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Credit risk optimization with Conditional Value-at-Risk criterion

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Abstract. This paper examines a new approach for credit risk optimization. The model is based on the Conditional Value-at-Risk (CVaR) risk measure, the expected loss exceeding Value-at-Risk. CVaR is also known as Mean Excess, Mean Shortfall, or Tail VaR. This model can simultaneously adjust all positions in a portfolio of financial instruments in order to minimize CVaR subject to trading and return constraints. The credit risk distribution is generated by Monte Carlo simulations and the optimization problem is solved effectively by linear programming. The algorithm is very efficient; it can handle hundreds of instruments and thousands of scenarios in reasonable computer time. The approach is demonstrated with a portfolio of emerging market bonds.

1. Introduction

Risk management is a core activity in asset allocation conducted by banks, insurance and investment companies, or any financial institution that evaluates risks. This paper examines a new approach for minimizing *portfolio credit risk*. Credit risk is the risk of a trading partner not fulfilling their obligations in full on the due date or at any time thereafter. Losses can result both from counterparty default, and from a decline in market value stemming from the credit quality migration of an issuer or counterparty. Traditionally used tools for assessing and optimizing market risk assume that the portfolio return-loss is normally distributed. With this assumption, the two statistical measures, mean and standard deviation, can be used to balance return and risk. The optimal portfolio is selected on the “efficient frontier”, the set of portfolios that have the best mean-variance profile [10]. In other words, this is the set of Pareto optimal points with two conflicting criteria: mean and variance.

Although this traditional approach has proven to be quite useful in various applications, it is inadequate for credit risk evaluations because credit losses are characterized by a large likelihood of small earnings, coupled with a small chance of losing a large amount of the investment. Thus the loss distributions are, in general, heavily skewed and standard optimization tools developed for market risk are inadequate. This, together

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with the lack of historical data to estimate credit correlations, poses significant modeling challenges, compared to market risk modeling and optimization.

To cope with skewed return-loss distributions, we consider *Conditional Value-at-Risk* (CVaR) as the risk measure. This measure is also called Mean Excess Loss, Mean Shortfall, or Tail VaR. By definition, β-CVaR is the expected loss exceeding *β-Value-at-Risk* (VaR), i.e., it is the mean value of the worst $(1-\beta)*100\%$ losses. For instance, at $\beta = 0.95$, CVaR is the average of the 5% worst losses. CVaR is a currency-denominated measure of significant undesirable changes in the value of the portfolio.

CVaR may be compared with the widely accepted VaR risk performance measure for which various estimation techniques have been proposed, see, e.g., [7,15]. VaR answers the question: what is the maximum loss with the confidence level $\beta * 100\%$ over a given time horizon? Thus, its calculation reveals that the loss will exceed VaR with likelihood $(1-\beta)*100\%$, but no information is provided on the amount of the excess loss, which may be significantly greater. Mathematically, VaR has serious limitations. In the case of a finite number of scenarios, it is a nonsmooth, nonconvex, and multiextremum function [11] (with respect to positions), making it difficult to control and optimize. Also, VaR has some other undesirable properties, such as the lack of sub-additivity [1,2].

By contrast, CVaR is considered a more consistent measure of risk than VaR. CVaR supplements the information provided by VaR and calculates the quantity of the excess loss. Since CVaR is greater than or equal to VaR, portfolios with a low CVaR also have a low VaR. Under quite general conditions, CVaR is a convex function with respect to positions [17], allowing the construction of efficient optimization algorithms. In particular, it has been shown in [17], that CVaR can be minimized using *linear programming (LP) techniques*. The *minimum CVaR approach* [17] is based on a new representation of the performance function that allows the simultaneous calculation of VaR and minimization of CVaR. A simple description of the approach for minimization of CVaR and optimization problems with CVaR constraints can be found in [21]. Since CVaR can be minimized by LP algorithms, a large number of instruments and scenarios can be handled. LP techniques are routinely used in financial planning applications, see, for instance, paper [4] which applies a penalty approach to controlling risks. However, comparing to the paper [4], we directly handle the quantile-based constraints with a specified confidence level.

Similar measures as CVaR have been earlier introduced in the stochastic programming literature, although not in financial mathematics context. The conditional expectation constraints and integrated chance constraints described in [14] may serve the same purpose as CVaR. The reader interested in other applications of optimization techniques in finance area can find relevant papers in [22].

In credit risk evaluations, we are interested in the losses experienced in the event of counterparty default or credit quality migration in the course of a day, a year or other standardized period. Several approaches are available for estimating credit risk [5,6,13, 18,19]. Probably, the most influential contribution in this field has been J.P. Morgan's CreditMetrics methodology [5].

Bucay and Rosen [3] conducted a case study and applied the CreditMetrics methodology to a portfolio of bonds issued in emerging markets. The portfolio consists of 197 bonds, issued by 86 obligors in 29 countries. Bond maturities range from a few months to 98 years and the portfolio duration is approximately five years. The mark-to-market value of the portfolio is \$8.8 billion. Mausser and Rosen [12] applied the regret optimization framework to minimize the credit risk of this portfolio. In this paper, we analyzed the same portfolio and minimized the credit risk using the Minimum CVaR approach. We have used the dataset of Monte Carlo scenarios generated at Algorithmics Inc. [3, 12]. Using the CreditMetrics methodology, a large number of scenarios is calculated based on credit events such as defaults and credit migrations. By evaluating the portfolio for each scenario, the loss distribution is generated. Monte Carlo simulation tools are widely used for evaluating credit risk and other risks of portfolios containing non-linear instruments, such as options (see, for instance, [11, 17, 15]).

The optimization analysis conducted for the portfolio of bonds may be briefly summarized as follows. First, each single position (a position corresponds to an obligor) is optimized. This provides the best hedge, i.e. the position that gives the minimum CVaR when holding the other positions fixed. Then, all portfolio positions are simultaneously adjusted to minimize the portfolio CVaR under trading and budget constraints. In this framework, the CVaR-return efficient frontier of the portfolios is also calculated. As mentioned above, the minimization of CVaR automatically leads to a significant improvement of VaR.

The remainder of this paper is organized as follows. First, we present the minimum CVaR approach [17]. Then, following [3, 12], we describe the bond portfolio. In this section, we give a brief description of the CreditMetrics methodology used to calculate the portfolio loss distribution. Then, we describe the optimization model and its parameters. Finally, we present the analysis and concluding remarks.

2. Minimum CVaR approach

This section describes the approach to minimization of CVaR. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to R$ be the loss function which depends upon the control vector $\mathbf{x} \in \mathbb{R}^n$ and the random vector $\mathbf{y} \in \mathbb{R}^m$. We use bold face for the vectors to distinguish them from scalars. We consider that the random vector $\mathbf{y}$ has the probability distribution function $p : \mathbb{R}^m \to \mathbb{R}$. However, the existence of the density is not critical for the considered approach, this assumption can be relaxed. Denote by $\Psi(\mathbf{x}, a)$ the probability function

$$\Psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y})\, d\mathbf{y} \,, \tag{1}$$

which, by definition, is the probability that the loss function $f(\mathbf{x}, \mathbf{y})$ does not exceed some threshold value α. The VaR (or percentile) function $\alpha(\mathbf{x}, \beta)$ is defined as follows

$$\alpha(\mathbf{x}, \beta) = \min\{\alpha \in \mathbb{R} : \Psi(\mathbf{x}, \alpha) \geq \beta\} \,. \tag{2}$$

Let us consider the following CVaR performance function $\Phi(\mathbf{x})$ which is the conditional expected value of the loss $f(\mathbf{x}, \mathbf{y})$ under the condition that it exceeds the quantile $\alpha(\mathbf{x}, \beta)$, i.e.,

$$\Phi(\mathbf{x}) = (1 - \beta)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha(\mathbf{x},\beta)} f(\mathbf{x}, \mathbf{y})\, p(\mathbf{y})\, d\mathbf{y} \,. \tag{3}$$

The decision vector $\mathbf{x}$ belongs to the feasible set $X \subset \mathbb{R}^n$. It was shown in [17] that the minimization of the excess loss function $\Phi(\mathbf{x})$ on the feasible set $X \subset \mathbb{R}^n$ can be reduced to the minimization of the function

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1 - \beta)^{-1} \int\limits_{\mathbf{y} \in \mathbb{R}^m} (f(\mathbf{x}, \mathbf{y}) - \alpha)^+ \; p(\mathbf{y}) \, d\mathbf{y} \; , \tag{4}$$

on the set $X \times \mathbb{R}$, where b^+ is the positive part of the number b, i.e., $b^+ = \max\{0, b\}$. The following equality is valid

$$\min_{\alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha) \; = \; \Phi(\mathbf{x}) \; ,$$

and the optimal solution α of this problem is VaR. This follows from the fact that the derivative of the function $F_\beta(\mathbf{x}, \alpha)$ with respect to α equals $1 + (1 - \beta)^{-1}(\Psi(\mathbf{x}, \alpha) - 1)$. Equating this derivative to zero gives (see details in [17])

$$\Psi(\mathbf{x}, \alpha) = \beta \; .$$

Consequently,

$$\min_{\mathbf{x} \in X, \alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha) \; = \; \min_{\mathbf{x} \in X} \min_{\alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha) \; = \; \min_{\mathbf{x} \in X} F_\beta(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \; = \; \min_{\mathbf{x} \in X} \Phi(\mathbf{x}) \; .$$

Thus, by minimizing the function $F_\beta(\mathbf{x}, \alpha)$ we can simultaneously find the VaR and optimal CVaR. Under general conditions, the function $F_\beta(\mathbf{x}, \alpha)$ is smooth [20] (key conditions: the density $p(\mathbf{y})$ and the loss function $f(\mathbf{x}, \mathbf{y})$ are smooth and the gradient of the function $f(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{y}$ is not equal zero).

The function $F_\beta(\mathbf{x}, \alpha)$, given by equation (4), is convex in α (discussions of properties of convex functions can be found, for instance, in [16,9]). $F_\beta(\mathbf{x}, \alpha)$ is convex in $\mathbf{x}$, if the function $f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$. We can use various approaches to calculate the integral function $F_\beta(\mathbf{x}, \alpha)$. If the integral in (4) can be calculated or approximated analytically, then we can use nonlinear programming techniques to optimize the function $F_\beta(\mathbf{x}, \alpha)$. In this paper, we approximate (4) using scenarios $\mathbf{y}_j$, $j = 1, \ldots, J$, which are sampled from the density function $p(\mathbf{y})$, i.e.,

$$\int\limits_{\mathbf{y} \in \mathbb{R}^m} \big(f(\mathbf{x}, \mathbf{y}) - \alpha \big)^+ \; p(\mathbf{y}) \, d\mathbf{y} \; \approx \; J^{-1} \sum_{j=1}^{J} \big(f(\mathbf{x}, \mathbf{y}_j) - \alpha \big)^+ \; .$$

If the loss function $f(\mathbf{x}, \mathbf{y}_j)$ is convex, and the feasible set X is convex, we can solve the convex optimization problem

$$\min_{\mathbf{x} \in X, \alpha \in \mathbb{R}} \quad \tilde{F}_\beta(\mathbf{x}, \alpha) \; , \tag{5}$$

where

$$\tilde{F}_\beta(\mathbf{x}, \alpha) \; \stackrel{def}{=} \; \alpha \; + \; \nu \sum_{j=1}^{J} \big(f(\mathbf{x}, \mathbf{y}_j) - \alpha \big)^+$$

and the constant ν equals $\nu = ((1-\beta)J)^{-1}$. By solving the (5) we find the optimal vector, $\mathbf{x}^*$, corresponding VaR, which equals α^*, and the optimal CVaR, which equals $\tilde{F}_\beta(\mathbf{x}^*, \alpha^*)$. Moreover, if the loss function $f(\mathbf{x}, \mathbf{y}_j)$ is linear in $\mathbf{x}$, and the set X is given by linear equalities and inequalities, we can reduce optimization problem (5) to the linear programming problem

$$\min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{z}\in\mathbb{R}^J, \alpha\in\mathbb{R}} \quad \alpha + \nu \sum_{j=1}^{J} z_j \tag{6}$$

$$\text{s.t.} \quad \mathbf{x} \in X\ , \tag{7}$$

$$z_j \geq f(\mathbf{x}, \mathbf{y}_j) - \alpha\ , \quad z_j \geq 0\ , \quad j = 1, \ldots, J, \tag{8}$$

where $z_j,\ \ j = 1, \ldots, J$ are dummy variables.

3. The bond portfolio

Bucay and Rosen [3] applied the CreditMetrics [5] methodology to a portfolio of corporate and sovereign bonds issued in emerging markets. They estimated the credit risk of this portfolio taking into account both defaults and credit migrations. Mausser and Rosen [12] conducted scenario optimization of this portfolio with the expected regret performance function. Compared to other academic and industrial studies, they optimized portfolio credit risk rather than just measuring credit risk. Below, following the papers [3, 12] we give a brief overview of the portfolio, the CreditMetrics methodology, and the most important portfolio statistics. Further, in this paper, we apply the CVaR optimization framework to the same portfolio and compare the results with [12].

3.1. Portfolio description

This test portfolio has been compiled by a group of financial institutions to asses the state-of-the-art of portfolio credit risk models. The portfolio consists of 197 emerging markets bonds, issued by 86 obligors in 29 countries. The date of the analysis is October 13, 1998 and the mark-to-market value of the portfolio is $8.8 billion. The portfolio mark-to-market value is simply a market based valuation of all the instruments in the portfolio. In the case of a single bond, the mark-to-market value is the sum of the discounted future cash flows obtained from a present value calculation. Most instruments are denominated in US dollars but 11 fixed rate bonds are denominated in seven other currencies; DEM, GBP, ITL, JPY, TRL, XEU, and ZAR[1]. Bond maturities range from a few months to 98 years and the portfolio duration is approximately five years.

[1] DEM – German Deutschemark, GBP – British Pound, ITL – Italian Lira, JPY – Japanese Yen, TRL – Turkish Lira, XEU – Euro, ZAR – South African Rand

3.2. *CreditMetrics*

CreditMetrics is a tool for assessing portfolio risks due to defaults and changes in the obligors' credit quality such as upgrades and downgrades in credit ratings. In this case study, we considered eight credit rating categories (including default). The modeling time horizon is one year. The first step in the CreditMetrics methodology establishes the likelihood of migrations between any possible credit quality states during the risk horizon for each individual obligor. The credit migration probabilities were obtained from Standard & Poor's transition matrix as of July 1998. This historically tabulated transition matrix is typically obtained by looking at time series of credit ratings of many firms. In order to get some desirable properties one wants a transition matrix to have an appropriate long-term behavior; this transition matrix can be adjusted somewhat to better approximate the long-term behavior as of a Markov Process. Then, the value of each obligor, i.e. the exposures, were calculated using the forward rates implied by today's term structure in each of the seven non-default states. In case of default, the value of each obligor is based on the appropriate recovery rate. The recovery rates are assumed to be constant and equal to 30% of the risk-free value for all obligors except two, which have lower rates. The one-year portfolio credit loss distribution is generated by the Monte Carlo simulation (20,000 scenarios of joint credit states of obligors and related losses). The sensitivity study with respect to the number of scenarios indicated that 20,000 scenarios scenarios is sufficient to estimate VaR and CVaR with sufficient precision. Joint default and migration correlations are driven by the correlations of the asset values of the obligors. Since the asset values are not observable, equity correlations of traded firms are used as a proxy for the asset correlations. More specifically, CreditMetrics maps each obligor to a country, region, or sector index that is more likely to affect its performance, and to a risk component that captures the firm-specific volatility. In each scenario, the portfolio mark-to-future value is obtained by summing up the exposures corresponding to this scenario. The valuation of a bond is derived from the zero-curve corresponding to the rating of the obligor. The mark-to-future value of the bond, or the forward price of the bond in one year from now, is derived by applying the zero-curve to the residual cash flows from year one to maturity. The credit loss distribution is then calculated by subtracting the portfolio mark-to-future values in each credit scenario from the mark-to-future value of the portfolio if no credit migration occurs. Figure 1 illustrates the portfolio loss distribution; it is skewed and has a long fat right tail. Some statistics from the one-year credit loss distribution are presented in Tables 1 and 2. Table 1 gives the portfolio *expected loss* which is the mean of the loss distribution and the *standard deviation* which measures the dispersion around the mean. Table 2 gives the VaR and CVaR of the loss distribution at different confidence levels. We examined the contribution of each individual asset to the risk of the portfolio. For a given obligor, we defined the risk contribution (for each risk measure) as the difference between the

Table 1. Mean and Standard Deviation for One-Year Loss Distribution

Expected loss	95
Standard deviation	232

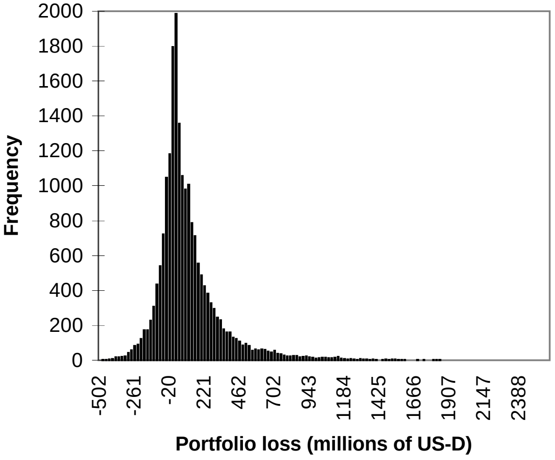

Fig. 1. One Year Credit Loss Distribution (millions of US-D)

risk for the entire portfolio and the risk of the portfolio without the given obligor. We expressed this contribution to the risk in percentage terms as the percentage decreases in the corresponding risk measure when the obligor is removed from the portfolio. Table 3 summarizes the contributions of the top risk obligors to the portfolio mark-to-market value, expected loss, standard deviation, VaR, and CVaR (prioritized according to CVaR contribution). This table provides the portfolio “Hot Spots”: twelve obligors that contribute most to the CVaR of the portfolio. Table 3 shows that the dominant risk contributors are bonds from Brazil, Russia, Venezuela, Argentina, Peru, and Colombia. To visualize these outputs, we plot the marginal risk (marginal CVaR as percentage of the market value) versus the market value (exposure) of each asset in Fig. 2. The product of the marginal risk and exposure approximately equals the risk contribution (e.g., Venezuela’s marginal risk equals 40% and the exposure equals $398 millions US-D, see Table 3, therefore, Venezuela CVaR contribution approximately equals 12% of the portfolio CVaR which is $1,320 millions of US-D, see Table 2). Points in the upper left part of the graph represent obligors with high marginal risk but whose exposure size is small. Points in the lower right corner represent large exposures with low marginal risk. To reduce the portfolio risk, a risk manager must suppress these dominant contributors, that is, suppress obligors with large exposures and high marginal risk. From Fig. 2, it is apparent that Brazil, Russia, and Venezuela have high marginal risks and also large

Table 2. VaR and CVaR for One-Year Loss Distribution in millions of US-D

β	VaR	CVaR
0.90	341	621
0.95	518	824
0.99	1,026	1,320
0.999	1,782	1,998

Table 3. Mark-to-Market Value (in millions of US-D), Expected Loss (μ), Standard Deviation (σ), VaR, CVaR. Values are expressed as the percentage decrease for the One-Year Loss Distribution ($\beta = 0.99$)

Obligor	Mark-to-market	μ (%)	σ (%)	VaR (%)	CVaR (%)
Brazil	880	14.5	17.1	20.4	19.4
Russia	756	9.8	12.2	14.3	15.4
Venezuela	398	6.2	14.1	12.4	12
Argentina	624	9.9	9.3	10.6	8.8
Peru	283	10.3	9.0	8.4	7.5
Colombia	605	2.3	3.0	3.3	4.0
Morocco	124	1.6	1.4	1.0	1.6
RussiaIan	48	1.3	2.5	2.0	1.5
MoscowTel	86	0.6	0.8	1.0	1.2
Romania	87	0.8	0.9	0.3	1.2
Mexico	488	9.2	2.0	1.8	0.9
Philippines	448	6.7	1.2	0.4	0.5

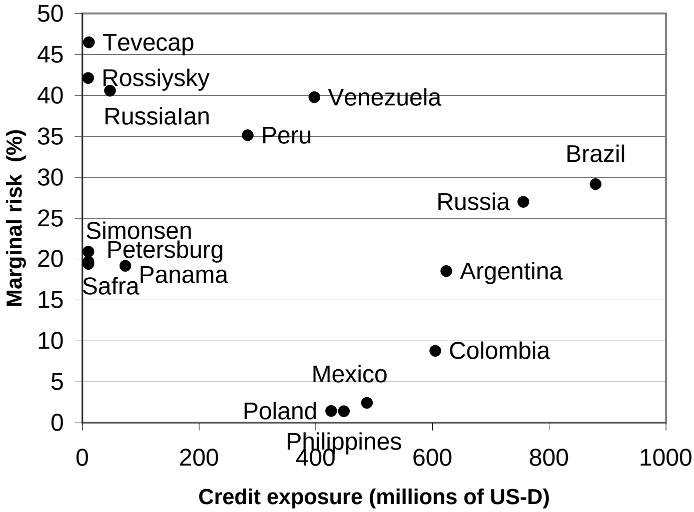

Fig. 2. Dominant Contributors to CVaR (Marginal Risk versus Credit Exposure, $\beta = 0.99$)

exposures. TeveCap, Rossiysky and RussianIan on the other hand have high marginal risk but small exposures. This concludes the portfolio description.

4. Optimization model

4.1. Problem statement

We consider an optimization model similar to [11], however, instead of minimizing a regret function we minimize the CVaR. Let $\mathbf{x} = (x_1, ..., x_n)$ be obligor weights (positions) expressed as multiples of current holdings, $\mathbf{b} = (b_1, ..., b_n)$ be future values of each instrument with no credit migration (benchmark scenario), and $\mathbf{y} = (y_1, ..., y_n)$

be the future (scenario-dependent) values with credit migration. The loss due to credit migration for the portfolio equals

$$f(\mathbf{x}, \mathbf{y}) = (\mathbf{b} - \mathbf{y})^T \mathbf{x} \ .$$

The CVaR optimization problem is

$$\min_{\mathbf{x} \in \mathbf{X} \subset \mathbb{R}^n} \ \Phi(\mathbf{x}) \ , \tag{9}$$

where $\mathbf{X}$ is the feasible set in $\mathbb{R}^n$. This set is defined in the following section (mean return constraint, box constraints on the positions of instruments, etc.). We approximated the performance function using scenarios $\mathbf{y}_j$, $j = 1, \dots, J$, which are sampled from the density function $p(\mathbf{y})$. Minimization of the CVaR function $\Phi(\mathbf{x})$ can be reduced to the following linear programming problem

$$\min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{R}^J, \alpha \in \mathbb{R}} \ \phi(\mathbf{z}, \alpha) \ = \ \alpha \ + \ \nu \sum_{j=1}^{J} z_j \tag{10}$$

$$\text{s.t.} \quad \mathbf{x} \in \mathbf{X} \ , \tag{11}$$

$$z_j \ \geq \ f(\mathbf{x}, \mathbf{y}_j) - \alpha \ , \quad z_j \ \geq \ 0 \ , \quad j = 1, \dots, J \ , \tag{12}$$

where $\nu = ((1 - \beta)J)^{-1}$. If $(\mathbf{x}^*, \alpha^*, \mathbf{z}^*)$ is an optimal solution of the optimization problem (10), then $\mathbf{x}^*$ is the optimal solution of the CVaR optimization problem (9), the function $\phi(\mathbf{z}^*, \alpha^*)$ equals the optimal CVaR, and α^* is VaR at the optimal point. Thus, by solving problem (10) we can simultaneously find approximations of the optimal CVaR and the corresponding VaR.

4.2. Constraints

In order to avoid unrealistic long or short positions in any of the holdings, we imposed the following constraints on the change in obligor weights

$$l_i \leq x_i \leq u_i \qquad i = 1, ..., n \ ,$$

where l_i is the lower trading limit and u_i is the upper trading limit (both expressed as multiples of current weighting). Further, we have a constraint that maintains the current value of the portfolio

$$\sum_{i=1}^{n} q_i x_i = \sum_{i=1}^{n} q_i \ ,$$

where $\mathbf{q} = (q_1, ..., q_n)$ are the current mark-to-market counterparty values. Alternatively, similar to the current value constraint, we can maintain the future portfolio value

$$\sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} b_i \ .$$

Finally, in order to achieve an expected portfolio return R, and to calculate the efficient frontier for the portfolio, we can include the constraint for the expected portfolio return with no credit migration as

$$\frac{\sum_{i=1}^{n}(q_i x_i) r_i}{\sum_{i=1}^{n} q_i x_i} \geq R \ ,$$

or equivalently as

$$\sum_{i=1}^{n} q_i (r_i - R) x_i \geq 0 \ ,$$

where r_i is the expected return for obligor i in the absence of credit migration. When calculating the efficient frontier for the portfolio, we also used the following additional constraints

$$x_i q_i \leq 0.20 \sum_{i=1}^{n} q_i, \qquad i = 1, ..., n \ ,$$

which imply that the value of each long individual position cannot exceed 20% of the current portfolio value.

4.3. Optimization problem

Combining the performance function and the constraints defined in the two previous sections yields the following linear programming problem

$$\min \ \ \phi(\mathbf{z}, \alpha) \ = \ \alpha \ + \ \nu \sum_{j=1}^{J} z_j \ , \tag{13}$$

$$\text{s.t.} \quad z_j \geq \sum_{i=1}^{n}((b_i - y_{ji}) x_i) - \alpha, \qquad j = 1, ..., J \ , \tag{14}$$

$$z_j \geq 0, \qquad j = 1, ..., J \ , \tag{15}$$

$$l_i \leq x_i \leq u_i, \qquad i = 1, ..., n \ , \tag{16}$$

$$\sum_{i=1}^{n} q_i x_i = \sum_{i=1}^{n} q_i \ , \qquad \textit{or} \qquad \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} b_i \ , \tag{17}$$

$$\sum_{i=1}^{n} q_i (r_i - R) x_i \geq 0 \ , \tag{18}$$

$$x_i q_i \leq 0.20 \sum_{i=1}^{n} q_i \ , \qquad i = 1, ..., n \ . \tag{19}$$

Solving (13)–(17) yields the optimal vector $\mathbf{x}^*$, corresponding VaR, which equals α^*, and the optimal CVaR, which equals $\phi(\mathbf{z}^*, \alpha^*)$. Solving (13)–(19) for different portfolio returns R yields the efficient frontier of the portfolio.

Table 4. "Best Hedge report", VaR, CVaR (in millions of US-D) and corresponding VaR and CVaR reductions (in %) for the Single Obligor Optimization ($\beta = 0.99$)

Obligor	Best Hedge	VaR	VaR (%)	CVaR	CVaR (%)
Brazil	−5.72	612	40	767	42
Russia	−9.55	667	35	863	35
Venezuela	−4.29	683	33	880	33
Argentina	−10.30	751	27	990	25
Peru	−7.35	740	28	980	26
Colombia	−45.07	808	21	1,040	21
Morocco	−88.29	792	23	1,035	22
RussiaIan	−21.25	777	24	989	25
MoscowTel	−610.14	727	29	941	29
Romania	−294.23	724	29	937	29
Mexico	−3.75	998	3	1,292	2
Philippines	−3.24	1,015	1	1,309	1

5. Analysis

5.1. Optimal hedging

As the first step to re-balancing of the portfolio, we changed the position of a single obligor, holding the other positions fixed. We minimized the portfolio credit risk, i.e. minimized CVaR, and obtained the size of the optimal contract. This is accomplished by conducting a one-instrument optimization of the model (13)–(15). The results of this optimization are presented in Table 4 for the twelve largest contributors to the risk, in terms of CVaR contribution. Table 4 shows that we can achieve the 40% reduction in VaR and the 42% reduction in CVaR if Brazil is given a weight of −5.72, i.e. going short about 6 times the current holdings. Thus, CVaR can reduced to $767 million from the original $1320 million. Similar conclusions can be reached about other obligors in Table 4.

Mausser and Rosen [12] conducted optimal hedging for this problem using the one-dimension VaR minimization. The optimal hedges obtained with our approach (see, Table 4) are very close to the hedges obtained in [12]. For instance, with our approach the best hedge for Brazil is −5.72 (VaR reduction is 40%) and with the minimum VaR approach [12] the best hedge is −5.02 (VaR reduction is 41%). Similar, for Russia, Venezuela, and Argentina we have −9.55 (VaR reduction is 35%), −4.29 (VaR reduction is 33%) and −10.33 (VaR reduction is 27%). For the same obligors paper [12] gives the following optimal hedges: −8.71 (VaR reduction is 35%), −3.32 (VaR reduction is 34%) and −7.81 (VaR reduction is 28%). Similar results hold for other risk dominant obligors in Table 4. For this example, the minimum CVaR hedge is always bigger that the minimum VaR hedge. From these numerical results, we can conclude that the one-dimensional minimization of VaR and CVaR virtually produces the same result (from a practical point of view). We have a similar conclusion for a portfolio of options when we compared the best hedges with the minimum VaR approach [11] and the best hedges with the minimum CVaR approach [17].

Table 5. VaR, CVaR (in millions of US-D) and corresponding VaR and CVaR reductions (in %) for the Multiple Obligor Optimization

Case	β	VaR	VaR (%)	CVaR	CVaR (%)
Original	0.900	340	–	621	–
	0.950	518	–	824	–
	0.990	1,026	–	1,320	–
	0.999	1,782	–	1,998	–
No Short	0.900	163	52	279	55
	0.950	239	54	359	56
	0.990	451	56	559	58
	0.999	699	61	761	62
Long and Short	0.900	149	56	264	58
	0.950	226	56	344	58
	0.990	433	58	542	59
	0.999	680	62	744	63

5.2. Minimization of CVaR

Further, we optimized all positions and solved the linear programming problem (13)–(17). In order to avoid unrealistically long or short positions in any obligor, the size of each position is bounded, see equation (16). We considered two cases:

- no short positions allowed, and the positions can be at most doubled in size;
- positions, both long and short, can be at most doubled in size.

The first case means that $l_i = 0$ and $u_i = 2$ in constraint (16), i.e.

$$0 \leq x_i \leq 2 \ .$$

And the second case implies that

$$-2 \leq x_i \leq 2.$$

We supposed that the re-balanced portfolio should maintain the future expected value, in absence of any credit migration, i.e. we included the second constraint (17). The result of this optimization, in the case of no short positions (No Short), and in the case of both long and short positions (Long and Short), are presented in Table 5. Further, we compared the risk profiles of the original and optimized portfolio. Table 5 shows that the two risk measures, VaR and CVaR, are significantly improved by the optimization. When no short positions are allowed, we reduced VaR and CVaR by about 60%. For example, at $\beta = 0.99$, we lowered CVaR to 559 million from the original 1320 million US-D. By allowing both short and long positions, we slightly improved reductions, but not significantly. Thus, we observe that we can reduce risk measures about 40% with the single obligor optimization and about 60% with the multiple obligor optimization. Table 6 compares expected loss and standard deviation of the original and optimized portfolios. Table 6 shows that the two risk measures, the expected loss and the standard deviation, also are dramatically improved when we minimized CVaR. For example, in the case of both long and short positions, the expected loss and standard deviation

Table 6. Expected Loss (in millions of US-D), Standard Deviation (in millions of US-D) and corresponding reductions (in %) for the Multiple Obligor Optimization

Case	β	Expected loss	(%)	Standard deviation	(%)
Original	–	95	–	232	–
No Short	0.90	50	47	107	54
	0.95	51	46	109	53
	0.99	60	37	120	48
	0.999	63	34	126	46
Long and Short	0.90	42	56	105	55
	0.95	44	54	107	54
	0.99	53	44	118	49
	0.999	58	39	124	47

Table 7. Positions (expressed as multiples of original holdings) for the Multiple Obligor Optimization (Minimization of CVaR with $\beta = 0.99$)

Obligor	Original	No Short	Short and Long
Brazil	1	0.08	0.18
Russia	1	0	0.09
Venezuela	1	0	−0.41
Argentina	1	0.35	0.47
Peru	1	0	−0.38
Colombia	1	0.89	1.00
Morocco	1	0.02	0.12
RussiaIan	1	0	−2.00
MoscowTel	1	1.52	1.99
Romania	1	0.45	1.33
Mexico	1	0.94	0.90
Philippines	1	1.04	0.94

are reduced about 50%. The corresponding position weights for the original twelve largest risk contributors are presented in Table 7. Table 7 shows that the positions of the largest risk contributors, Brazil, Russia, Argentina, and Colombia, were reduced or removed from the portfolio. Table 8 summarizes the contribution of the obligors to the portfolio mark-to-market value, expected loss, standard deviation, VaR and CVaR after the optimization. This table can be compared with Table 3 which describes the risk contributors before the optimization. From Table 8, we can conclude that the risk after the optimization is more spread out and not so concentrated on few obligors. The largest risk contributors with respect to CVaR are Colombia and Poland. Figure 3 illustrates the dominant risk contributors of the optimized portfolio. This figure can be compared with Fig. 2, which shows dominant risk contributors for the original portfolio. The risk outliers in the original portfolio, Brazil, Russia and Venezuela, are no longer dominant in the optimized portfolio. The optimal portfolio reduces the marginal risk contributors. For example, the highest marginal risk contribution in the portfolio is reduced to about 7% (Colombia) from the original 45% (TeveCap). The largest risk contributors in the optimized portfolio are now Colombia and Poland, but with

Table 8. Mark-to-Market Value (in millions of US-D), Expected Loss (μ), Standard Deviation (σ), VaR, CVaR. Values are expressed as the percentage decrease in Risk Measure for the Optimal Portfolio (No Short Positions, $\beta = 0.99$)

Obligor	Mark-to-Market	μ (%)	σ (%)	VaR (%)	CVaR (%)
Colombia	538	3.2	3.2	1.9	6.7
Poland	683	8.2	4.1	5.2	4.9
Mexico	459	13.7	7.0	5.0	4.2
Philippines	466	11.0	4.8	4.6	3.6
China	556	3.7	2.4	3.6	3.3
Bulgaria	315	0.5	11.0	3.6	3.3
Argentina	218	5.5	3.2	3.4	3.0
Kazakhstan	329	4.6	1.4	2.5	2.9
Jordan	263	9.3	2.7	1.8	2.2
Croatia	301	1.7	1.3	1.9	2.1
Israel	675	1.5	0.5	0.4	1.4
Brazil	70	1.8	1.1	0.3	1.3

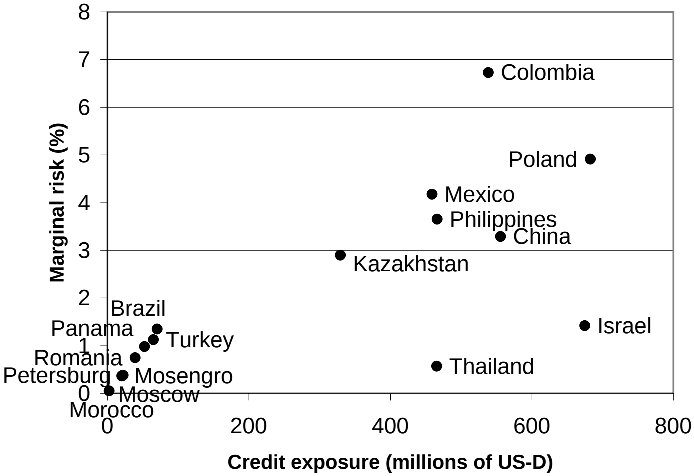

Fig. 3. Dominant Contributors to CVaR for the Optimized Portfolio when no Short Positions are allowed (Marginal Risk versus Credit Exposure, $\beta = 0.99$)

much smaller marginal risks than the largest contributors in the original portfolio. We compared our calculations with the minimum expected regret approach [12]. With the minimum expected regret approach, the risk is defined as an average loss exceeding some specified in advance threshold (unacceptable loss). Table 9 reproduced from [12] shows calculation results with the minimum expected regret approach for No Short case with $\beta = 0.990$ and $\beta = 0.999$. It appears that with the minimum expected regret approach, by doing sensitivity analysis with respect to the regret threshold, it is possible to achieve similar reductions in VaR and CVaR as with CVaR minimization approach (see, Table 5, lines 8 and 9). However, with respect to CVaR value, our approach always outperformed the minimum expected regret approach. Also, in nine out of ten runs

Table 9. Calculation results with the Minimum Expected Regret Approach reproduced from [12], Regret Threshold, VaR, CVaR (in millions of US-D) and corresponding VaR and CVaR reductions (in %) for the Multiple Obligor Optimization, No Shorts

Case	Regret Threshold	VaR	VaR (%)	CVaR	CVaR (%)
No Short, $\beta = 0.990$	0	495	52	727	45
	250	408	60	598	55
	500	461	55	561	57
	750	511	50	604	54
	1,000	650	37	735	44
No Short, $\beta = 0.999$	0	1,074	40	1,370	31
	250	999	44	1,152	42
	500	696	61	791	60
	750	750	58	772	61
	1,000	876	51	931	53

presented in Table 9, our approach outperformed or gave the same reduction in VaR (except one case where there was a few percent underperformance) as the minimum expected regret approach. Moreover, our approach conducted calculations in "one shot" (without sensitivity analysis with respect to the threshold value).

5.3. Risk-Return

When optimizing CVaR, we focused only on credit risk reductions without considering the expected portfolio return. In order to achieve the desired portfolio return and to observe the Risk-Return trade offs, we calculated the efficient frontier of the portfolio. We included in the model the following constraints:

- constraints (14) and (15) for dummy variables;
- no short positions are allowed, constraint (16) with $l_i = 0$ and $u_i = \infty$, $i = 1, ..., n$;
- the current mark-to-market value must be maintained, the first constraint in (17);
- the return constraint (18) with various values of return R;
- the long position of an individual counterparty cannot exceed 20% of the current portfolio value, constraint (19).

First, we considered that $\beta = 0.99$. We assumed that the expected returns for each obligor are given by the one-year forward returns of their holdings, assuming no credit migration. Figure 4 shows the efficient frontier for the portfolio and the relative position of the original portfolio which has an expected portfolio return 7.26%. We can see from this figure that the original portfolio is inefficient. More specifically, we can achieve the same expected portfolio return 7.26%, but with only one fourth of the risk of the original portfolio. It is interesting to compare the risk profile of the original portfolio (expected return 7.26%), with the optimal portfolio having the same return. Table 10 demonstrates that the optimization reduces all risk measures. In the optimal portfolio, we reduced the expected loss by almost 100%, standard deviation by 34%, VaR and CVaR by 80%. Finally, Table 11 shows the position weights for the 12 largest instruments in the portfolio after the Risk-Return optimization with the expected portfolio return 7.62%.

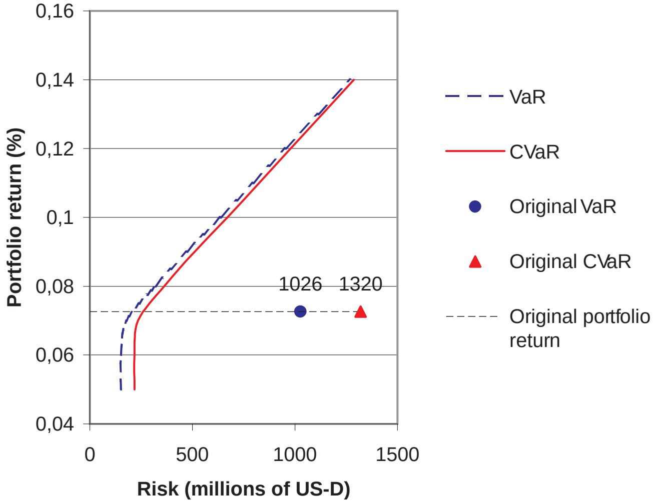

Fig. 4. Efficient Frontier ($\beta = 0.99$)

Table 10. Expected Loss (μ), Standard Deviation (σ), VaR, CVaR (in millions US-D) and corresponding reductions (in %) for the Risk-Return Optimization with the Expected Portfolio Return 7.26% ($\beta = 0.99$)

Case	μ	μ(%)	σ	σ (%)	VaR	VaR (%)	CVaR	CVaR (%)
Original	95	–	232	–	1026	–	1320	–
Optimized	0.005	100	152	34	210	80	263	80

The improvements in the risk measures stem from the relaxing of the trading constraints. For instance, the largest individual position change is in ThailandAAA, of approximately 150 times the original position. Such dramatical change may be infeasible and constraints on the position change may need to be imposed. Conducting the Risk-Return analysis, we followed the setup of the [12] but with different performance function. As the risk measure we used CVaR and the paper [12] used the Expected Regret. It is difficult to compare the efficient frontiers for different performance functions. However, we see that the portfolio with the expected return 7.62% on the efficient CVaR-Return frontier obtained with our approach outperforms in CVaR and VaR the portfolio on the Minimum Expected Regret frontier. This is not surprising, because, we minimized CVaR rather than Expected Regret. Also, we conducted, the Risk-Return analysis for the model with the confidence levels $\beta = 0.95$ and $\beta = 0.90$. The findings for these confidence levels are similar to the ones with $\beta = 0.99$. Here, we present only graphs of efficient frontiers with $\beta = 0.95$ (Fig. 5) and $\beta = 0.90$ (Fig. 6).

6. Calculation time and iterations

Although the considered Credit Risk optimization problem is modeled with a large number of scenarios, we easily solved it using linear programming techniques. Table 12

Table 11. Positions (expressed as multiples of original holdings) for the Risk-Return Optimization with the Expected Return 7.26% ($\beta = 0.99$)

Obligor	Original Position	Optimized Position
ThailandAAA	1	148.74
TelekomMalaysia	1	28.61
TelChile	1	24.64
Malayanbanking	1	21.81
MalysiaPetrol	1	20.41
ChileVapores	1	19.58
TeleComarg	1	17.23
Metrogas	1	17.18
IndFinCorp	1	15.20
KoreaElectric	1	11.07
Vietnam	1	10.73
Lithuania	1	10.72

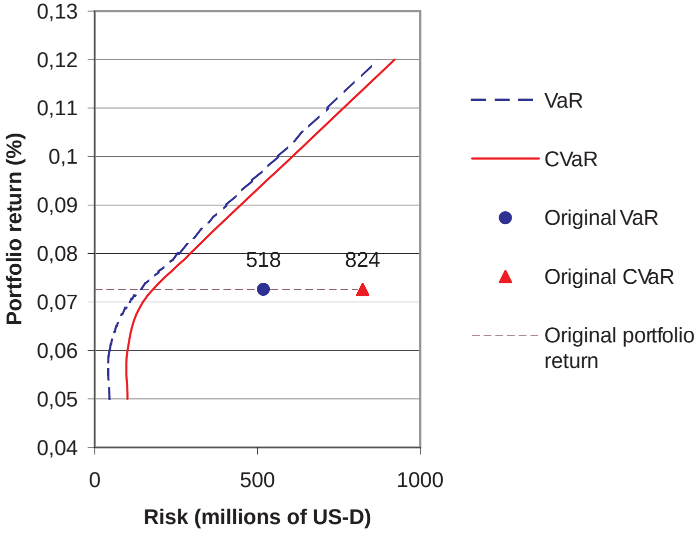

Fig. 5. Efficient Frontier ($\beta = 0.95$)

presents the average times and number of iterations using the CPLEX solver in GAMS on the Sun Ultra 1 140 MHz processor. This table presents the solving times in the case of single obligor optimization (Single), multiple obligor optimization (No Short and Long and Short), and Risk-Return analysis (Risk-Return).

7. Conclusions

We conducted a case study on optimization of Credit Risk of the portfolio of bonds. While we considered a specific bond portfolio, the CVaR model extends naturally to securities and trading constraints of more general nature. Using the CVaR optimiza-

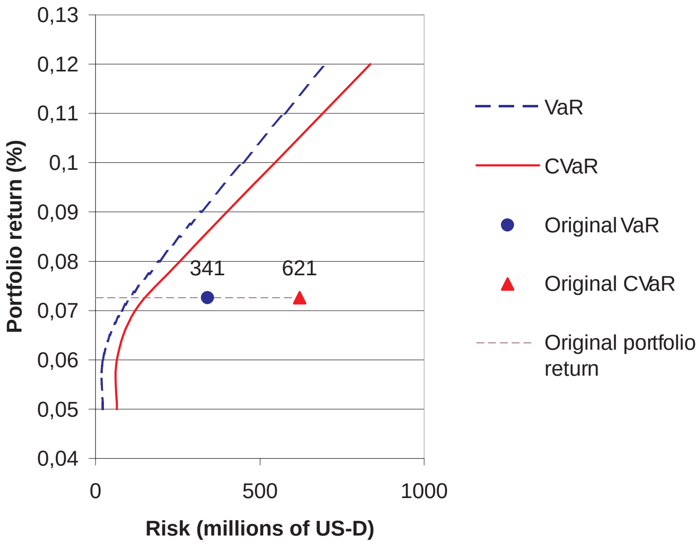

Fig. 6. Efficient Frontier ($\beta = 0.90$)

Table 12. Number of iterations and solving time with CPLEX Solver on the Sun Ultra 1 140 MHz processor ($\beta = 0.99$)

Case	Iterations	Time (min)
Single	39000	2.2
No Short	23000	20
Long and Short	29000	37
Risk-Return	42000	44.5

tion framework we simultaneously adjusted two closely related risk measures: CVaR and VaR. Although we used CVaR as a performance function, the optimization leads to reductions of all risk measures considered in this paper: CVaR, VaR, the expected loss, and the standard deviation. From a bank perspective, this approach looks quite attractive. The bank should have reserves to cover expected loss and capital to cover unexpected loss. For the considered portfolio, the expected and unexpected loss are 95 and 931 million US-D (unexpected loss is the maximum loss at some quantile, i.e. VaR, minus the expected loss). We have managed to reduce the expected loss by almost 100%, the unexpected loss about 80%, and still achieved the same expected portfolio return. Our results are quite similar to the results obtained with the Minimum Expected Regret Approach [11]. However, unlike the Minimum Expected Regret Approach we do not need to conduct sensitivity studies with respect to the regret threshold. The Minimum CVaR approach automatically finds the threshold (i.e., VaR) corresponding to a specified confidence level β. Our approach relies on linear programming techniques which allows one to cope with large portfolios and large numbers of scenarios.

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