CVaR norm $\langle\!\langle X \rangle\!\rangle^S_\alpha$ for $X \propto Exp(1)$, as a function of α.

For $\alpha = 0$,

$$\langle\!\langle X \rangle\!\rangle_\alpha = \frac{1}{\lambda} = EX = E|X| = L_1(X).$$

For $\alpha = 1$,

$$\langle\!\langle X \rangle\!\rangle_\alpha = 0.$$

Figure 5 shows a plot of the function $\langle\!\langle X \rangle\!\rangle_\alpha$.

Risk Quadrangle defines risk $\mathcal{R}(X)$, deviation $\mathcal{D}(X)$, regret $\mathcal{V}(X)$, error $\mathcal{E}(X)$ and statistic $\mathcal{S}(X)$, satisfying some axioms, see [9]. Considered functionals can be regular or non-regular. further work that if $\mathcal{R}(X)$ is a regular Measure of Risk, than $\mathcal{R}(|X|)$ is a norm and a regular Measure of Error. This paper proves that $\langle\langle X \rangle\rangle_\alpha$ is a regular Measure of Error and finds the corresponding functions $\mathcal{R}(X)$, $\mathcal{D}(X)$, $\mathcal{V}(X)$ and $\mathcal{S}(X)$ in quadrangle generated by the Measure of Error $\mathcal{E}(X) = \langle\langle X \rangle\rangle_\alpha$ (see Section 2.4).

This paper defines also non convex functions closely related to CVaR norm. In deterministic case, by definition, CVaR norm is the average of the biggest by absolute value $(1-\alpha)n$ components of a vector. The *negative CVaR function* is defined as an average of the smallest by absolute value αn components of a vector. We define *non-scaled* version of *negative CVaR function* as the difference of l_1 and $\langle\langle \cdot \rangle\rangle_\alpha$ norms:

$$r_\alpha^-(\mathbf{x}) = l_1(\mathbf{x}) - \langle\langle \mathbf{x} \rangle\rangle_\alpha.$$

From the definition of $\langle\langle \mathbf{x} \rangle\rangle_\alpha$ follows:

$$r_{\alpha_j}^-(\mathbf{x}) = (|x|_{(1)} + \ldots + |x|_{(j)})/n, \text{ for } \alpha = \alpha_j = j/n,$$

$$r_0^-(\mathbf{x}) = 0, \text{ for } \alpha = 0,$$
$$r_1^-(\mathbf{x}) = l_1(\mathbf{x}), \text{ for } \alpha = 1.$$

We also define *scaled* version of *negative CVaR function* as follows

$$r_\alpha^{-,S}(\mathbf{x}) = \frac{1}{\alpha}(l_1(\mathbf{x}) - (1-\alpha)\langle\langle \mathbf{x} \rangle\rangle_\alpha^S).$$

From the definition of $\langle\langle \mathbf{x} \rangle\rangle_\alpha^S$ follows:

$$r_{\alpha_j}^-(\mathbf{x}) = (|x|_{(1)} + \ldots + |x|_{(j)})/j, \text{ for } \alpha = \alpha_j = j/n,$$

$$r_0^-(\mathbf{x}) = \min_i |x_i|, \text{ for } \alpha = 0,$$
$$r_1^-(\mathbf{x}) = l_1(\mathbf{x}), \text{ for } \alpha = 1.$$

A general definition of the negative CVaR function, both in deterministic and stochastic cases, is considered in Section 3.

Figure 6 shows level-sets of $\langle\langle \mathbf{x} \rangle\rangle_\alpha^S$ and $r_\alpha^{-,S}(\mathbf{x})$ in $\mathbb{R}^2$ for different values of α. The function $r_\alpha^{-,S}$ is a natural extension of $\langle\langle \cdot \rangle\rangle_\alpha^S$. When α variates from 0 to 1, the function $r_\alpha^{-,S}(\mathbf{x})$ changes from $\min_i |x_i|$ to $l_1(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n |x_i|$, and the function $\langle\langle \mathbf{x} \rangle\rangle_\alpha^S$ changes from $l_1(\mathbf{x})$ to $\max_i |x_i|$.

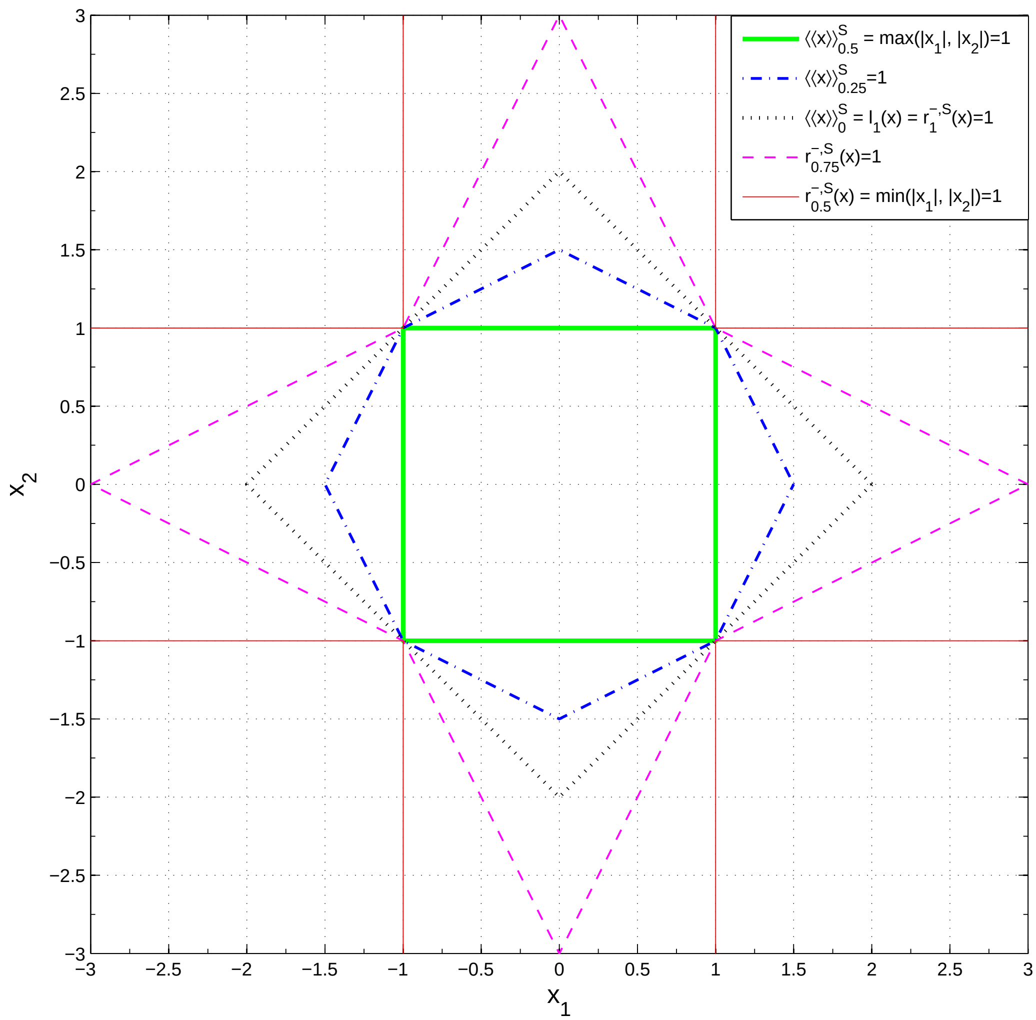

Figure 6: Level-sets of *scaled CVaR norm* $\langle\langle \mathbf{x} \rangle\rangle_\alpha^S$ for $\alpha = 0, 0.25, 0.5$ and level-sets of scaled negative CVaR function $r_\alpha^{-,S}(\mathbf{x})$ for $\alpha = 0.5, 0.75, 1$ in $\mathbb{R}^2$ space. For $\alpha \in [0.5, 1]$ norm $\langle\langle \mathbf{x} \rangle\rangle_\alpha^S = \max_i |x_i|$. For $\alpha \in [0, 0.5]$ function $r_\alpha^{-,S}(\mathbf{x}) = \min_i |x_i|$. Equality $\langle\langle \mathbf{x} \rangle\rangle_0^S = l_1(\mathbf{x}) = r_1^{-,S}(\mathbf{x})$ holds.

2. CVaR Norm in Stochastic Case

This section gives a formal definition of CVaR norm in stochastic case and proves various properties of the norm.

2.1. CVaR Norm Definition and Properties

Let us denote

$$[x]^+ = \max\{0, x\},\ [x]^- = \max\{0, -x\}.$$

Consider cumulative distribution function $F_X(x) = P(X \le x)$. If, for a probability level $\alpha \in (0, 1)$, there is a unique x such that $F_X(x) = \alpha$, then this x is called the α-quantile $q_\alpha(X)$. In general, however, the value x is not unique, or may not even exist any. There are two values to consider as extremes:

$$q^+_\alpha(X) = \inf\{x|F_X(x) > \alpha\}, \quad q^-_\alpha(X) = \sup\{x|F_X(x) < \alpha\}.$$

We will call by the *quantile* the entire interval between the two extreme values,

$$q_\alpha(X) = [q^-_\alpha(X), q^+_\alpha(X)]. \tag{2}$$

We will use notation $\int q_p(X)dp \equiv \int q^-_p(X)dp$, which is a reasonable since $\int q^+_p(X)dp = \int q^-_p(X)dp$.

Further we provide two general definitions of *CVaR norm*, following from the two equivalent general definitions of CVaR (see [8]). We also show that definitions for the discrete case and continuous case, made in introduction, are special cases of these general definitions.

Definition 1. *Let X be a random variable with $E|X| < \infty$. Then, CVaR norm of X with parameter $\alpha \in [0, 1)$ is defined as follows:*

$$\langle\langle X \rangle\rangle^S_\alpha = \min_c \left\{ c + \frac{1}{1-\alpha} E[|X| - c]^+ \right\}.$$

If also X is essentially finite (i.e., exists $C \in \mathbb{R} : |X| < C$), then for $\alpha = 1$
$\langle\langle X \rangle\rangle^S_1 = \sup |X|$.

Definition 2. *Let X be a random variable with $E|X| < \infty$. Then CVaR norm of X with parameter $\alpha \in [0, 1)$ is defined as follows:*

$$\langle\langle X \rangle\rangle^S_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_p(|X|)dp.$$

If also X is essentially finite (i.e., exists $C \in \mathbb{R} : |X| < C$), then for $\alpha = 1$
$\langle\langle X \rangle\rangle^S_1 = \sup |X|$.

It immediately follows from the definitions of CVaR (see [8]) that the Definitions 1 and 2 are equivalent.

Proposition 1. *Let X be a continuous random variable, i.e., its cumulative distribution function is continuous. Then $\langle\langle X \rangle\rangle^S_\alpha = E\left(|X| \,\big|\, |X| > q_\alpha(|X|)\right)$.*

Proof. If cumulative distribution function F_X of random variable X is continuous, then $F_{|X|}$ is continuous, $q_p(|X|) = F^{-1}_{|X|}(p)$ and

$$\langle\!\langle X \rangle\!\rangle^S_\alpha = \frac{\int_\alpha^1 F^{-1}_{|X|}(p)dp}{1-\alpha} = \frac{\int_{F^{-1}_{|X|}(\alpha)}^\infty x dF_{|X|}(x)}{P\left(|X| > F^{-1}_{|X|}(\alpha)\right)} = E\left(|X| \,\middle|\, |X| > F^{-1}_{|X|}(\alpha)\right).$$

□

Let X be a discrete random variable, i.e., it takes values $\{x_i\}_{i=1}^N$ with positive probabilities $\{p_i\}_{i=1}^N$ (N also can be ∞).

Let us denote by the sequence $\{|x|_{(i)}\}_{i=1}^N$ an ordered sequence $\{|x_i|\}_{i=1}^N$, i.e., $|x|_{(i)} \leq |x|_{(i+1)}$. $\{|x_i|\}_{i=1}^N$ exists, such that if $|x|_{(i)} \leftrightarrow |x_j|$, then $|x|_{(i)} = |x_j|$. We also denote by $\{|p|_{(i)}\}_{i=1}^N$ a corresponding to the $\{|x|_{(i)}\}_{i=1}^N$ sequence of probabilities from the $\{p_i\}_{i=1}^N$.

Note that ordered sequence $\{|x|_{(i)}\}$ exists only for special sets of $\{x_i\}$. In particular, it exists in following cases: set $\{x_i\}$ is finite; sequence $\{x_i\}_{i=1}^\infty$ has no converging subsequences; all converging subsequences of sequence $\{|x_i|\}_{i=1}^\infty$ converge to $\bar{x} = \sup\{|x_i|\}$. If ordered sequence $\{|x|_{(i)}\}$ exists, then the following proposition holds.

Proposition 2. *Let X be a discrete random variable, i.e., it takes values $\{x_i\}_{i=1}^N$, with positive probabilities $\{p_i\}_{i=1}^N$, where $\sum_{i=1}^N p_i = 1$ and $N \in \mathbb{N} \cup \infty$. exists, such that if $|x|_{(i)} \leftrightarrow |x_j|$, then $|x|_{(i)} = |x_j|$. Let us denote by $\{|p|_{(i)}\}_{i=1}^N$ a corresponding to the $\{|x|_{(i)}\}_{i=1}^N$ sequence of values from the $\{p_i\}_{i=1}^N$, i.e. if $|x|_{(i)} \leftrightarrow |x_j|$, then $|p|_{(i)} = |p_j|$. Then*

- *for $N < \infty$,*

  $$\langle\!\langle X \rangle\!\rangle^S_1 = |x|_{(N)},$$

  for $N = \infty$,

  $$\langle\!\langle X \rangle\!\rangle^S_1 = \lim_{i\to\infty} |x|_{(i)}.$$

- *for $N < \infty$ and $j = 0, \dots, N-1$*

  $$\langle\!\langle X \rangle\!\rangle^S_{\alpha_j} = \frac{1}{1-\alpha_j} \sum_{i=j+1}^N |x|_{(i)}|p|_{(i)},$$

  where $\alpha_j = \sum_{i=1}^j |p|_{(i)}$,
  for $N = \infty$ and $j \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$,

  $$\langle\!\langle X \rangle\!\rangle^S_{\alpha_j} = \frac{1}{1-\alpha_j} \sum_{i=j+1}^\infty |x|_{(i)}|p|_{(i)},$$

- *for $\alpha_j < \alpha < \alpha_{j+1}$,*

  $$\langle\!\langle X \rangle\!\rangle^S_\alpha = (1-\lambda)\frac{1-\alpha_j}{1-\alpha}\langle\!\langle X \rangle\!\rangle^S_{\alpha_j} + \lambda\frac{1-\alpha_{j+1}}{1-\alpha}\langle\!\langle X \rangle\!\rangle^S_{\alpha_{j+1}},$$

  where $\lambda = (\alpha - \alpha_j)/(\alpha_{j+1} - \alpha_j)$.

Proof. Let us proceed with the proof bullet by bullet.

- It follows directly from the definition that $\text{CVaR}_1(|X|) = \sup\{|x_i|\}_{i=1}^N$.
  Since $|x|_{(i)} < |x|_{(i+1)}$, then in case $N < \infty$,

$$\langle\!\langle X \rangle\!\rangle_1^S = \sup\{|x_i|\}_{i=1}^N = \max_i |x|_{(i)} = |x|_{(N)}.$$

  Consider the case $N = \infty$. Since $\{|x|_{(i)}\}_{i=1}^\infty$ is a non decreasing sequence, then it has a finite or an infinite limit $\lim_{i\to\infty} |x|_{(i)}$. Since also all probabilities $|p|_{(i)} > 0$, then

$$\langle\!\langle X \rangle\!\rangle_1^S = \sup\{|x|_{(i)}\}_{i=1}^\infty = \lim_{i\to\infty} |x|_{(i)}.$$

- If X is a discrete random variable, then $F_{|X|}(x)$ is a step function. Then, $q_p(|X|)$ is a step function of p. Length of a step number i is $\alpha_i - \alpha_{i-1} = |p|_{(i)}$. Height of the step number i is $q_{\alpha_i}(|X|) = |x|_{(i)}$. It implies that the area under the step equals to the following integral:

$$\int_{\alpha_{i-1}}^{\alpha_i} q_p(|X|)dp = |p|_{(i)}|x|_{(i)}.$$

  Then,

$$\langle\!\langle X \rangle\!\rangle_{\alpha_j}^S = \frac{1}{1-\alpha_j}\int_{\alpha_j}^1 q_p(|X|)dp =$$

$$= \frac{1}{1-\alpha_j}\sum_{i=j+1}^N \int_{\alpha_{i-1}}^{\alpha_i} q_p(|X|)dp = \frac{1}{1-\alpha_j}\sum_{i=j+1}^N |p|_{(i)}|x|_{(i)}.$$

- Since $q_p(|X|)$ is a step function of p, then it is a constant function of α on the interval (α_j, α_{j+1}). Then, the integral $\int_\alpha^1 q_p(|X|)dp$ is a linear function of α for $\alpha_j < \alpha < \alpha_{j+1}$. It implies that this integral is the linear combination

$$\int_\alpha^1 q_p(|X|)dp = (1-\lambda)\int_{\alpha_j}^1 q_p(|X|)dp + \lambda\int_{\alpha_{j+1}}^1 q_p(|X|)dp,$$

  where $\lambda = (\alpha - \alpha_j)/(\alpha_{j+1} - \alpha_j)$. Then,

$$\langle\!\langle X \rangle\!\rangle_\alpha^S = (1-\lambda)\frac{1-\alpha_j}{1-\alpha}\langle\!\langle X \rangle\!\rangle_{\alpha_j}^S + \lambda\frac{1-\alpha_{j+1}}{1-\alpha}\langle\!\langle X \rangle\!\rangle_{\alpha_{j+1}}^S.$$

□

Definition 2 immediately implies that $\langle\!\langle X \rangle\!\rangle_0^S = L_1(X)$, $\langle\!\langle X \rangle\!\rangle_1^S = L_\infty(X)$. Furthermore, we prove that $\langle\!\langle X \rangle\!\rangle_\alpha^S$ is a norm for $\alpha \in (0,1)$.

Proposition 3. *Let X be a random variable. $\langle\!\langle X \rangle\!\rangle_\alpha^S$ is a norm in the space of random variables.*

Proof. By definition, function $d(X)$ is a norm if

1. $d(X) = 0 \Leftrightarrow X \equiv 0$,
2. $d(\lambda X) = |\lambda| d(X)$,
3. $d(X+Y) \leq d(X) + d(Y)$.

Since $\text{CVaR}_\alpha(X)$ is a regular Measure of Risk (see [9]), then for $\lambda > 0$

$$\text{CVaR}_\alpha(\lambda X) = \lambda \text{CVaR}_\alpha(X),$$

and

$$\text{CVaR}_\alpha(X+Y) \leq \text{CVaR}_\alpha(X) + \text{CVaR}_\alpha(Y). \tag{3}$$

By definition, $f(X)$ is monotonic if $X \leq Y$ implies $f(X) \leq f(Y)$. Since $\text{CVaR}_\alpha(X)$ is monotonic (see [9]), then for $X \geq Y$

$$\text{CVaR}_\alpha(X) \geq \text{CVaR}_\alpha(Y). \tag{4}$$

Let us prove the axioms of a norm for $\langle\!\langle X \rangle\!\rangle_\alpha^S = \text{CVaR}_\alpha(|X|)$.

1. $\text{CVaR}_\alpha(|X|) = 0 \Leftrightarrow X \equiv 0$.
   The statement $X \equiv 0 \Rightarrow \text{CVaR}_\alpha(|X|) = 0$ is obvious.
   If $X \neq 0$, then $q_p(|X|) > 0$ for $p \in (0,1)$, and

   $$\text{CVaR}_\alpha(|X|) = \frac{1}{1-\alpha} \int_\alpha^1 q_p(|X|)dp > 0.$$

   If $\alpha = 0$ and $X \neq 0$, then $\text{CVaR}_0(|X|) = E|X| > 0$.
   If $\alpha = 1$ and $X \neq 0$, then $\text{CVaR}_1(|X|) = \sup(|X|) > 0$.
2. $\text{CVaR}_\alpha(|\lambda X|) = |\lambda| \text{CVaR}_\alpha(|X|)$. Since $|\lambda| > 0$, then

   $$\text{CVaR}_\alpha(|\lambda X|) = \text{CVaR}_\alpha(|\lambda||X|) = |\lambda| \text{CVaR}_\alpha(|X|).$$

3. $\text{CVaR}_\alpha(|X+Y|) \leq \text{CVaR}_\alpha(|X|) + \text{CVaR}_\alpha(|Y|)$.
   Since $|X+Y| \leq |X| + |Y|$, using (4) we have

   $$\text{CVaR}_\alpha(|X+Y|) \leq \text{CVaR}_\alpha(|X| + |Y|).$$

   Finally,

   $$\text{CVaR}_\alpha(|X| + |Y|) \leq \text{CVaR}_\alpha(|X|) + \text{CVaR}_\alpha(|Y|),$$

   which follows from (3).

□

The next proposition provides an alternative way to calculate $\langle\!\langle X \rangle\!\rangle_\alpha^S$.

Proposition 4. *Let X be a random variable. Let Y be a random variable defined as follows*

$$Y = \begin{cases} X, & \text{with probability } \frac{1}{2}, \\ -X, & \text{with probability } \frac{1}{2}. \end{cases}$$

Then,

$$\langle\!\langle X \rangle\!\rangle_\alpha^S = \text{CVaR}_{(1+\alpha)/2}(Y).$$

Proof. Minimization form definition of CVaR, see [8] or Definition 1, implies that

$$\mathrm{CVaR}_{(1+\alpha)/2}(Y) = \min_c \left\{ c + \frac{1}{(1-\alpha)/2} E[Y-c]^+ \right\}, \tag{5}$$

since $1/(1-(1+\alpha)/2) = 1/((1-\alpha)/2)$. Define

$$c_Y = \arg\min_c \left\{ c + \frac{1}{(1-\alpha)/2} E[Y-c]^+ \right\}.$$

Notice that Y is symmetric, therefore $0 \in q_Y(1/2)$. Notice also and $(1+\alpha)/2 \geq 1/2$, since $\alpha \geq 0$. Since optimal solution in CVaR definition (5) is the quantile $q_{(1+\alpha)/2}(Y)$, then $c_Y \geq 0$. Then,

$$\mathrm{CVaR}_{(1+\alpha)/2}(Y) = \min_c \left\{ c + \frac{1}{(1-\alpha)/2} E[Y-c]^+ \right\} = \tag{6}$$

$$= c_Y + \frac{1}{(1-\alpha)/2} E[Y-c_Y]^+ = \tag{7}$$

$$= \min_{c\geq 0} \left\{ c + \frac{1}{(1-\alpha)/2} E[Y-c]^+ \right\} = \tag{8}$$

$$= \min_{c\geq 0} \left\{ c + \frac{1}{(1-\alpha)/2} E[Y^+-c]^+ \right\}, \tag{9}$$

where equality between (6) and (7) follows from definition of c_Y; equality between (7) and (8) follows from $c_Y \geq 0$; equality between (8) and (9) follows from $[Y-c]^+ = [Y^+-c]^+$ for $c \geq 0$. Note that

$$Y^+ = \begin{cases} |X|, & \text{with probability } \frac{1}{2}, \\ 0, & \text{with probability } \frac{1}{2}. \end{cases}$$

Therefore,

$$\mathrm{CVaR}_{(1+\alpha)/2}(Y) = \min_{c\geq 0} \left\{ c + \frac{1}{(1-\alpha)/2} \frac{1}{2} E[|X|-c]^+ \right\} =$$

$$= \min_{c\geq 0} \left\{ c + \frac{1}{1-\alpha} E[|X|-c]^+ \right\} = \langle\langle X \rangle\rangle_\alpha^S. \tag{10}$$

Last equation in (10) follows from CVaR norm Definition 1 and from

$$\arg\min_c \left\{ c + \frac{1}{1-\alpha} E[|X|-c]^+ \right\} \geq 0,$$

which holds since arg min is a quantile $q_\alpha(|X|)$ and $|X| \geq 0$, therefore $q_\alpha(|X|) \geq 0$. □

2.2. CVaR Norm Properties With Respect to α

Let us remind some general properties of integrals of quantile.

Proposition 5. *Let X be a random variable. For $0 \leq \alpha \leq 1$,*

- $\frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp$ *is a continuous increasing function of α,*

- $\lim_{\alpha\to 1} \frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp = \sup X$,
- $\frac{1}{\alpha}\int_0^\alpha q_p(X)dp$ *is a continuous non-decreasing function of* α,
- $\int_0^\alpha q_p(X)dp$ *is a convex function of* α,
- $\int_0^\alpha q_p(X)dp$ *is a piecewise-linear function of* α *for discretely distributed* X.

Proof. We provide references for these statements bullet by bullet.

- CVaR_α is a continuous increasing function of α, see [8]. Then, integral form definition of $\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp$ implies that the integral $\frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp$ is a continuous increasing function of α.
- $\lim_{\alpha\to 1}\mathrm{CVaR}_\alpha(X) = \sup X$, see [8]. Therefore, $\lim_{\alpha\to 1}\frac{1}{1-\alpha}\int_\alpha^1 q_p(X)dp = \sup X$.
- Notice that $q_p(-X) = -q_{1-p}(X)$, then

  $$\frac{1}{\alpha}\int_{1-\alpha}^1 q_p(-X)dp = -\frac{1}{\alpha}\int_{1-\alpha}^1 q_{1-p}(X)dp = -\frac{1}{\alpha}\int_0^\alpha q_p(X)dp.$$

  The first bullet of this proposition implies that $\frac{1}{\alpha}\int_{1-\alpha}^1 q_p(-X)dp$ is a decreasing continuous function of α, therefore, integral $\frac{1}{\alpha}\int_0^\alpha q_p(X)dp$ is a continuous non-decreasing function of α.
- Integral $\int_0^\alpha q_p(X)dp$ is called the *Absolute Lorenz Curve*, which is known to be convex w.r.t. α, e.g. [6].
- Consider X having an atom x with probability p. If $\alpha_1 = F_X(x)$ and $\alpha_2 = \alpha_1 + p$, then $q_\alpha(X) = x$ for $\alpha \in (\alpha_1, \alpha_2)$. Therefore, $\int_0^\alpha q_p(X)dp$ is linear on $\alpha \in (\alpha_1, \alpha_2)$. Since $\int_0^\alpha q_p(X)dp$ is also continuous, and X is dicretely distributed, then $\int_0^\alpha q_p(X)dp$ is a piecewise-linear function of α.

□

Let X be a random variable. Take constant C, such that $L_1(X) \le C \le L_\infty(X)$. The following corollary assures that there exists a single α such that $\mathrm{CVaR}_\alpha(|X|) = C$. Therefore, for any p there is a single α such that $\mathrm{CVaR}_\alpha(|X|) = L_p(X)$.

Corollary 1. *Let* X *be a random variable. The norm* $\langle\langle X\rangle\rangle_\alpha^S$ *is a continuous increasing function of* α.

Proof. Consider $Y = |X|$ and apply Proposition 5 to Y. □

The following proposition establishes properties of non-scaled CVaR norm with respect to parameter α.

Definition 3. *Let* X *be a random variable with* $E|X| < \infty$*. Then, CVaR norm of* X *with parameter* $\alpha \in [0, 1]$ *is defined as follows:*

$$\langle\langle X\rangle\rangle_\alpha = \begin{cases} (1-\alpha)\langle\langle X\rangle\rangle_\alpha^S, & \text{for } \alpha \in [0,1), \\ 0, & \text{for } \alpha = 1. \end{cases}$$

Corollary 2. *Let X be a random variable. The norm $\langle\langle X\rangle\rangle_\alpha$ is a concave and decreasing function of α. Furthermore, if X is discretely distributed, $\langle\langle X\rangle\rangle_\alpha$ is a piecewise-linear function of α.*

Proof. By definition,

$$\langle\langle X\rangle\rangle_\alpha = (1-\alpha)\mathrm{CVaR}_\alpha(|X|) = \int_\alpha^1 q_p(|X|)dp.$$

If $\alpha_1 < \alpha_2$, then

$$\langle\langle X\rangle\rangle_{\alpha_1} = \int_{\alpha_1}^1 q_p(|X|)dp \geq \int_{\alpha_2}^1 q_p(|X|)dp = \langle\langle X\rangle\rangle_{\alpha_2}.$$

Therefore, $\langle\langle X\rangle\rangle_\alpha$ is a decreasing function of α.
To prove that $\langle\langle X\rangle\rangle_\alpha$ is a concave function of α, consider $Y = |X|$ and apply Proposition 5 to Y. Since $\int_0^\alpha q_p(|X|)dp$ is convex, then $\langle\langle X\rangle\rangle_\alpha = E|X| - \int_0^\alpha q_p(|X|)dp$ is a concave function of α.

For piecewise-linearity, take $Y = |X|$, note that $\langle\langle X\rangle\rangle_\alpha = EY - \int_0^\alpha q_p(Y)dp$, and apply Proposition 5. □

Corollary 3. *Let X be a random variable. $(1-\alpha)\mathrm{CVaR}_\alpha(X)$ is a concave function of α. Furthermore, it is a piecewise-linear function of α if X is discretely distributed.*

Proof. $(1-\alpha)\mathrm{CVaR}_\alpha(X) = \int_\alpha^1 q_p(X)dp = EX - \int_0^\alpha q_p(X)dp$, therefore, it is concave w.r.t. α, and it is piecewise-linear for discretely distributed X, see Proposition 5. □

2.3. Dual Norm to CVaR Norm and CVaR Normed Space

Definition 4. *Let $\mathbb{X}$ be a normed space over $\mathbb{R}$ with norm $\|\cdot\|$ (i.e., $\|X\| \in \mathbb{R}$ for $X \in \mathbb{X}$). Then, the dual (or conjugate) normed space $\mathbb{X}^*$ is defined as the set of all continuous linear functionals from $\mathbb{X}$ into $\mathbb{R}$. For $f \in \mathbb{X}^*$, the dual norm $\|\cdot\|^*$ of f is defined by*

$$\|f\|^* = \sup\{|f(x)| : x \in \mathbb{X}, \|x\| \leq 1\} = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in \mathbb{X}, x \neq 0\right\}.$$

The asterisk $*$ denotes the dual norm to a norm. Therefore, $\langle\langle Y\rangle\rangle_\alpha^{S*}$ denotes the norm dual to the CVaR norm $\langle\langle X\rangle\rangle_\alpha^S$.

Proposition 6. *Let X be a random variable. The norm $\langle\langle Y\rangle\rangle_\alpha^{S*} = \max\{E|Y|, (1-\alpha)\sup|Y|\}$ is dual to the norm $\langle\langle X\rangle\rangle_\alpha^S$ for $\alpha \in (0,1)$.*

Proof. Paper [9] proved that $\mathrm{CVaR}_\alpha(X) = \sup_{Q\in\mathcal{Q}} EXQ$, where

$$\mathcal{Q} = \left\{Q \,\middle|\, 0 \leq Q \leq \frac{1}{1-\alpha}, EQ = 1\right\}.$$

Then, $\langle\langle X\rangle\rangle_\alpha^S = \mathrm{CVaR}_\alpha(|X|) = \sup_{Q\in\mathcal{Q}} E|X|Q$. Let us prove that

$$\sup_{Q\in\mathcal{Q}} E|X|Q = \sup_{Y\in\mathcal{Y}} EXY,$$

where $\mathcal{Y} = \left\{Y \,\middle|\, |Y| \leq \frac{1}{1-\alpha}, E|Y| \leq 1\right\}$.
First,

$$(I(X>0) - I(X<0))Q \in \mathcal{Y} \Rightarrow \sup_{Q \in \mathcal{Q}} E|X|Q \leq \sup_{Y \in \mathcal{Y}} EXY,$$

since $X(I(X>0) - I(X<0))Q = |X|Q$.
Second,

$$\sup_{Y \in \mathcal{Y}} EXY \leq \sup_{Y \in \mathcal{Y}} E|X||Y| \leq \sup_{Q \in \mathcal{Q}} E|X|Q.$$

Finally,

$$\sup_{Y \in \mathcal{Y}} EXY = \sup_{Q \in \mathcal{Q}} E|X|Q = CVaR_\alpha(|X|) = \langle\!\langle X \rangle\!\rangle_\alpha^S.$$

Then, $\mathcal{Y}^* = \{Y | EXY \leq \langle\!\langle X \rangle\!\rangle_\alpha^S\}$ must be a convex hull of $\mathcal{Y}$. The set $\mathcal{Y}$ is closed:

$$Y_k \to Y, \ |Y_k| \leq \frac{1}{1-\alpha} \Rightarrow |Y| \leq \frac{1}{1-\alpha},$$

$$Y_k \to Y, \ E|Y_k| \leq 1 \Rightarrow E|Y| \leq 1,$$

and convex:

$$Y_1, Y_2 \in \mathcal{Y} \Rightarrow |\lambda Y_1 + (1-\lambda)Y_2| \leq \lambda|Y_1| + (1-\lambda)|Y_2| \leq \frac{1}{1-\alpha}(\lambda + (1-\lambda)),$$

$$E|\lambda Y_1 + (1-\lambda)Y_2| \leq \lambda E|Y_1| + (1-\lambda)E|Y_2| \leq 1 \cdot (\lambda + (1-\lambda)).$$

Then, $\mathcal{Y}^* = \mathcal{Y}$ and $\mathcal{Y}$ is a unit ball in the dual norm to the CVaR norm

$$\langle\!\langle Y \rangle\!\rangle_\alpha^{S*} = \sup_{X \neq 0} \frac{EXY}{\langle\!\langle X \rangle\!\rangle_\alpha^S},$$

since $\langle\!\langle Y \rangle\!\rangle_\alpha^{S*} \leq 1 \Leftrightarrow EXY \leq \langle\!\langle X \rangle\!\rangle_\alpha^S$ for all X. Then, the unit sphere in the dual norm is the set

$$\left\{Y \,\middle|\, \sup|Y| = \frac{1}{1-\alpha}, E|Y| \leq 1\right\} \cup \left\{Y \,\middle|\, \sup|Y| \leq \frac{1}{1-\alpha}, E|Y| = 1\right\}.$$

Then, the dual norm equals $\langle\!\langle Y \rangle\!\rangle_\alpha^{S*} = \max\{E|Y|, (1-\alpha)\sup|Y|\}$. □

Definition 5. *A Banach space is a vector space $\mathbb{X}$ over $\mathbb{R}$, which is equipped with a norm $\|\cdot\|$ and which is complete with respect to that norm. By definition, completeness means that for every Cauchy sequence $\{x_n\}_{n=1}^\infty$ in $\mathbb{X}$ (i.e., for every $\varepsilon > 0$ exists N such that $\|x_m - x_n\| < \varepsilon$ for all $m, n > N$), there exists an element x in $\mathbb{X}$ such that*

$$\lim_{n\to\infty} x_n = x, \quad i.e., \quad \lim_{n\to\infty} \|x_n - x\| = 0.$$

The next statement follows directly from Definition 5 of a Banach space.

Proposition 7. *Let L be a norm, $\mathcal{X}_L = \{X | L(X) < \infty\}$ and space $(\mathcal{X}_L, L)$ is a Banach space. Let $\bar{L}$ be a norm such that*

$$\textit{exist such } C_-, C_+ > 0 \textit{ that } C_- L(X) \leq \bar{L}(X) \leq C_+ L(X) \textit{for all } X.$$

Then, $\mathcal{X}_{\bar{L}} = \mathcal{X}_L$ and $(\mathcal{X}_L, \bar{L})$ is a Banach space.

Corollary 4. *The norm $\langle\langle X\rangle\rangle_\alpha^S$ generates a Banach space for $\alpha \in [0,1]$.*

Proof.
$$E|X| \le \langle\langle X\rangle\rangle_\alpha^S \le \frac{1}{1-\alpha}E|X|,$$
for $\alpha < 1$. $E|X| = L_1(X)$ and it is known that L_1-norm generates a Banach space. $\langle\langle X\rangle\rangle_1^S = \sup|X| = L_\infty(X)$ and it is known that L_∞-norm generates a Banach space. □

2.4. Risk Quadrangle With CVaR Norm (CVaR Norm Quadrangle)

Risk Quadrangle (see [9]) defines risk $\mathcal{R}(X)$, deviation $\mathcal{D}(X)$, regret $\mathcal{V}(X)$, error $\mathcal{E}(X)$ and statistic $\mathcal{S}(X)$ related by the following equations:

$$\mathcal{V}(X) = EX + \mathcal{E}(X),\ \mathcal{R}(X) = EX + \mathcal{D}(X), \tag{11}$$
$$\mathcal{D}(X) = \min_C\{\mathcal{E}(X-C)\},\ \mathcal{R}(X) = \min_C\{C + \mathcal{V}(X-C)\}, \tag{12}$$
$$\mathcal{S}(X) = \arg\min_C\{\mathcal{E}(X-C)\} = \arg\min_C\{C + \mathcal{V}(X-C)\}. \tag{13}$$

Measure of risk $\mathcal{R}(X)$ is regular if

- $\mathcal{R}(X) \in (-\infty, \infty]$,
- $\mathcal{R}(X)$ is closed convex,
- $\mathcal{R}(C) = C$ for any constant C,
- $\mathcal{R}(X) > EX$ for any nonconstant X.

Measure of deviation $\mathcal{D}(X)$ is regular if

- $\mathcal{D}(X) \in [0, \infty]$,
- $\mathcal{D}(X)$ is closed convex,
- $\mathcal{D}(C) = 0$ for any constant C,
- $\mathcal{D}(X) > 0$ for any nonconstant X.

Measure of error $\mathcal{E}(X)$ is regular if

- $\mathcal{E}(X) \in [0, \infty]$,
- $\mathcal{E}(X)$ is closed convex,
- $\mathcal{E}(0) = 0$,
- $\mathcal{E}(X) > 0$ for any $X \neq 0$,
- for sequence of random variables $\{X_k\}_{k=1}^\infty$
$$\lim_{k\to\infty}\mathcal{E}(X_k) = 0 \Rightarrow \lim_{k\to\infty} EX_k = 0,$$
which is equivalent to $\mathcal{E}(X) \ge \psi(EX)$ with a convex function ψ on $(-\infty, \infty)$ having $\psi(0) = 0$ but $\psi(t) > 0$ for $t \neq 0$.

Measure of regret $\mathcal{V}(X)$ *is regular* if

- $\mathcal{V}(X) \in (-\infty, \infty]$,
- $\mathcal{V}(X)$ is closed convex,
- $\mathcal{V}(0) = 0$,
- $\mathcal{V}(X) > 0$ for any $X \neq 0$,
- for sequence of random variables $\{X_k\}_{k=1}^{\infty}$

$$\lim_{k\to\infty} [\mathcal{V}(X_k) - EX_k] = 0 \Rightarrow \lim_{k\to\infty} EX_k = 0.$$

The *quadrangle* $(\mathcal{R}, \mathcal{D}, \mathcal{E}, \mathcal{V}, \mathcal{S})$ *is regular* if axioms 11– 13 are hold and if also $\mathcal{R}(X)$ is a regular measure of risk, $\mathcal{D}(X)$ is a regular measure of deviation, $\mathcal{V}(X)$ is a regular measure of regret, and $\mathcal{E}(X)$ is a regular measure of error.
Quadrangle Theorem (see [9]) implies that if axioms 11– 13 are hold for functions $\mathcal{R}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{V}$, $\mathcal{S}$, and if also $\mathcal{E}(X)$ is a regular measure of error, then $(\mathcal{R}, \mathcal{D}, \mathcal{E}, \mathcal{V}, \mathcal{S})$ is a regular quadrangle.
We will prove that $\langle\!\langle X \rangle\!\rangle_{\alpha}^{S}$ is a regular measure of error (it will imply that the quadrangle, generated by CVaR norm as a measure of error is regular).
We also prove that if $\mathcal{E}(X) = \langle\!\langle X \rangle\!\rangle_{\alpha}^{S} = \mathrm{CVaR}_{\alpha}(|X|)$ and quadrangle axioms 11– 13 hold, then the risk measure is $\mathcal{R}(X) = \frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X)$ and the statistic is $\mathcal{S}(X) = \frac{1}{2}\left(q_{(1-\alpha)/2}(X) + q_{(1+\alpha)/2}(X)\right)$.

Proposition 8. $\mathcal{E}(X) = \langle\!\langle X \rangle\!\rangle_{\alpha}^{S}$ *is a regular measure of error.*

Proof. We further prove axioms of the regular measure of error.

- $\mathcal{E}(X) \in [0, \infty]$, follows from the fact that $\langle\!\langle X \rangle\!\rangle_{\alpha}^{S}$ is a norm.
- Let us prove that $\mathcal{E}(X)$ is closed and convex. $\langle\!\langle X \rangle\!\rangle_{\alpha}^{S}$ is a norm, therefore, it is a convex function.
  Closeness is equivalent to the following statement. Let us consider a sequence of random variables $\{X_k\}_{k=1}^{\infty}$ and a random variable X such that expectations $\mu(X_k - X) \to 0$ and variances $\sigma(X_k - X) \to 0$ for $k \to \infty$ and $\mathcal{E}(X_k) \leq C$ for all k. Then, under these conditions, $\mathcal{E}(X) \leq C$.
  Here is the proof of this statement.

$$\begin{aligned}\mathcal{E}(X_k) - \mathcal{E}(X) \leq \mathcal{E}(X_k - X) \leq \frac{1}{1-\alpha} E|X_k - X| \leq \frac{1}{1-\alpha}\sqrt{E|X_k - X|^2} = \\ = \frac{1}{1-\alpha}\sqrt{\sigma^2(X_k - X) - \mu^2(X_k - X)} \to 0,\end{aligned} \tag{14}$$

  for $k \to \infty$. It implies that

$$\mathcal{E}(X) - \mathcal{E}(X_k) \leq \mathcal{E}(X - X_k) = \mathcal{E}(X_k - X) \to 0, \tag{15}$$

  for $k \to \infty$. Combining (14) and (15) we have

$$|\mathcal{E}(X) - \mathcal{E}(X_k)| \to 0 \Leftrightarrow \mathcal{E}(X_k) \to \mathcal{E}(X) \Rightarrow \mathcal{E}(X) \leq C.$$

- $\mathcal{E}(0) = 0$, follows from the fact that $\langle\langle X \rangle\rangle_\alpha^S$ is a norm.
- $\mathcal{E}(X) > 0$ for any $X \neq 0$, follows from the fact that $\langle\langle X \rangle\rangle_\alpha^S$ is a norm.
- Let us prove that $\mathcal{E}(X) \geq \psi(EX)$ with a convex function ψ on $(-\infty, \infty)$ having $\psi(0) = 0$ but $\psi(t) > 0$ for $t \neq 0$.
  Assume $\psi(x) = |x|$. Since $\mathrm{CVaR}_\alpha(X)$ is a regular measure of risk, it satisfies the following inequality: $\mathrm{CVaR}_\alpha(X) \geq EX$. Therefore,
  $$\langle\langle X \rangle\rangle_\alpha^S = \mathrm{CVaR}_\alpha(|X|) \geq E|X| \geq |EX| = \psi(EX).$$

$\square$

Proposition 9. *Let X be a random variable.*

$$\arg\min_d \langle\langle X - d \rangle\rangle_\alpha^S = \frac{1}{2}\left(q_{(1-\alpha)/2}(X) + q_{(1+\alpha)/2}(X)\right),$$

$$\min_d \langle\langle X - d \rangle\rangle_\alpha^S = \frac{1}{1-\alpha}\left(\frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X) + \frac{1-\alpha}{2}CVaR_{(1+\alpha)/2}(X) - EX\right).$$

Proof. According to Definition 2 of $\langle\langle X \rangle\rangle_\alpha^S$:

$$\langle\langle X \rangle\rangle_\alpha^S = \min_c \left\{c + \frac{1}{1-\alpha}E[|X| - c]^+\right\},$$

then

$$\min_d \langle\langle X - d \rangle\rangle_\alpha^S = \min_d \min_c \left\{c + \frac{1}{1-\alpha}E[|X - d| - c]^+\right\}.$$

Notice that optimal $c^* \geq 0$, because $c^* = q_\alpha(|X - d|)$ and $\inf |X - d| \geq 0$.
The following chain of equalities is valid

$$\begin{aligned}
&E[|X - d| - c]^+ = E(|X - d| - c)I(|X - d| - c \geq 0) = \\
=&E(X - d - c)I(X > d)I(|X - d| - c \geq 0) + E(d - X - c)I(X \leq d)I(|X - d| - c \geq 0) = \\
=&E(X - (d + c))I(X > d)I(X \geq (d + c)) + E((d - c) - X)I(X \leq d)I(X \leq (d - c)) = \\
=&E(X - (d + c))I(X \geq (d + c)) - E(X - (d - c))I(X \leq (d - c)) = \\
=&E[X - (d + c)]^+ - E(X - (d - c)) + E(X - (d - c))I(X > (d - c)) = \\
=&(d - c) + E[X - (d + c)]^+ + E[X - (d - c)]^+ - EX.
\end{aligned} \tag{16}$$

Notice that

$$(1 - \alpha)c + (d - c) = \frac{1-\alpha}{2}(d + c) + \frac{1+\alpha}{2}(d - c). \tag{17}$$

Combining (16) and (17) we have

$$\begin{aligned}
\langle\langle X - d \rangle\rangle_\alpha^S = \frac{1}{1-\alpha}\min_d \min_c \Bigg[&\frac{1-\alpha}{2}\left((d + c) + \frac{1}{(1-\alpha)/2}E[X - (d + c)]^+\right) + \\
+ &\frac{1+\alpha}{2}\left((d - c) + \frac{1}{(1+\alpha)/2}E[X - (d - c)]^+\right) - EX\Bigg].
\end{aligned} \tag{18}$$

Notice that if $Q(d,c) = G(d+c) + H(d-c)$, then

$$\min_d \min_c Q(d,c) = \min_{d,c} Q(d,c) = \min_{d,c} G(d+c) + H(d-c) =$$
$$= \min_{(d+c),(d-c)} G(d+c) + H(d-c) = \min_{d+c} G(d+c) + \min_{d-c} H(d-c). \quad (19)$$

Applying (19) to (18), we have

$$\begin{aligned}\langle\langle X-d\rangle\rangle_\alpha^S =& \frac{1}{1-\alpha} \min_{(d+c)} \left[\frac{1-\alpha}{2}\left((d+c) + \frac{1}{(1-\alpha)/2} E[X-(d+c)]^+\right)\right] + \\ &+ \frac{1}{1-\alpha} \min_{(d-c)} \left[\frac{1+\alpha}{2}\left((d-c) + \frac{1}{(1+\alpha)/2} E[X-(d-c)]^+\right)\right] - \frac{1}{1-\alpha}EX = \\ =& \frac{1}{1-\alpha}\left[\frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X) - EX\right],\end{aligned}$$

where

$$(d+c)^* = q_{(1+\alpha)/2}(X),\ (d-c)^* = q_{(1-\alpha)/2}(X),$$

which implies

$$d^* = \frac{1}{2}\left(q_{(1-\alpha)/2}(X) + q_{(1+\alpha)/2}(X)\right) = \arg\min_d \langle\langle X-d\rangle\rangle_\alpha^S.$$

□

Proposition 10. CVaR Norm Quadrangle. *Error measure $\mathcal{E}(X) = \langle\langle X\rangle\rangle_\alpha$ generates the following regular quadrangle:*

$$\begin{aligned}\mathcal{S}(X) &= \frac{1}{2}\left(q_{(1-\alpha)/2}(X) + q_{(1+\alpha)/2}(X)\right), \\ \mathcal{R}(X) &= \frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X), \\ \mathcal{D}(X) &= \frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X-EX) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X-EX), \\ \mathcal{V}(X) &= \langle\langle X\rangle\rangle_\alpha + EX, \\ \mathcal{E}(X) &= \langle\langle X\rangle\rangle_\alpha.\end{aligned}$$

Proof. It was proved in [9] that if $\mathcal{E}(X)$ is a regular measure of error, then $\lambda\mathcal{E}(X)$ is a regular measure of error for any positive λ. Since $\langle\langle X\rangle\rangle_\alpha^S$ is a regular measure of error and $\langle\langle X\rangle\rangle_\alpha = (1-\alpha)\langle\langle X\rangle\rangle_\alpha^S$, then $\langle\langle X\rangle\rangle_\alpha$ is a regular measure of error. From Proposition 9 and equality $\mathrm{CVaR}_\alpha(X) - EX = CVaR_\alpha(X-EX)$ follows that quadrangle axioms 11– 13 hold. Regularity follows from Proposition 8 and Quadrangle Theorem (see [9]). □

CVaR Norm Quadrangle from Proposition 10 is similar to Mixed-Quantile-Based quadrangle (see [9]) for

$$\alpha_1 = (1+\alpha)/2,\ \alpha_2 = (1-\alpha)/2,\ \lambda_1 = (1-\alpha)/2,\ \lambda_2 = (1+\alpha)/2. \quad (20)$$

Define

$$\mathcal{E}_{\alpha_k}(X) = E\left[\frac{\alpha_k}{1-\alpha_k}X^+ + X^-\right],\ \mathcal{V}_{\alpha_k}(X) = \frac{1}{1-\alpha_k}EX^+.$$

With parameters from (20) we obtain following Mixed-Quantile-Based quadrangle

$$\begin{aligned}
\mathcal{S}(X) &= \frac{1-\alpha}{2}q_{(1+\alpha)/2}(X) + \frac{1+\alpha}{2}q_{(1-\alpha)/2}(X),\\
\mathcal{R}(X) &= \frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X),\\
\mathcal{D}(X) &= \frac{1-\alpha}{2}\mathrm{CVaR}_{(1+\alpha)/2}(X-EX) + \frac{1+\alpha}{2}\mathrm{CVaR}_{(1-\alpha)/2}(X-EX),\\
\mathcal{V}(X) &= \min_{B_1,B_2}\left\{\lambda_1\mathcal{V}_{\alpha_1}(X-B_1) + \lambda_2\mathcal{V}_{\alpha_2}(X-B_2) | \lambda_1 B_1 + \lambda_2 B_2 = 0\right\},\\
\mathcal{E}(X) &= \min_{B_1,B_2}\left\{\lambda_1\mathcal{E}_{\alpha_1}(X-B_1) + \lambda_2\mathcal{E}_{\alpha_2}(X-B_2) | \lambda_1 B_1 + \lambda_2 B_2 = 0\right\}.
\end{aligned}$$

Note that CVaR Norm Quadrangle and Mixed-Quantile-Based quadrangle have the same deviation and risk measures. Therefore, suppose one is optimizing measure of error over some parametric family $X(\theta)$:

$$\min_{\theta} \quad \mathcal{E}_i(X(\theta)), \tag{21}$$

where $i = 1$ for error from CVaR Norm Quadrangle, and $i = 2$ for error from Mixed-Quantile-Based quadrangle. Assume that $X(\theta) = \theta_0 + Y(\tilde{\theta})$, where $\theta = (\theta_0, \tilde{\theta})$, and θ_0 is a free parameter. Define $\theta_i^* = \arg\min_\theta \quad \mathcal{E}_i(X(\theta))$. Then $\tilde{\theta}_1^* = \tilde{\theta}_2^* = \arg\min_{\tilde{\theta}} \mathcal{D}(Y(\tilde{\theta})) = \tilde{\theta}^*$. Therefore, $Y(\tilde{\theta}_1^*) = Y(\tilde{\theta}_2^*)$ and two optimal points $X(\theta_1^*)$ and $X(\theta_2^*)$ for problems (21) can be obtained from each other by adding constant shift

$$X(\theta_1^*) = (\theta_0^*)_1 + Y(\tilde{\theta}^*),\ X(\theta_2^*) = (\theta_0^*)_2 + Y(\tilde{\theta}^*),\ X(\theta_1^*) - X(\theta_2^*) = (\theta_0^*)_1 - (\theta_0^*)_2.$$

3. Negative CVaR Function

Paper [4] considers a class of functions defined similar to L_p norms, but for $p \in [0, 1)$. These functions are not norms and they are concave for some regions of the space they are defined[4]. Such norms are used in optimization problems to achieve a sparsity of a solution vector. We will define similar functions in terms of CVaR concept.

First, let us consider a classic $\mathrm{CVaR}_\alpha(X)$ for $\alpha < 0$ or $\alpha > 1$. According to CVaR definition in minimization form (see [8] or definition 2),

$$\mathrm{CVaR}_\alpha(X) = \min_c\{c + \frac{1}{1-\alpha}E[X-c]^+\}. \tag{22}$$

Let us prove that $\mathrm{CVaR}_\alpha(X) = -\infty$ for $\alpha \in (-\infty, 0) \cup (1, \infty)$.
Consider $c < \inf X$ (c may be $-\infty$), then expression under minimization in formula (22) equals to

$$c + \frac{1}{1-\alpha}E[X-c]^+ = c\frac{-\alpha}{1-\alpha} + \frac{1}{1-\alpha}EX. \tag{23}$$

[4]For $p \in [0, 1)$ there is $l_p(\mathbf{x})$ in $\mathbb{R}^n$ and $L_p(X)$ in the space of random variables. Concavity holds, for example, for region $\mathbf{x} \geq 0$ in $\mathbb{R}^n$ and for region $X \geq 0$ in the space of random variables.

If $\alpha < 0$ or $\alpha > 1$, then $\frac{-\alpha}{1-\alpha} > 0$ and the expression (23) tends to $-\infty$ for $c \to -\infty$. We see that definition (22) for $\alpha < 0$ and $\alpha > 1$ makes no sense.

Further we define *negative CVaR function.*

Definition 6. *Negative CVaR function $R_\alpha^-(X)$ is defined as follows:*
for $\alpha \in (0, 1]$,

$$R_\alpha^-(X) = \frac{1}{\alpha}(E|X| - (1-\alpha)\mathrm{CVaR}_\alpha(|X|)),$$

for $\alpha = 0$,

$$R_0^-(X) = \inf |X|.$$

$R_\alpha^-(X)$ can be interpreted as an expectation of $|X|$ in left α-tail. Note that $R_\alpha^-(X)$ is then the average quantile of the random variable $|X|$.

Definition 7. *Negative CVaR function $R_\alpha^-(X)$ is defined as follows:*
for $\alpha \in (0, 1]$,

$$R_\alpha^-(X) = \frac{1}{\alpha}\int_0^\alpha q_p(|X|)dp,$$

for $\alpha = 0$,

$$R_0^-(X) = \inf |X|.$$

Two definitions are equivalent since $E|X| = \int_0^1 q_p(|X|)dp$ and $\langle\langle X\rangle\rangle_\alpha = \int_\alpha^1 q_p(|X|)dp$.

The following proposition gives an alternative definition of the negative CVaR function, similar to the definition of CVaR norm.

Proposition 11.

$$R_\alpha^-(X) = \max_c\{c - \frac{1}{\alpha}E[|X| - c]^-\}.$$

Proof.

$$R_\alpha^-(X) = \frac{1}{\alpha}(E|X| - (1-\alpha)\mathrm{CVaR}_\alpha(|X|)) = \frac{1}{\alpha}(E|X| - \min_c\{(1-\alpha)c + E[|X| - c]^+\} =$$
$$= \frac{1}{\alpha}\max_c\{E|X| - c + \alpha c - E[|X| - c]^+\} = \max_c\{c - \frac{1}{\alpha}E[|X| - c]^-\}.$$

□

For $p \in (0, 1)$ the following inequality holds $L_p(X) \le L_1(X)$, where $L_p(X) = (E|X|^p)^{1/p}$. Since x^p is a concave function for $0 < p < 1$, using Jensen's inequality we have $E|X|^p \le (E|X|)^p$, therefore, $(E|X|^p)^{1/p} \le E|X|$. Similar statement is valid for the negative CVaR function.

Proposition 12.

$$0 \le R_\alpha^-(X) \le L_1(X) = E|X|.$$

The following proposition establishes the properties of negative CVaR function similar to the properties of CVaR norm.

Proposition 13. *The negative CVaR function satisfies the following properties:*

- $R_\alpha^-(\lambda X) = |\lambda| R_\alpha^-(X)$,

- *$R^-_\alpha(0) = 0$, but also $R^-_\alpha(X) = 0$ for some $X \neq 0$,*
- *for X, Y such that $XY \geq 0$ inequality holds*

$$R^-_\alpha(\lambda X + (1-\lambda)Y) \geq \lambda R^-_\alpha(X) + (1-\lambda)R^-_\alpha(Y),$$

- *function $R^-_\alpha(X)$ is concave in the subspace of positive random variables $X \geq 0$.*

Proof. We prove the properties one by one.

- $R^-_\alpha(\lambda X) = |\lambda| R^-_\alpha(X)$ follows from the fact, that $E|X|$ and $(1-\alpha)\mathrm{CVaR}_\alpha(|X|)$ are norms.
- Assume $X = 0$ with probability 0.5 and $X = 1$ with probability 0.5. Then, for $\alpha \in [0, 0.5]$ function $R^-_\alpha(X) = 0$.
- Notice that if $XY \geq 0$, then $|X+Y| = |X| + |Y|$, therefore

$$\max_c\{c - \frac{1}{\alpha}E[|X+Y| - c]^-\} = \max_c\{c - \frac{1}{\alpha}E[|X| + |Y| - c]^-\} \geq$$
$$\geq c_X + c_Y - \frac{1}{\alpha}E[|X| + |Y| - c_X - c_Y]^-,$$

  where $c_X = \arg\min_c\{c - \frac{1}{\alpha}E[|X| - c]^-\}$, $c_Y = \arg\min_c\{c - \frac{1}{\alpha}E[|Y| - c]^-\}$. Considering that $[x]^-$ is a concave function, we obtain

$$R^-_\alpha(X+Y) \geq c_X + c_Y - \frac{1}{\alpha}(E[|X| - c_X]^- + E[|Y| - c_Y]^-) = R^-_\alpha(X) + R^-_\alpha(Y).$$

- If $X \geq 0$ and $Y \geq 0$, then $XY \geq 0$, therefore $R^-_\alpha(X+Y) \geq R^-_\alpha(X) + R^-_\alpha(Y)$, i.e., $R^-_\alpha(X)$ is concave in subspace of positive random variables $X \geq 0$.

□

Proposition 13 states that $R^-_\alpha(X)$ is a concave function for $X \geq 0$. Notice that this property cannot be strengthened to concavity in the whole space of random variables. Consider a function $g(X)$ such that $g(X) \geq 0$, $g(0) = 0$ and $g(X) \not\equiv 0$. Assume that $g(X)$ is concave in the space of random variables. Since $g(X) \not\equiv 0$, then exists X such that $g(X) > 0$. Then

$$g(X) + g(-X) > 0 = g(0) = g(X - X),$$

which implies that $g(X)$ is not a concave function.

Let us prove some properties of CVaR negative function with respect to parameter α.

Corollary 5. *Let X be a random variable. Then*

- *$R^-_\alpha(X)$ is a continuous non-decreasing function w.r.t. α,*
- *$\alpha R^-_\alpha(X)$ is a convex non-decreasing function w.r.t. α.*

Proof. We will prove negative CVaR function properties one by one.

- Consider $Y = |X|$ and apply Proposition 5 to Y.

- $\alpha R^-_\alpha(X) = L_1(X) - (1-\alpha)\mathrm{CVaR}_\alpha(|X|) = L_1(X) - \langle\!\langle X \rangle\!\rangle_\alpha$. Since $\langle\!\langle X \rangle\!\rangle_\alpha$ is a concave, non-increasing function w.r.t. α, then $-\langle\!\langle X \rangle\!\rangle_\alpha$ is a convex non-decreasing function of α. $L_1(X)$ does not depend upon α, therefore $L_1(X) - \langle\!\langle X \rangle\!\rangle_\alpha$ is also a convex non-decreasing function of α.

□

4. Case Study

We illustrate CVaR Norm Quadrangle, see Proposition 10, with the following case study. The case study results are posted at this link[5].

Let us consider a linear regression problem with CVaR norm error. Let $\mathbf{X}$ be a $n \times d$ design matrix, where n is a number of observations, d is a number of explanatory variables. Let $\mathbf{y} \in \mathbb{R}^n$ be a vector of observations on the dependent variable. Let $\mathbf{e} \in \mathbb{R}^n$ be a vector of ones. Define extended matrix $\tilde{\mathbf{X}} = [\mathbf{e}, \mathbf{X}]$, including additional constant term. Let us consider linear regression: $\hat{\mathbf{y}} = \tilde{\mathbf{X}}\mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^{d+1}$ is a vector of parameters. We will minimize CVaR norm of vector of residuals $\mathbf{y} - \hat{\mathbf{y}}$:

$$\min_{\mathbf{a} \in \mathbb{R}^{d+1}} \quad \langle\!\langle \mathbf{y} - \tilde{\mathbf{X}}\mathbf{a} \rangle\!\rangle_\alpha. \tag{24}$$

We consider the dataset from the case study «Estimation of CVaR through Explanatory Factors with Mixed Quantile Regression»[6]. The data contains returns of the Fidelity Magellan Fund as a dependent variable. Russell Value Index (RUJ), Russell 1000 Value Index (RLV), Russell 2000 Growth Index (RUO) and Russell 1000 Growth Index (RLG) are taken as independent variables. Data include 1,264 observations.

The CVaR norm is minimized with Portfolio Safeguard [1] software package. Confidence level α in CVaR norm equals $\alpha = 0.9$. We minimized CVaR instead of CVaR norm, according to Proposition 4. Denote $\bar{\mathbf{y}} = [\mathbf{y}; -\mathbf{y}] \in \mathbb{R}^{2n}$ and $\bar{\mathbf{X}} = [\tilde{\mathbf{X}}; -\tilde{\mathbf{X}}] \in \mathbb{R}^{2n \times d}$. Proposition 4 implies

$$\langle\!\langle \mathbf{y} - \tilde{\mathbf{X}}\mathbf{a} \rangle\!\rangle^S_\alpha = \mathrm{CVaR}_{(1+\alpha)/2}(\bar{\mathbf{y}} - \bar{\mathbf{X}}\mathbf{a}).$$

Then, problem (24) is equivalently stated as follows

$$\min_{\mathbf{a} \in \mathbb{R}^{d+1}} \quad \mathrm{CVaR}_{(1+\alpha)/2}(\bar{\mathbf{y}} - \bar{\mathbf{X}}\mathbf{a}).$$

Optimization results for this problem are in Table 1.

CVaR Norm Quadrangle is a regular quadrangle, see Proposition 10. According to the Regression Theorem, see [9], the interceipt, obtained in regression, equals to the Statistic of a modified residuals. In CVaR Norm Quadrangle, Statistic equals $\mathcal{S}(X) = (q_{(1+\alpha)/2}(X) + q_{(1-\alpha)/2}(X))/2$. Denote the optimal vector of parameters obtained in regression by $\mathbf{a}^* = [c^*, \mathbf{b}^*]$, where $c^* \in \mathbb{R}$ is an optimal interceipt. According to the Regression

[5] http://www.ise.ufl.edu/uryasev/research/testproblems/advanced-statistics/cvar-norm-regression/

[6] http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/estimation-of-cvar-through-explanatory-factors-with-mixed-quantile-regression/

rlv	rlg	ruj	ruo	intercept	objective
0.578	0.484	-0.07	-0.008	-0.002	0.015

Table 1: Optimal vector of parameters and objective for linear regression with CVaR norm.

Theorem, $c^* \in \mathcal{S}(\mathbf{y} - \mathbf{X}\mathbf{b}^*)$ (we write $\in$ because, in general, quantile $q_p(X)$ is an interval, see (2), therefore $\mathcal{S}(X)$ is also an interval). At the optimal point, $c^* = -0.002$, $q^-_{0.05}(\mathbf{y} - \mathbf{X}\mathbf{b}^*) = -0.013$, $q^-_{0.95}(\mathbf{y} - \mathbf{X}\mathbf{b}^*) = 0.009$. Therefore,

$$\mathcal{S}(\mathbf{y} - \mathbf{X}\mathbf{b}^*) \approx (q^-_{0.05}(\mathbf{y} - \mathbf{X}\mathbf{b}^*) + q^-_{0.95}(\mathbf{y} - \mathbf{X}\mathbf{b}^*))/2 = (-0.013 + 0.009)/2 = -0.002 = c^*.$$

Numerical experiment confirm theoretical results for CVaR Norm Quadrangle.

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