Use of Conditional Value-at-Risk in Stochastic Programs with Poorly Defined Distributions[1]

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Abstract

On the example of the Weapon-Target Assignment (WTA) problem, we present risk management procedures for military applications that address uncertainties in distributions. In the considered formulation, the cumulative damage to the targets is maximized, which leads to Mixed-Integer Programming problems with non-linear objectives. By using a relaxation technique that preserves integrality of the optimal solutions, we developed LP formulations for the deterministic and

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two-stage stochastic WTA problems. The risk of incorrect second-stage decisions due to errors in specified distributions of the second-stage targets is controlled using the Conditional Value-at-Risk risk measure. An LP formulation for the two-stage SWTA problem with uncertainties in distributions has been developed, which produces integer optimal solutions for the first-stage decision variables, and also yields a tight lower bound for the corresponding MIP problem.

Keywords: Weapon-target assignment problem, risk management, stochastic programming

Introduction

This paper applies a general methodology of risk management in military applications [2] to a stochastic version of the Weapon-Target Assignment [1, 3, 4] problem. The approach suggested in [2] is built on the recently developed technique [5, 6] for risk management using the Conditional Value-at-Risk (CVaR) risk measure. The general framework was developed for specific military applications such as surveillance, planning, and scheduling, which require robust decision making in a dynamic, distributed, and uncertain environment. The focus of the suggested approach has been on the development of robust and efficient procedures for decision-making in stochastic framework with multiple risk factors and uncertainties in the distributions of stochastic parameters.

In a preceding paper the authors tested the developed methodology of risk management using Conditional Value-at-Risk on one-stage and two-stage stochastic versions of a Weapon-Target Assignment (WTA) problem. In the WTA problem formulation developed in [2], an optimal decision minimized the total cost of the mission, including battle damage, while ensuring that all targets are destroyed with the prescribed probability level. In such a setup, the WTA problem could easily be formulated as a linear programming (LP) problem, or integer programming (IP) problem with linear objective and constraints.

In this paper, we consider the WTA problem in a more realistic formulation, where the cumulative damage to the targets is maximized. Though this setup has some advantages over the previous formulation (for example, it allows for prioritizing the targets by importance and achieving a desirable tradeoff between assigning more weapons to high-priority targets and fewer weapons to low-priority ones), it leads to an integer programming problem with nonlinear objective and linear constraints.

In this paper, the nonlinear integer programming problem will be transformed into a (convex) linear programming problem, and the corresponding LP relaxations for deterministic and two-stage stochastic WTA problems will be developed. Further, a formulation of a two-stage stochastic WTA problem with uncertainties in the distributions of second-stage scenario parameters will be presented. We employ the Conditional Value-at-Risk risk measure [5, 6] in order to constrain the risk of generating an incorrect decision.

The paper is organized as follows. The next section introduces a generic nonlinear model for the WTA problem, and demonstrates how an LP relaxation can be constructed for the original IP problem with a nonlinear objective. Section 2 presents a fast algorithm based on an LP relaxation for the two-stage stochastic WTA problem. Section 3 considers the two-stage SWTA problem with uncertainties in distributions. A case study for the problem is presented in Section 4.

1 Deterministic Weapon-Target Assignment Problem

In the preceding paper (see also [2]) we have considered a formulation for the Weapon-Target Assignment problem where the total cost of the mission is minimized while satisfying some probabilistic constraints on the target destruction. An advantage of this formulation is the linearity of the mathematical programming problems it reduces to. Now, following [2], we consider another setup for the WTA problem, where the total damage to the targets is minimized with constraints on weapons availability. Though this formulation results in an integer programming problem with nonlinear objective and linear constraints, we will demonstrate how a linear relaxation of this problem can be developed.

First, consider a deterministic formulation of the WTA problem. Let N denote the number of targets to be destroyed, and M be the total number of weapons available. Assume that the weapons (aircraft, missiles, etc.) are identical in their capabilities of destroying targets. Let each target have a value (priority) V_j, $j = 1, \ldots, N$. Define the probability p_j of destroying the target j by a single weapon as Bernoulli trial with independent outcomes:

$$\mathsf{P}\,[\text{target } j \text{ is destroyed by a single weapon}] = p_j$$

$$\mathsf{P}\,[\text{ target } j \text{ is not destroyed by a single weapon}] = q_j = 1 - p_j$$

Introducing the decision variables x_j, $j = 1, ..., N$, as the number of weapons assigned to each target $j = 1, ..., N$, we write the deterministic WTA problem as an integer programming problem with nonlinear objective:

$$\min \quad \sum_{j=1}^{N} V_j \, q_j^{x_j} \tag{1a}$$

$$\text{s. t.} \quad \sum_{j=1}^{N} x_j = M, \tag{1b}$$

$$x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, N,$$

where $\mathbb{Z}_+$ is the set of non-negative integer numbers. The objective function (1a) represents the weighted cumulative probability of survival of the set of targets. Constraint (1b) is the munitions capacity constraint, where the equality sign means that all munitions have to be utilized during the mission. denBroeder et al. [1] showed that this problem could be solved using a greedy algorithm in $O(N + M \log N)$ time. However, we wish to extend the model and unfortunately, this strategy will no longer hold. Hence, an alternative strategy for quick solving of this problem is required.

The optimization problem (1) has a special structure: the objective of (1a) is a linear combination of univariate nonlinear functions, where the j^{th} function has argument x_j. Taking this into account, we replace every nonlinear summand $q_j^{x_j}$ in the objective of problem (1a) by a piecewise linear function $\varphi_j(x_j)$ such that

$$\forall \, x_j \in \mathbb{Z}_+ : \quad \varphi_j(x_j) = q_j^{x_j}, \quad j = 1, \ldots, N,$$

i.e., all vertices of function $\varphi_j(x_j)$ are located in integer points $x_j \in \mathbb{Z}_+$, and for integer values of the argument function $\varphi_j(x_j)$ equals $q_j^{x_j}$. The corresponding IP problem is

$$\min \quad \sum_{j=1}^{N} V_j \varphi_j(x_j) \tag{2a}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} x_j = M, \tag{2b}$$

$$x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, N.$$

Clearly, problems (1) and (2) have the same optimal solutions.

Now consider a linear relaxation of (2) obtained by relaxing the integrality of the decision variables x_j and representing the piecewise convex functions $\varphi_j(x_j)$ by the maximum of M linear functions

$$\varphi_j(x_j) = \max_{x_j} \{l_{j,0}(x_j),\, \ldots, l_{j,M-1}(x_j)\},$$

where $l_{j,m}(x_j)$ contains a linear segment of $\varphi_j(x_j)$ at $x_j \in [m, m+1]$, and correspondingly has the form

$$l_{j,m}(x_j) = q_j^m \left[(1-q_j)\,(m-x_j)+1\right], \quad j=1,...,N, \;\; m=0,...,M-1.$$

This relaxation yields the following LP formulation:

$$\min \quad \sum_{j=1}^{N} V_j\, z_j \tag{3a}$$

$$\text{s .t.} \quad z_j \geq q_j^m \left[(1-q_j)(m-x_j)+1\right], \quad j=1,\ldots,N, \quad m=0,\ldots,M-1, \tag{3b}$$

$$\sum_{j=1}^{N} x_j = M, \tag{3c}$$

$$x_j \geq 0, j=1,...,N, \tag{3d}$$

where z_j are auxiliary variables.

Proposition 1 *Linear programming problem* (3) *has an optimal solution, which is integer in variables* $x_1, \ldots, x_N$.

Proof. According to the basics of linear programming theory, an optimal solution of LP problem with bounded feasible set

$$\min_{\mathbf{y}\in\mathbb{R}^n} \{\mathbf{c}\,\mathbf{y} \mid A\,\mathbf{y} \leq 0\}$$

is achieved in extreme points (vertices) of the feasible region $A\,\mathbf{y} \leq 0$. To be a vertex of a polyhedral $A\,\mathbf{y} \leq 0$, point $\mathbf{y}^* \in \mathbb{R}^n$ has to satisfy n different equations of the system $A\,\mathbf{y} = 0$ (indeed, a point in $\mathbb{R}^n$ can only be defined as the intersection of n hyperplanes).

Therefore, to show that problem (3) has an optimal solution with integer-valued variables $x_1, \ldots, x_N$, we have to demonstrate that all vertices of the feasible region (3b)–(3d) have integer coordinates x_i, $i = 1, \ldots, N$.

Without loss of generality, we assume that the feasible set of (3) is bounded (we may impose constraints $z_i \le C$ for some large $C > 0$ without affecting the set of optimal solutions). Then, consider a feasible point

$$\mathbf{y}^* = (x_1, \dots, x_N, z_1, \dots, z_N) \in \mathbb{R}^{2N}$$

that has K non-integer components $x_{i_1}, \dots, x_{i_K}$, where $2 \le K \le N$. This point may satisfy at most $2N - K + 1$ different equalities that define the boundary of the feasible set (3b)–(3d). Indeed, each of $N - K$ integer-valued components $x_{j_0} = m_0 \in \{1, \dots, M-1\}$ and the corresponding z_{j_0} may satisfy 2 equalities (3b)

$$\begin{aligned} & z_{j_0} = q_{j_0}^{m_0} \left[(1 - q_{j_0})(m_0 - x_{j_0}) + 1 \right] \\ \text{and} \quad & z_{j_0} = q_{j_0}^{m_0 - 1} \left[(1 - q_{j_0})(m_0 - 1 - x_{j_0}) + 1 \right], \end{aligned}$$

(if $m_0 = 0$, the corresponding x_{j_0} and z_{j_0} satisfy 1 equality $x_{j_0} = 0$ from set (3d) and 1 equality $z_{j_0} = 1$ from set (3b); the case $m_0 = M$ is treated similarly). Each non-integer x_{i_k} and the corresponding z_{i_k} may satisfy at most 1 equality (3b)

$$z_{i_k} = q_{i_k}^{m} \left[(1 - q_{i_k})(m - x_{i_k}) + 1 \right].$$

Additionally, point $\mathbf{y}^*$ satisfies constraint (3c)

$$\sum_{j=1}^{N} x_j = M.$$

Thus, a feasible point with $2 \le K \le N$ non-integer components $x_{i_1}, \dots, x_{i_K}$ may satisfy at most $2(N - K) + K + 1 = 2N - K + 1 \quad < \quad 2N$ different equalities that define the boundary of the feasible region (3b)–(3d), and therefore cannot be an extreme point of the feasible region. $\square$

Proposition 2 *The set of optimal solutions of problem* (1) *coincides with the set of integer optimal values of variables* $x_1, \dots, x_N$ *in problem* (3)*. Optimal values of objective functions of* (1) *and* (3) *coincide as well.*

Proof. Observe that:

(i) The set $S \subset \mathbb{Z}_+^N$ of feasible values $x_1, \dots, x_N$ of problem (1) is a subset of the feasible region of (3).

(ii) By construction of problem (3), objective functions (1a) and (3a) take identical values on S.

(iii) Proposition 1 implies that objective function (3a) achieves global minimum on S.

From (i)–(iii) follows the statement of the Proposition 2. □

As it has been mentioned, dealing with an LP problem instead of IP problem dramatically increases the speed and robustness of computations, especially in large-scale instances.

2 Two-Stage Stochastic WTA Problem

In reality, many of the parameters of models (1) or (3) are not known with certainty. In this section, we consider the uncertain parameter is the number of targets to be destroyed.

Without lost of generality, assume that there are K *categories of targets*. The targets are categorized by their survivability and importance, so that all the targets within category k have the same probability of survival q_k and priority V_k. Assume that there are n_k detected targets and ξ_k undetected targets in each category $k = 1, \ldots, K$, where $\{\xi_k \mid k = 1, \ldots, K\}$ are random numbers. The undetected targets are expected to appear at some time in the future. Thus, we have two clearly identified stages in our problem: in the first stage one has to destroy the already detected targets, and in the second stage one must destroy the targets that might be detected beyond the current time horizon, but before some end time T. Consequently, one has to make an assignment of weapons in the first stage that allows enough remaining weapons to attack the possible second-stage targets. This type of problem is well known as two-stage recourse problem.

According to the stochastic programming approach, the uncertain number of targets at the second stage is modeled by the set of scenarios $\{(\xi_1^s, \ldots, \xi_K^s) \mid s = 1, \ldots, S\}$, where ξ_k^s is the number of the second-stage targets in category k under scenario s. Let x_{ki} be equal to the number of weapons assigned to a first-stage target i in category k, and y_{ki} be the number of weapons assigned to a second-stage target i in category k, then the recourse form of the two-stage Stochastic WTA (SWTA) problem is

$$\min \quad \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k\, q_k^{x_{ki}} + \mathsf{E}_\xi [\, Q(\mathbf{x}, \xi)] \right\} \tag{4a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} \leq M, \tag{4b}$$

$$x_{ki} \in \mathbb{Z}_+, \quad k = 1, \ldots, K, \quad i = 1, \ldots, n_k.$$

Here, the recourse function $Q(\mathbf{x}, \xi)$ is the solution to the problem

$$Q(\mathbf{x}, \xi) \;=\; \min \quad \left\{ \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} V_k\, q_k^{y_{ki}^s} \right\} \tag{5a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k + \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} y_{ki}^s = M, \tag{5b}$$

$$y_{ki}^s \in \mathbb{Z}_+, \;\; k = 1, \ldots, K, \;\; s = 1, \ldots, S, \;\; i = 1, \ldots, \xi_k^s.$$

Inequality in the first-stage munitions capacity constraint (4b) protects against weapon depletion at the first stage, whereas equality (5b) ensures full weapon utilization at the second stage.

The two-stage SWTA problem (4)–(5) can be linearized in the same way as described in the preceding section. After the linearization, the extensive form of the two-stage SWTA problem reads as

$$\begin{aligned}
\min \quad & \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} V_k u_{ki}^s \right\} \\
\text{s. t.} \quad & z_{ki} \geq (q_k)^m \left[(1 - q_k)(m - x_{ki}) + 1 \right], \\
& \quad k = 1, \ldots, K, \;\; i = 1, \ldots, n_k, \;\; m = 0, \ldots, M-1, \\
& \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} \leq M, \\
& u_{ki}^s \geq (q_k)^m \left[(1 - q_k)(m - y_{ki}^s) + 1 \right], \\
& \quad k = 1, \ldots, K, \;\; i = 1, \ldots, \xi_k^s, \;\; m = 0, \ldots, M-1, \;\; s = 1, \ldots, S, \\
& \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} + \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} y_{ki}^s = M, \quad s = 1, \ldots, S, \\
& x_{ki}, \; y_{ki}^s, \; z_{ki}, \; u_{ki}^s \geq 0.
\end{aligned} \tag{6}$$

Proposition 3 *The LP formulation* (6) *of the two-stage SWTA problem has an optimal solution, which is integer in variables* x_{ki} *and* y^s_{ki}.

Proof is analogous to that of Proposition 1. □

The objective (6) was found to unreasonably favor assignments to targets with large numbers in a category. An alternative objective, which scales assignments by n_k (ξ^s_k for the second stage) tends to provide more realistic solution:

$$\min \left\{ \sum_{k=1}^{K} (n_k)^{-1} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} (\xi^s_k)^{-1} \sum_{i=1}^{\xi^s_k} V_k u^s_{ki} \right\}.$$

3 Two-Stage WTA Problem with Uncertainties in Specified Distributions

The preceding section discussed a classic setup for the two-stage stochastic WTA problem, which assumes that the number of the second-stage targets is uncertain now, but will become known with certainty after the first-stage decision is made and before time T.

In some situations, however, the number of second-stage targets may still remain uncertain until completion of the second stage. As an example, consider a combat reconnaissance mission where a combat unit (e.g., a UAV) has first to liquidate all known (previously detected, or *first-stage*) targets, and then perform area search in order to find and destroy all targets that have not been detected yet or survived the first-stage attack (the *second-stage* targets). Suppose that at any moment of this search there is a non-zero probability of detecting a new target, hence the total number of second-stage targets remains unknown until the mission is finished. Therefore, rather than assuming a certain number of second-stage targets, it is more appropriate to deal with a probability distribution for the number of targets. This distribution may depend on the battle situation, weather conditions etc. and consequently may not be known in advance (before the beginning of the mission). However, we assume that upon completion of the first stage of the mission, the battle unit is able to determine the true distribution of the second-stage targets (for example, by analyzing the volume of jamming, etc.)

In accordance to the described setup we propose a two-stage stochastic WTA problem, where a second-stage scenario s specifies not the number of targets in a category, but a probability distribution of the number of second-stage targets. The first- and second-stage decision variables x_{ki} and y_k^s determine the number of shots to be fired at a target in category k under scenario s. Note that as the number of the second-stage targets in category k under scenario s is unknown, variables y_k^s do not contain subscript i. Thus, the second-stage decision prescribes the number of weapons to be used for each target detected in category k, given the realization of scenario s.

Consider a set of scenarios $s = 1, \ldots, S$ that specifies the family of distributions Θ_k^s for random variables ξ_k^s representing the number of the second-stage targets in categories $1, \ldots, K$:

$$P_{\Theta_k^s}[\,\xi_k^s = i\,] = \theta_{ki}^s, \quad \sum_i \theta_{ki}^s = 1.$$

For simplicity, assume that the maximum possible number of second-stage targets $I_{\max}$ is the same for all categories and all scenarios, i.e., random variables ξ_k^s have the same support $\{0, 1, \ldots, I_{\max}\}$, but different measures Θ_k^s on this support set. Also, we assume that variables ξ_k^s are independent for $k = 1, \ldots, K$.

Having an uncertain number of targets at the second stage, we have to take into account the risk of munitions depletion, and, consequently, failure to destroy all the detected targets. One way to hedge against shortage of munitions is to perform a worst-case analysis, e.g., to require that

$$\sum_{k=1}^{K} \left(\sum_{i=1}^{n_k} x_k + I_{\max} y_k^s \right) \leq M, \quad s = 1, \ldots, S. \tag{7}$$

However, constraint of type (7) may be too conservative and restricting, especially when $I_{\max}$ is a large number and the probability $\mathsf{P}[\xi_k^s = I_{\max}]$ is relatively small. Indeed, the event of encountering the largest possible number of targets in every category at the second stage should have very low probability.

Replacing $I_{\max}$ in (7) with the expected number of the second-stage targets $\mathsf{E}[\xi_k^s]$ may be also inappropriate, especially for distributions Θ_k^s with "heavy tails".

To circumvent the possibility of running out of ammo at the second stage, we propose to use a munitions constraint where the average munitions utilization in, say, 10% of "worst cases" (i.e., when too many second-stage targets are detected) does not exceed the munitions limit M. This type of constraint can be formulated using the Conditional Value-at-Risk operator:

$$\mathsf{CVaR}_\alpha \left[\sum_{k=1}^{K} \left(\sum_{i=1}^{n_k} x_k + \xi_k^s y_k^s \right) \right] \leq M, \quad s = 1, \ldots, S, \tag{8}$$

where α is the confidence level. Inequality (8) constrains the (weighted) average of munitions utilization in $(1-\alpha)\cdot 100\%$ of worst cases.

To calculate the Conditional Value-at-Risk of the function

$$f(\xi^s, \mathbf{y}^s) = \sum_{k=1}^{K} \left(\sum_{i=1}^{n_k} x_k + \xi_k^s y_k^s \right),$$

we introduce the following scenario model for the number of the second-stage targets in all categories:

$$\left\{ (\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \in \{0, 1, \ldots, I_{\max}\}^K \mid j = 1, \ldots, J \right\}, \quad J = (1 + I_{\max})^K,$$

where the collection of vectors $\{(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \mid j = 1, \ldots, J\}$ spans all possible combinations of the number of second-stage targets in categories $1, \ldots, K$. Without loss of generality[2], the probability of encountering $\tilde{\xi}_{1j}$ targets in the first category, $\tilde{\xi}_{2j}$ targets in the second category etc., under scenario s equals to

$$P[(\xi_1^s, \xi_2^s, \ldots, \xi_K^s) = (\tilde{\xi}_{1j}\tilde{\xi}_{2j} \ldots, \tilde{\xi}_{Kj})] = \prod_{k=1}^{K} P[\xi_k^s = \tilde{\xi}_{kj}] = \prod_{k=1}^{K} \theta^s_{k,\tilde{\xi}_{kj}} = \pi_j^s. \tag{9}$$

Note that scenario set $\{(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \mid j = 1, \ldots, J\}$ is the same for all scenarios $1, \ldots, S$. A scenario s assigns probability π_j^s to each vector $(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj})$ from this collection. Naturally,

$$\sum_{j=1}^{J} \pi_j^s = 1, \quad s = 1, \ldots, S.$$

Thus, the two-stage stochastic WTA problem with uncertainties in distri-

[2] The necessary condition of stochastic independence requires variables $\xi_1^s, \ldots, \xi_K^s$ to be mutually uncorrelated, which imposes a limitation on using the multiplicative rule in (9). Expression (9) for probabilities π_j^s may still be used if we assume that scenario s defines a joint probability distribution for number of targets over all categories.

butions reads as

$$\begin{aligned} \min \quad & \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k (q_k)^{x_{ki}} + \mathsf{E}_\Theta \left[\mathsf{E}_\xi [Q(\mathbf{x}, \xi)] \right] \right\} \\ \text{s. t.} \quad & \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k \le M, \\ & x_{ki} \in \mathbb{Z}_+, \quad k = 1, \dots, K, \quad i = 1, \dots, n_k, \end{aligned} \tag{10}$$

where the recourse function Q equals to

$$\begin{aligned} Q(\mathbf{x}, \xi^s) \; = \; \min_{y} \quad & \sum_{k=1}^{K} V_k (q_k)^{y_k^s} \\ \text{s. t.} \quad & \mathsf{CVaR}_\alpha \left[\sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k + \sum_{k=1}^{K} \xi_k^s y_k^s \right] \le M, \\ & y_k^s \in \mathbb{Z}_+, \quad k = 1, \dots, K, \quad s = 1, \dots, S. \end{aligned} \tag{11}$$

The linearized extensive form of the recourse problem (10)–(11) is as follows:

$$\min \quad \left\{ \sum_{k=1}^{K} (n_k)^{-1} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} V_k u_k^s \right\} \tag{12a}$$

$$\text{s. t.} \quad z_{ki} \ge (q_k)^m \left[(1 - q_k)(m - x_{ki}) + 1 \right], \quad k = 1, \dots, K, \quad , i = 1, \dots, n_k, \quad m = 0, \dots, M-1, \tag{12b}$$

$$\sum_{k=1}^{K} \sum_{i=1}^{n_k} n_k x_{ki} = M_1, \tag{12c}$$

$$u_k^s \ge (q_k)^m \left[(1 - q_k)(m - y_k^s) + 1 \right], \quad k = 1, \dots, K, \quad m = 0, \dots, M-1, \quad s = 1, \dots, S, \tag{12d}$$

$$w_j^s \ge M_1 + \sum_{k=1}^{K} \tilde{\xi}_{kj} y_k^s - \zeta^s, \quad s = 1, \dots, S, \quad j = 1, \dots, J, \tag{12e}$$

$$\zeta^s + (1 - \alpha)^{-1} \sum_{j=1}^{J} \pi_j^s w_j^s \le M, \tag{12f}$$

$$M_1 \le M, \tag{12g}$$

$$x_{ki},\, y_k^s,\, z_{ki},\, u_k^s,\, w_j^s \ge 0; \quad \zeta^s \in \mathbb{R}, \quad M_1 \in \mathbb{Z}_+. \tag{12h}$$

In accordance to the general principle of diversification of risk, the CVaR constraint (8), represented by inequalities (12e)–(12f), does not allow for an integer-valued optimal solution for the second-stage decision variables, which in turn precludes integrality of the first-stage decision. However, it is possible to achieve an integer-valued solution at the first stage by introducing an integer variable M_1 in constraints (12c), (12e), or, equivalently, solving $M+1$ problems (12), where $M_1 \in \{0, \ldots, M\}$ is a parameter.

If the primary interest is the first-stage weapon-target assignment, and the integrality of the second-stage decision is not critical, the linear programming formulation (12) yields a fast algorithm for the two-stage stochastic WTA problems with uncertainties. The corresponding optimal solution may be regarded as the one that allows for destruction of the detected targets while preserving sufficient resources for destroying the possible future targets.

Integer-valued optimal solution of (10)–(11) is achieved by declaring the variables y_k^s as integer. In this case, solution of the LP problem (12) represents a lower bound for the optimal solution of the MIP problem (10)–(11).

4 Case Study

In this section we test the developed algorithms for the stochastic WTA problem, and compare solutions of the classic two-stage stochastic WTA problem (6) and the two-stage stochastic WTA problem with uncertainties in distributions (12). However, reporting and discussing the optimal solutions of the considered problems would be a rather unwieldy task even in small instances, since formulations (6) and (12) contain individual decision variables for each first-stage and, in case (6), second-stage target. Therefore, to make the presentation compact, we consider and report solutions of the following MIP analogs of problems (6) and (12) correspondingly:

$$\min \quad \left\{ \sum_{k=1}^{K} V_k(q_k)^{x_k} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} V_k(q_k)^{y_k^s} \right\} \tag{13a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} n_k x_k \le M, \tag{13b}$$

$$\sum_{k=1}^{K} (n_k x_k + \xi_k^s y_k^s) = M, \quad s = 1, \ldots, S, \tag{13c}$$

$$x_k, y_k^s \in \mathbb{Z}_+, \tag{13d}$$

and

$$\min \quad \left\{ \sum_{k=1}^{K} V_k(q_k)^{x_k} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} V_k(q_k)^{y_k^s} \right\} \tag{14a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} n_k x_k \leq M, \tag{14b}$$

$$w_j^s \geq \sum_{k=1}^{K} (n_k x_k + \tilde{\xi}_{kj} y_k^s) - \zeta^s, \quad s = 1, \ldots, S, \quad j = 1, \ldots, J, \tag{14c}$$

$$\zeta^s + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j^s w_j^s \leq M, \tag{14d}$$

$$x_k,\ y_k^s \in \mathbb{Z}_+; \quad w_j^s \geq 0; \quad \zeta^s \in \mathbb{R}. \tag{14e}$$

In the formulation (13) and (14) variables x_k and y_k^s represent the number of weapons used for every first-stage and second-stage target in a category k, respectively. In fact, the linear programming formulations (6) and (12) are the linearizations of the MIP formulations (13) and (14) correspondingly. Solution of the LP problem (12) was used as the lower bound for the MIP problem (14).

The developed models for the stochastic Weapon-Target Assignment problems were implemented in C++ and we used CPLEX 7.0 solver to find solutions to the corresponding MIP and LP problems. The setup of the test problem was as follows:

- There are two categories of targets ($K = 2$)
- Number of weapons $M = 25$
- Targets in the second category are more important than those in the first category: $V_1 = 1$, $V_2 = 3$
- Also, the second-category targets are harder to kill: $q_1 = 0.20$, $q_2 = 0.36$
- The numbers of detected targets in each category are $n_1 = 3, \quad n_2 = 1$
- There are 3 scenarios for distribution of second-stage targets in each category ($S = 3$), and all the distributions have support $\{0, 1, \ldots, 5\}$ ($I_{\max} = 5$); at each scenario, the expected number of the second-stage targets in categories I and II is given in Table 1

- The expected values presented in Table 1 were used as the scenario information for the classical two-stage stochastic WTA problem (13) and (6).

Table 1: The expected values for the number of the second-stage targets in two categories for scenarios $s = 1, 2, 3$.

	$s=1$	$s=2$	$s=3$
Category I	2	2	3
Category II	1	3	4

Table 2 compares the optimal solutions of two-stage stochastic WTA problem in different formulations. The values in columns corresponding to $\alpha = 0.01, 0.50, 0.99$ present the optimal solution of the stochastic WTA problem (14) with the indicated confidence level α in the CVaR constraint. Column with $\alpha = 0.00$ displays the optimal solution of the classical two-stage SWTA problem (13), where the scenario values for the number the second-stage targets were taken from Table 1.[3] The last column, $\alpha = 1.00$, displays the solution of problem (14), where the CVaR constraint is replaced by the "worst-case" constraint (5a).

Table 2: Solution of the MIP problem and problem (14a) for different values of confidence level α. The case $\alpha = 1.00$ corresponds to the case when CVaR constraints in (14a) is replaced with (7). Column $\alpha = 0.00$ presents the solution of problem (6).

		$\alpha=0.00$		$\alpha=0.01$		$\alpha=0.50$		$\alpha=0.99$		$\alpha=1.00$	
Category		I	II	I	II	I	II	I	II	I	II
First stage		2	4	2	4	2	6	2	4	2	4
Second stage	$s=1$	4	7	4	6	2	4	1	2	1	2
	$s=2$	3	3	2	3	2	2	1	2	1	2
	$s=3$	1	3	2	2	1	2	1	2	1	2

[3] When α approaches zero, Conditional Value-at-Risk of a stochastic value becomes equal to its expectation.

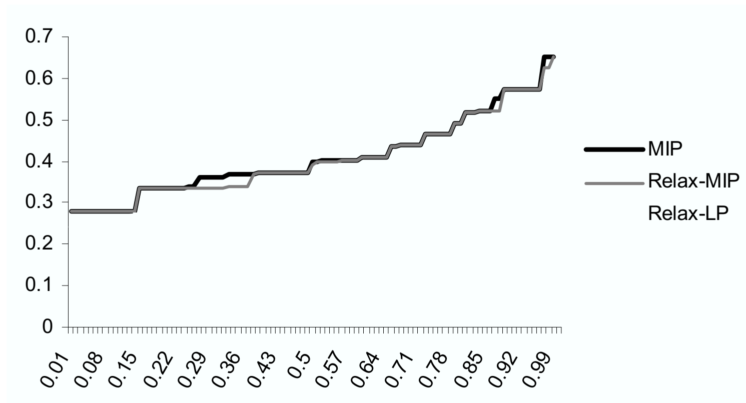

Figure 1: MIP curve represent the optimal objective values of the MIP problem (14); Relax-LP curve correspond to the solution of (12) with all continuous variables; Relax-MIP points show the solution of the problem (12) where the second-stage decision variables y_k^s are integer.

Table 2 shows that the classic two-stage stochastic WTA problem that uses only the expected values of the second-stage target distributions yields the solution, which is quantitatively very similar to the solution of the two-stage stochastic WTA problem with uncertainties in distributions (14) for low confidence level α in CVaR constraint. These solutions allocate ample munitions for destruction of each of the encountered second-stage targets, which may lead to several munitions shortage at worst-case scenarios. As confidence level α increases, and, correspondingly, worst-case scenarios gain more weight in the optimization problem (14), the number of weapons to be reserved for each of the second-stage targets decreases (such solutions are more robust with respect to encountering “more-than-expected” second-stage targets). Finally, at high confidence levels ($\alpha = 0.99$), problem (14) produces very conservative solution that coincides with the solution obtained by replacing the CVaR constraint by the “worst-case” constraint (7).

Figure 1 displays the degradation of the optimal objective value of the stochastic WTA problem with increasing of the confidence level α. On this figure, the MIP curve corresponds to the optimal objective value of problem (14), Relax-LP curve shows the optimal objective of the LP problem (12), and Relax-MIP curve displays the objective of the problem (12) with integer variables y_k^s. Evidently, solution of the LP problem (12) presents a good lower bound for both the solutions of problem (14) and MIP problem (13).

Conclusions

We have considered several formulations for the two-stage stochastic Weapon-Target Assignment problem, where the cumulative damage to the targets is maximized. In the original formulation, the WTA problem is presented as an integer programming problem with a nonlinear objective. We have developed a linear relaxation for both the deterministic and traditional two-stage stochastic formulations of the WTA problem. In the scope of the two-stage stochastic WTA problem we considered the setup where probability distributions for the number of the second-stage targets are unknown. To control the risks of generating an incorrect decision due to the uncertainties in distributions, we with the uncertainties in distributions, we applied the risk-management techniques based on the Conditional Value-at-Risk (CVaR) risk measure. For the two-stage SWTA problem with uncertainties in distributions we developed an LP formulation that yields tight lower bound for the corresponding MIP problem, and has integer optimal solution for the first-stage variables.

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