CASH FLOW MATCHING PROBLEM WITH CVaR CONSTRAINTS: A CASE STUDY WITH PORTFOLIO SAFEGUARD

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Abstract

Bond immunization is an important topic in portfolio management. This project demonstrates a scenario based optimization framework for solving a cash flow matching problem where the time horizon of the liabilities is longer than the maturities of available bonds and the interest rates are uncertain. Bond purchase decisions are made each period to generate cash flow for covering the obligations in future. Since cash flows depend upon future prices of bonds, which are not addressed precisely, some risk management approach needs to be used to handle uncertainties in cash flows. We use Conditional Value-at-Risk (CVaR) to measure risk of shortfalls. We calculate the optimal bond positions every period such that initial cost of the cash flow matching problem is minimized. We use Portfolio Safeguard (PSG) decision support tool to solve the optimization problem.

1 Introduction

Bond immunization, including duration matching and cash flow matching, is a topic which has been studied for a long time in financial applications. Given the stream of

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liabilities, the objective of cash flow matching problem is to match the future cash flow stream of liabilities with some asset cash flows.

Let p_0 be the vector whose components are the initial prices of the bonds available in market, x_0 be the vector whose components are the initial purchasing amounts of the bonds, l_t and c_t be the liability and payment vector at period t respectively. The classic cash flow matching problem can be formulated as follows:

$$\begin{aligned} &\min_{x_0}\ p_0^T x_0 \\ &\text{subject to} \\ &\quad l_t - c_t^T x_0 \le 0, \quad t = 1, ..., N, \\ &\qquad\ -x_0 \le 0. \end{aligned} \tag{1}$$

In the classic cash flow matching problem (1), the liabilities have a shorter (or equal) time horizon comparing to the maturities of the bonds available in the market. So the resulting portfolio can be truly immunized to changes in interest rates.

However, if the liabilities have a longer time horizon when compared to the maturities of the bonds, the problem may not have a solution. Hiller and Eckstein [3], Zenios [9], and Consigli and Dempster [2] proposed stochastic programming-based approaches for the cash flow matching problem.

Iyengar and Ma [5] used Conditional Value-at-Risk (CVaR)[8] to formulate and solve the bond cash matching problem. Considered bonds of various maturities pay coupons as well as face values in different time periods. But unlike classic cash flow matching problem, the liabilities have a longer time horizon than maturities of the bonds currently available in market. Hence some purchases of bonds should be made in future periods. Moreover, it is considered that the future prices of the bonds are uncertain. Therefore, the resulting portfolio cannot be truly immunized to changes of interest rates. The objective of the model constructed in the article is to design portfolios providing the necessary cash flow with high probability and to minimize the total initial portfolio cost.

This paper is based on the model suggested by Iyengar and Ma [5]. Section 2 describes the mathematical problem formulation. Section 3 describes the numerical implementation of the optimization model in detail. We conduct calculations with the new financial optimization software, Portfolio Safeguard (PSG) [1]. PSG provides compact and intuitive problem formulations and codes. We used a set of bond prices scenarios provided for benchmarking purposes by Professor Ken Kortanek.

2 Problem Description and Notations

The considered cash flow matching problem has longer duration than the maturities of the bonds available in market at initial time. Hence purchasing bonds in the initial period could not generate a cash flow with long enough duration to cover the stream of liabilities. Purchases of bonds in later periods are necessary. Therefore the first issue to address is to generate the future prices appropriately. This project is based on the interest rate scenarios generated by K. Kortanek with Hull and White [4] interest rate model. You can read Rebonato [6] for a comprehensive introduction to the calibration of interest rate models.

Various researchers have studied CVaR, sometimes under different names (expected shortfall, Tail-VaR). We will use notations from Rockafellar and Uryasev [7].

Suppose random variable X determines some financial outcome such as future loss (or return with minus sign) of some investment. By definition, Value-at-Risk at level α is the α-quantile of the distribution of X:

$$\mathrm{VaR}_\alpha(X) = \inf\{z | F_X(z) > \alpha\},$$

where F_X denotes the probability distribution function of random variable X.

Conditional Value-at-Risk (CVaR) for continuous distributions equals to the expected loss exceeding VaR:

$$\mathrm{CVaR}_\alpha(X) = E[X | X \geq VaR_\alpha(X)].$$

This formula underlies the name of CVaR as conditional expectation. For the general case the definition is more complicated, and can be found in Rockafellar and Uryasev [8].

In this paper we find the optimal portfolio by minimizing the cost while controlling the downside risk at a specified confidence level. Similar to Iyengar and Ma [5] we used Conditional Value-at-Risk (CVaR) to control the risk by requesting that CVaR is less than or equal to zero.

There are N periods, $t = \{1, ..., N\}$, where t is the index of time periods. In the case study we considered the case with 120 time steps, where a time step equals to 0.5 years.

We assume that there is a constant collection of bonds available for investment at each time moment, and there are M bonds in this collection, $j \in \{1, ..., M\}$. We denote by p_0^j the price of bond j at time step 0. $c_{t,l}^j$ is the cash flow at time step l from bond j purchased at time step t, $t \in \{1, ..., l-1\}$, $l \in \{1, ..., N\}$, $j \in \{1, ..., M\}$, and $c_t = (c_t^1, ..., c_t^M)^T$ is the column vector obtained by transposing the row vector $(c_t^1, ..., c_t^M)$, $t \in \{1, ..., N\}$. In the considered case study, 11 bonds are available, and Table 1 shows the maturities, coupon rates and the initial prices of those bonds.

There are K simulations, $k = \{1, ..., K\}$ of bond prices. Every simulation provides prices of all bonds for all time periods. We have done numerical calculations with 200 simulations (the paper [5] considered 1000 simulations).

$p_t^{k,j}$ is the price of bond j at time step t during simulation k, $k \in \{1, ..., K\}$, $j \in \{1, ..., M\}$ and $t \in \{1, ..., N\}$.

p_t denotes a random price vector whose components are the prices of the bonds available in the market at time t, $t \in \{1, ..., N\}$.

Let l_t be the liability at time step t, $t \in \{1, ..., N\}$. Similar to Iyengar and Ma [5]

Table 1: Details of Treasury Bonds.

Bond Index	Name	Maturity	Coupon Rate (%)	Current Price
1	T-bill	0.5	0	95.8561
2	T-note	1	4.5	96.1385
3	T-note	2	4.5	92.6873
4	T-note	3	4.5	89.5784
5	T-note	4	4.5	86.7610
6	T-note	5	4.5	84.1959
7	T-bond	10	5.0	77.5948
8	T-bond	15	5.0	71.9232
9	T-bond	20	5.0	68.1357
10	T-bond	25	5.0	65.5990
11	T-bond	30	5.0	63.8989

we consider the following stream of liabilities:

$$l_t = \begin{cases} 100 & \text{if } t/2 = 0, ..., 10, \\ 110 - 2.2 \times (\frac{t}{2} - 10) & \text{if } t/2 = 11, ..., 60, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose x_t^j is the number of units of bond j purchased at time step t, $j \in \{1, ..., M\}$, $t \in \{0, ..., N\}$, and $x_t = (x_t^1, ..., x_t^M)^T$ is the column vector obtained by transposing the row vector $(x_t^1, ..., x_t^M)$, $t \in \{0, ..., N\}$. Optimal values of these variables will be determined by solving an optimization problem.

Let L_t be the underperformance of replicating portfolio versus the liability at the end of time period t, i.e.

$$L_t = l_t + p_t^T x_t - \sum_{s=0}^{t-1} c_{s,t}^T x_s, \quad t = 1, ..., N.$$

Let us denote by CVaR_max_risk the following function:

$$\text{CVaR\_max\_risk}_\alpha(L_1, ..., L_N) = \text{CVaR}_\alpha(\max_{0 \le t \le N} L_t). \tag{2}$$

Following Iyengar and Ma [5] we formulate the cash flow matching problem:

$$\begin{gathered} \min_{x_0} \; p_0^T x_0 \\ \text{subject to} \\ \text{CVaR\_max\_risk}_\alpha(L_1, ..., L_N) \le 0, \quad t = 1, ..., N, \\ -x_t \le 0, \quad t = 0, ..., N. \end{gathered} \tag{3}$$

Problem (3) reduces to the classic one if we force $x_t = 0$ for $t > 0$. The rationale for imposing the CVaR constraint is to manage the downside risk with confidence level measured by α. The protection from the downside risk increases as the value of α increases. The formulation also manages the implicit reinvestment risk in the reinvestment strategy x_t related to the uncertainties in the bond prices at time t for $t > 0$.

3 Numerical Implementation

The problem was optimized numerically with the financial optimization software, PSG. We used PSG because it contains preprogrammed functions needed for solving the considered optimization problems. With the preprogrammed functions, the PSG code is quite simple and transparent.

3.1 Objective

We used linear function (included in PSG library) to construct the objective of the optimization problem. According to the PSG specifications, we provided the matrix `matrix_0` containing the initial prices of the bonds. We denote this linear function by Linear_1(`matrix_0`). Matrix `matrix_0` contains 2 rows (including the header line). Column `id` contains 1, column Scenario_Benchmark (SB) contains 0, and column `x0_j` contains initial price of bond j. The matrix `matrix_0` is defined as follows:

id	x0_1	x0_2	⋯	x0_11	SB
1	p_0^1	p_0^2	⋯	p_0^{11}	0

3.2 Constraint

We used $\text{CVaR\_max\_risk}_\alpha(L_1, ..., L_N)$ (included in PSG library) to construct the constraint of the optimization problem. According to PSG specifications, we need to provide the confidence level α and matrices `matrix_1`, ..., `matrix_120` containing the simulated future prices of bonds, liabilities and the amounts of receipts from period 1 to period 120 to define the function CVaR_max_risk. The confidence level is set to $\alpha = 0.9$. We denote the considered CVaR_max_risk function (see (2)), as follows:

$$\text{CVaR\_max\_risk\_1}(\alpha, \texttt{matrix\_1}, \texttt{matrix\_2}, ..., \texttt{matrix\_N})\,.$$

In the n-th matrix, named `matrix_n`, column `id` is the counter of simulations, column SB contains the amount of the liability at n-th period, column `xt_j` contains the n-th period amounts of receipt from bond j purchased at time t when $t < n$, where n is the counter of the periods, and for $t = n$ the column `xt_j` contains the negatives of the simulated prices of bond j at time t. Therefore, the matrix `matrix_n` is arranged as follows:

id	x0_1	x0_2	$\cdots$	x0_11	x1_1	$\cdots$	xt_j	$\cdots$	xn − 1_11	xn_1	xn_2	$\cdots$	xn_11	SB
1	$c^1_{0,n}$	$c^2_{0,n}$	$\cdots$	$c^{11}_{0,n}$	$c^1_{1,n}$	$\cdots$	$c^j_{t,n}$	$\cdots$	$c^{11}_{n-1,n}$	$p^1_{n,1}$	$p^2_{n,1}$	$\cdots$	$p^{11}_{n,1}$	l_n
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
200	$c^1_{0,n}$	$c^2_{0,n}$	$\cdots$	$c^{11}_{0,n}$	$c^1_{1,n}$	$\cdots$	$c^j_{t,n}$	$\cdots$	$c^{11}_{n-1,n}$	$p^1_{n,200}$	$p^2_{n,200}$	$\cdots$	$p^{11}_{n,200}$	l_n

The n-th matrix `matrix_n` has $11 \times (n+1) + 2$ columns, and 201 rows (including the first row as the title line). In the column `xt_j` of `matrix_n` ($t < n$), the elements are equal to:

$$c(t,j) = \begin{cases} 100 & \text{if } j = 1 \text{ and } t = n-1, \\ 2.25 & \text{if } 2 \le j \le 6 \text{ and } n - 2 \times (j-1) + 1 \le t \le n-1, \\ 102.25 & \text{if } 2 \le j \le 6 \text{ and } t = n - 2 \times (j-1), \\ 2.5 & \text{if } 7 \le j \le 11 \text{ and } n - 10 \times (j-5) + 1 \le t \le n-1, \\ 102.5 & \text{if } 7 \le j \le 11 \text{ and } t = n - 10 \times (j-5), \\ 0 & \text{otherwise.} \end{cases}$$

where $j = \mod(m-2, 11) + 1$, $t = [(m-2)/11]$, and m is the counter of columns.

To improve the efficiency of PSG calculations we excluded zero columns (i.e., columns with all zero elements). The previous matrices were provided before excluding the zero columns.

For $t = n$, the 11 columns include the simulated prices. Besides, the last column elements are equal to:

$$\text{SB}(n) = \begin{cases} 100 + \frac{n}{2} & \text{if } \mod(n,2) = 0 \text{ and } 0 \le n \le 20, \\ 110 + 2.2 \times (\frac{n}{2} - 10) & \text{if } \mod(n,2) = 0 \text{ and } 20 < n \le 120, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Problem Statement in PSG Format

The optimization problem statement in PSG format is as follows:

```
problem: problem_Cash_flow_matching_BIG, type = minimize
objective: objective_cvar_max_risk
linear_1(matrix_0)
constraint: constraint_cvar_max, upper_bound = 0
cvar_max_risk_1(0.9,matrix_1,...,matrix_120)
box_of_variables: lowerbounds = 0
Solver: VAN, precision = 5
```

The objective name is `objective_cvar_max_risk` and the constraint name is `constraint_cvar_max`. These names are not used by the current code and they are provided for importing purposes to PSG Shell Environment. The box constraints on variables are defined in `box_of_variables`. We have non-negativity constraints on variables, therefore, `lowerbounds` = 0 .

3.4 Implementation with PSG Run-File Environment

PSG Run-File Environment provides a simple possibility to optimize a problem. You should put the file with the problem statement and the files with matrices containing data to some folder. You should run the executable file Run-File.exe and obtain the optimization results placed in the folder containing the optimization problem data.

Here is the content of the resulting solution file:

```
Problem:  solution_status = optimal
Timing:  Data_loading_time=21.87,Preprocessing_time=0.28,Solving_time=224.8
Variables:  optimal_point = point_problem_Cash_flow_matching_BIG
Objective:  objective_cvar_max_risk = 1172.700073
Constraint:  constraint_cvar_max = 2.969229e-006 [2.969229e-006]
Function:  linear_1(matrix_0) = 1.172700e+003
Function:  cvar_max_risk_1(0.9e+00,matrix_1,...,matrix_120) = 2.969229e-006
```

The total problem solving time equals to 224.8 seconds. The objective function value (which is the cost of bonds purchased at the initial period) equals to 1172.7 . The risk constraint value equals to 2.9692e-6. According to the problem statement the constraint value should be non-positive. We see that the constraint is satisfied with the high precision.

The resulting solution vector (positions in bonds at different time periods) is shown in Figure 1.

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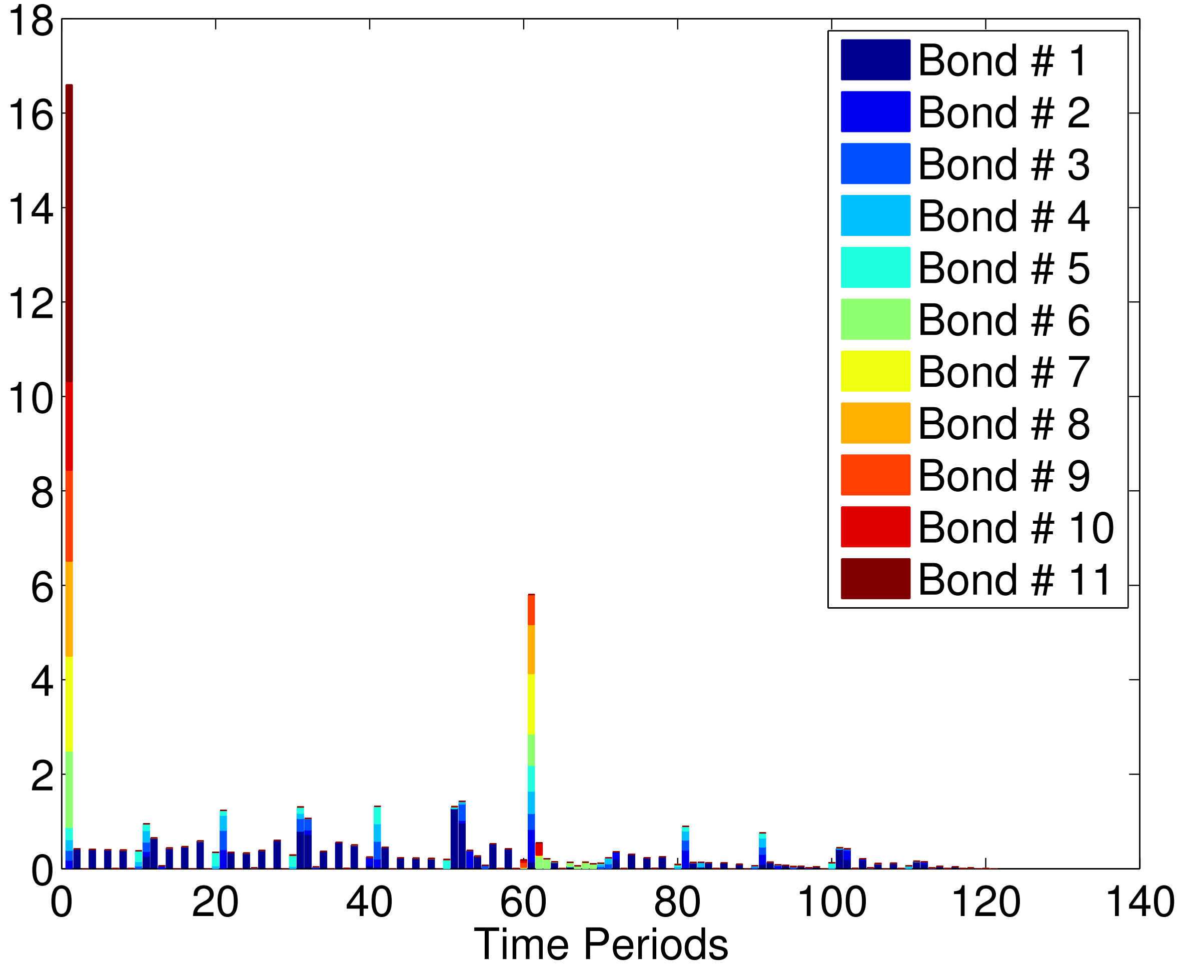

Figure 1: Optimal Bond Portfolio Strategy over Time. For instance, the first bar shows the portfolio at time period 0; and the top brown sub-bar in this first bar shows the amount of bond 11 purchased at time period 0 (top brown sub-bar begins around 10.4 and ends around 16.7 i.e., the position of bond 11 in the initial period approximately equals to 6.3).