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Journal of Banking & Finance xxx (2007) xxx–xxx
Journal of BANKING & FINANCE
www.elsevier.com/locate/jbf

Equilibrium with investors using a diversity of deviation measures

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Abstract

It has been argued that investors who optimize their portfolios with attention paid only to mean and standard deviation will all end up choosing some multiple of a certain master fund portfolio. Justification for the capital asset pricing model of classical portfolio theory, which relates individual assets to such a master fund, has come from this direction in particular. Attempts have been made to provide solid mathematical support by showing that the imputed behavior of investors is a consequence of price equilibrium in a market in which assets are traded subject to budget constraints, and optimization is carried out with respect to utility functions that depend only on mean and standard deviation.

In recent years, reliance on standard deviation has come under increasing criticism because of inconsistencies in its effect on portfolio preferences. One response has been to introduce generalized measures of deviation which lead to alternative master funds. The market implications of such extensions of theory have hitherto been unclear, but in this paper the existence of equilibrium is established in circumstances where nonstandard deviations are admitted. Equilibrium is guaranteed even when different investors use different measures of deviation and thereby end up with portfolios scaled from different master funds. Whether they employ the same measure or not, they may impose caps on deviation, which likewise may be different.

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doi:10.1016/j.jbankfin.2007.04.002

Please cite this article in press as: Rockafellar, R.T. et al., Equilibrium with investors using a diversity ..., J. Bank Finance (2007), doi:10.1016/j.jbankfin.2007.04.002

JEL classification: C62; C68; D58; G11

Keywords: Financial equilibrium; Portfolio analysis; Generalized deviation measures; Generalized master funds; Risk management; CAPM-like models

1. Introduction

In the broad picture of economic theory, market equilibrium is concerned with the existence of prices such that, when agents called consumers buy and sell goods according to their preferences, subject to budget constraints, and other agents called producers buy inputs and sell outputs so as to maximize profits, supply will balance demand. Sophisticated results have been obtained even when the goods in question have uncertain consequences for the consumers who hold them and could, in part, represent financial assets.

In finance, where uncertainty is the dominant characteristic, the question about equilibrium that has most attracted attention is more particular and is tied to practical considerations like justifying the widespread use of the capital asset pricing model (CAPM). That model, coming from classical portfolio theory as pioneered by Markowitz (1952), is based on the premise that an investor, in choosing a mix of assets and looking at the random variable describing the future of that mix, will primarily pay attention to the expectation and variance, or standard deviation, of that random variable. An "efficient frontier" is derived which displays the trade-off between expectation and standard deviation. Under broad assumptions, it is shown that a special mix of assets, called a master fund, exists with the property that any investor who wishes to minimize standard deviation subject to a constraint on expectation will end up choosing some multiple of that master fund, scaled up or down.

The issue that inevitably arises then is whether the master fund, because of its universal involvement, somehow stems from a kind of market equilibrium. It seemingly would have to, if the notion that all investors are prone to act in this same manner is to make sense, along with the usual capital asset pricing model.

Much research has gone into this over the years, from the early work of Sharpe (1964) and Mossin (1966) to the subsequent papers of Nielsen (1988), Nielsen (1989), Nielsen (1990), Nielsen (1992), Allingham (1991), Sharpe (1991) and later Konno and Shirakawa (1995), among others. In most of this work, an investor seeks to maximize a utility expression of the expected value and standard deviation of the future value of a portfolio (in a model with a single future period). That future value comes from random variables giving the future values of various assets and therefore is independent of current market prices. Konno and Shirakawa, however, replace future value by the rate of return (relative to present value) in both places. They also investigated in Konno and Shirakawa (1994) whether equilibrium could be achieved if every investor used mean absolute deviation instead of standard deviation.

The question of whether standard deviation, while convenient, is the ideal tool for an investor to use appraising discrepancies between future values and their expectations has long been asked. Markowitz, already in 1959, recognized that standard deviation has the drawback of penalizing outcomes higher than expectations just as much as outcomes lower than expectations. He suggested using a downside version of standard deviation as an alternative. Standard deviation has come under heavier attack more recently with the introduction of "coherent risk measures" by Artzner et al. (1999) (cf. Föllmer and Schied, 2002) for a comprehensive exposition. The preferences forced on investors by optimizing relative to standard deviation can have "incoherent" consequences. These criticisms have led to the study of other measures of deviation which might be adopted as substitutes, for instance ones based in one way or another on conditional value-at-risk (Rockafellar and Uryasev, 2000; Rockafellar and Uryasev, 2002). A systematic study was undertaken by us in Rockafellar et al. (2002a) and Rockafellar et al. (2006a).

Our purpose in this paper is to demonstrate that equilibrium in a financial market can be established when investors, with preferences expressed by a utility function of mean and deviation, use any of these alternative measures without being limited to standard deviation, its downside version, or to mean absolute deviation. Moreover we will show that equilibrium is guaranteed even when different investors, belonging to investor classes with different attitudes toward risk, take different approaches to deviation. Nothing like this has previously been proved, or even contemplated. It raises new questions about market behavior in which several master funds might be active simultaneously.

As a framework for the generalized measures of deviation which will be central to our endeavor, it is useful to take the view that the uncertain outcome associated with an asset or mix of assets is a random variable X regarded as a function on a set Ω of future states ω furnished with a probability structure; $X(\omega)$ is the outcome in state ω. We focus on the space of such X for which EX and $\sigma(X)$ are well defined (finite), which we denote for simplicity by $\mathscr{L}^2(\Omega)$. We use C to stand for either a constant in $\mathbb{R}$ (the real numbers) or the corresponding constant random variable $X \equiv C$.

A *deviation measure* $\mathscr{D}$ assigns to each random variable X in the designated space $\mathscr{L}^2(\Omega)$ a number in $[0,\infty]$ in such a manner that

(a) $\mathscr{D}(X) = 0$ for constant X, but $\mathscr{D}(X) > 0$ for nonconstant X,
(b) $\mathscr{D}(\lambda X) = \lambda\mathscr{D}(X)$ for $\lambda \geqslant 0$,
(c) $\mathscr{D}(X + X') \leqslant \mathscr{D}(X) + \mathscr{D}(X')$.

This definition appears simpler than the one we originally proposed in 2002 in Rockafellar et al. (2002a) and repeated in Rockafellar et al. (2006a), Rockafellar et al. (2005) and Rockafellar et al. (2006b). But it is equivalent to it by Rockafellar et al. (2006a, Proposition 1), according to which the combination of (a) with the convexity of $\mathscr{D}$ coming from (b) and (c) implies that

$$\mathscr{D}(X + C) = \mathscr{D}(X) \quad \text{for all constants } C, \text{ so } \mathscr{D}(X) = \mathscr{D}(X - EX) \quad \text{for all } X.$$

In line with the developments in those earlier works of ours, it will further be assumed that the deviation measures we deal with in this paper have the following property of lower semicontinuity with respect to sequences $\{X_k\}_{k=1}^{\infty}$: if $E[X_k - X] \to 0$, $\sigma(X_k - X) \to 0$ and $\mathscr{D}(X_k) \leqslant c$ as $k \to \infty$, then $\mathscr{D}(X) \leqslant c$.

Standard deviation, the case of $\mathscr{D}(X) = \sigma(X)$, obeys these rules, but there are numerous other examples of interest which, unlike standard deviation, can have $\mathscr{D}(-X) \neq \mathscr{D}(X)$. For these, and a more technically detailed development, we refer to Rockafellar et al. (2006a). A connection with coherent risk measures $\mathscr{R}$ is brought out there, in particular: deviation measures $\mathscr{D}$ satisfying the additional condition that $\mathscr{D}(X) \leqslant EX - \inf X$ correspond one-to-one with coherent risk measures $\mathscr{R}$ that satisfy the additional condition (not part of the definition in Artzner et al. (1999)) that $\mathscr{R}(X) > E[-X]$ for nonconstant X, under the rule that $\mathscr{D}(X) = \mathscr{R}(X - EX)$. Standard deviation does not satisfy $\sigma(X) \leqslant EX - \inf X$ and thus does not correspond in this way to a coherent risk measure.

A key fact in the theory of generalized deviations, which motivates and supports our efforts here, is that the classical portfolio analysis of how investors should react to the trade-off between mean and standard deviation can thoroughly be extended to the context of a generalized deviation measure $\mathcal{D}$ being substituted for σ. Moreover a corresponding extension of the concept of master fund then plays a significant role; see Rockafellar et al. (2005). Our aim is to draw on those results in an exploration of whether the existence of financial equilibrium and the ways that master funds might enter it.

We depart from most of the other work on equilibrium not only through this broader perspective, but also by taking the random variables that characterize the risky assets to be *rates of returns* on market values, rather than future values directly. In this we follow the lead of Konno and Shirakawa (1994) and Konno and Shirakawa (1995), but in contrast to them we still take the preferences of the investors to be based on the future values of portfolios as determined by applying rates of return to current prices. These preferences are able then to reflect differences in attitudes to risk which might come from the current wealth of an investor. Of course, no approach to financial equilibrium can ever be fully satisfactory in a model in which there is only a present and a single future. Without some dynamical extension into later stages, such a model is bound to be artificial. However, we can hope at least that our contribution here offers progress in understanding the issues and reveals new challenges for the development of pricing schemes.

Another distinctive feature of our approach is that we treat the preferences of investors more broadly than before by allowing an investor to insist on a *cap* on deviation, i.e., an upper bound constraint on the deviation value of a portfolio. Different investors, even when agreeing on the measure of deviation to be utilized, can impose caps at different levels. Until now, no model of financial equilibrium has admitted such constraints.

This paper does not directly provide any indication of which measures of deviation the participants in a financial market might actually choose to employ. By showing, though, that the effect of these choices lies entirely in the associated master funds, it paves the way toward possibly reconstructing the choices through market analysis. Under the hypothesis, for example, that all investors make the same choice, which might not be standard deviation but results in a master fund corresponding to the S&P 500, studies could be carried out to see which deviation measure from a collection of possibilities most closely produces that result. Other indices like the Russell 2000 could also be considered, especially in pursuing the notion that several different classes of investors might be characterized by different deviation measures. Various ideas could be exploited through solving inverse portfolio problems (to restore the parameters of such problems) or through statistical analysis. For such analysis, it would likely be helpful to extend traditional statistics, with symmetric measures of error, to error expressions reflecting the lack of symmetry between gains and losses in the decisions of investors. Preliminary elements of such developments, utilizing general deviation measures, are already in place in Rockafellar et al. (2002b), but much remains to be investigated.

2. Assets, investors and portfolios

In the market we study, there are assets indexed by $j = 0, 1, \ldots, n$ having rates of return denoted by $r_0, r_1, \ldots, r_n$. Asset 0 is assumed to be riskless, so r_0 is a constant, but the other assets are risky; their rates of return are nonconstant random variables.

The investors in our model are indexed by $i = 1, \ldots, m$. All the investors know, and agree on, the statistics of all the assets. A deviation measure $\mathscr{D}_i$ serves investor i in evaluating risk. These deviation measures may be the same or different; $\mathscr{D}_i = \sigma$ is just one of the possibilities.

A risky portfolio for investor i corresponds to a vector $x_i = (x_{i1}, \ldots, x_{in})$ in which x_{ij} is the amount invested in asset j. The future value of such a portfolio is the random variable $\sum_{j=1}^n x_{ij}(1 + r_j)$, with expectation $\sum_{j=1}^n x_{ij}(1 + Er_j)$ and deviation $\mathscr{D}_i\left(\sum_{j=1}^n x_{ij}(1 + r_j)\right) = \mathscr{D}_i\left(\sum_{j=1}^n x_{ij} r_j\right)$. As long as $x_i \neq (0, \ldots, 0)$, risk is present and this deviation value is positive under our assumptions. We suppose that $\mathscr{D}_i(r_j) < \infty$ and $\mathscr{D}_i(-r_j) < \infty$ for all i and j, in order to guarantee, by way of the deviation axioms already listed, that $\mathscr{D}_i\left(\sum_{j=1}^n x_{ij} r_j\right) < \infty$ as well.

The generalized master funds that arise in this setting come out of a fundamental problem of optimization in which investor i, in selecting x_i, correspondingly allocates $1 - \sum_{j=1}^n x_{ij}$ to the riskless asset, so that the mixed portfolio thereby obtained represents one unit of investment, in total. This mixed portfolio has future value $\left(1 - \sum_{j=1}^n x_{ij}\right) r_0 + \sum_{j=1}^n x_{ij}(1 + r_i)$ and rate of return $r_0 + \sum_{j=1}^n x_{ij}(r_i - r_0)$, for which the expected value is $r_0 + \sum_{j=1}^n x_{ij}(Er_i - r_0)$.

The fundamental problem in the selection of $x_i = (x_{i1}, \ldots, x_{in})$ that we concern ourselves with here, as a prelude to utility optimization in our equilibrium model, is to

$$\mathscr{P}_i(\Delta_i) \quad \text{minimize } \mathscr{D}_i\left(\sum_{j=1}^n x_{ij} r_j\right) \quad \text{subject to } x_{ij} \geqslant 0, \quad \sum_{j=1}^n x_{ij}(Er_j - r_0) \geqslant \Delta_i,$$

where the parameter $\Delta_i > 0$ gives the amount of "gain" above the risk-free rate that is demanded of the x_i-portfolio.

The constraint $x_i \geqslant 0$ in $\mathscr{P}_i(\Delta_i)$ requires $x_{ij} \geqslant 0$ for $j = 1, \ldots, n$, but this does not necessarily keep the amount $1 - \sum_{j=1}^n x_{ij}$ invested in the risk-free asset from perhaps being negative, which would correspond to taking on a debt of this magnitude. Thus, *we exclude short positions in the risky assets*, but on the other hand *we allow leveraging*, i.e., borrowing in order to invest in those assets.

In the classical mean-variance portfolio theory behind the capital asset pricing model, short positions have usually been allowed, although this has presented difficulties. In disallowing short positions here, we follow the precedent of Konno and Shirakawa (1994) and Konno and Shirakawa (1995), in their equilibrium work on standard deviation and mean absolute deviation. However, they also disallowed debt (leveraging), whereas we do not.

In our earlier efforts in Rockafellar et al. (2005) on extending the central facts in classical portfolio theory to generalized deviation measures, we did allow shorting, but the result that will needed in the present paper, a generalized "one-fund theorem", can readily be adapted to the absence of shorting. It is stated as Theorem 2.1 below. The conditions on the random variables r_j that go into it, along with a "relevance" condition needed later in our equilibrium analysis, are the following.

Assumptions 2.1 (*Risky assets*). We suppose that the expected rates of return Er_j for the risky assets j are not identical, and

$$Er_j > r_0 \quad \text{for } j = 1, \ldots, n. \tag{1}$$

We assume that no linear combination of $r_1, \ldots, r_n$ with coefficients not all zero is a constant (i.e., riskless). Furthermore, we assume that none of the risky assets is *irrelevant to the market*, in the context of the deviation measures $\mathcal{D}_i$ being used, in the sense of not being present in a positive amount in any solution to any problem $\mathcal{P}_i(\Delta_i)$.

The relevance assumption merely amounts to a harmless kind of "normalization" of our framework, since an asset having no interest to any of the investors under consideration ought simply to be left out. The linear-independence-type assumption on the risky assets likewise has this character, as explained in Rockafellar et al. (2005, Proposition 1), and similarly the conditions on Er_j.

Theorem 2.1. *In terms of*

$$\begin{cases} \bar{\delta}_i = \textit{minimum deviation value in } \mathcal{P}_i(\Delta_i) \textit{ for } \Delta_i = 1, \\ \overline{X}_i = \textit{set of minimizing vectors in } \mathcal{P}_i(\Delta_i) \textit{ for } \Delta_i = 1, \end{cases} \tag{2}$$

the following facts hold:

$$\bar{\delta}_i \textit{ is positive}, \quad \overline{X}_i \textit{ is nonempty, convex and compact}, \tag{3}$$

and furthermore

$$\textit{for} \quad \mathcal{P}_i(\Delta_i) \quad \textit{in general} \begin{cases} \textit{the maximum value is} \quad \delta_i = \Delta_i \bar{\delta}_i, \\ \textit{the solution set is} \quad X_i = \{\Delta_i \bar{x}_i | \bar{x}_i \in \overline{X}_i\}. \end{cases} \tag{4}$$

Also, the inequality constraint in $\mathcal{P}_i(\Delta_i)$ *always holds as an equation at optimality.*

Proof. The argument provided in Rockafellar et al. (2005, Proposition 5) for the case with shorting allowed carries over with only the obvious adjustments in wording. The provision in (1) ensures the feasibility of attaining any level of gain $\Delta_i > 0$ above the risk-free rate. □

Following our terminology in Rockafellar et al. (2005), we call $\bar{\delta}_i$ the *basic deviation value* associated with $\mathcal{D}_i$ and $r_0, r_1, \ldots, r_n$. It gives the rate at which the deviation of a portfolio has to increase as the targeted future value is increased. We say that the vectors $\bar{x}_i \in \overline{X}_i$ give the *basic funds* associated with $\mathcal{D}_i$ and $r_0, r_1, \ldots, r_n$. (Although we know there is at least one such vector, there might be more than one, in general; see Rockafellar et al. (2005).) According to (4), investor i, in solving $\mathcal{P}_i(\Delta_i)$ with $\Delta_i > 0$, will always get a portfolio that is some multiple of a basic fund portfolio.

The *master fund* portfolios associated with the deviation measure $\mathcal{D}_i$ in this framework are obtained by rescaling so as to have unit investment instead of unit "gain". In other words, they correspond to the vectors $\tilde{x}_i = (\tilde{x}_{i1}, \ldots, \tilde{x}_{in})$ having the form

$$\tilde{x}_i = \bar{x}_i / (\bar{x}_{i1} + \cdots + \bar{x}_{in}) \quad \text{with } \bar{x}_i \in \overline{X}_i. \tag{5}$$

Here $\bar{x}_{i1} + \cdots + \bar{x}_{in} > 0$ because $\bar{x}_i \geqslant 0$ and $\bar{x}_i \neq 0$, the latter coming from the equation $\sum_{j=1}^{n} \bar{x}_{ij}(Er_i - r_0) = 1$ that holds for solutions to $\mathcal{P}_i(1)$. Through this relationship, choosing a multiple of a basic fund portfolio is equivalent to choosing a multiple of a master fund portfolio.

Working with basic funds is more convenient for us than working with master funds in the context of optimization, because of their immediate connection with the optimization

Please cite this article in press as: Rockafellar, R.T. et al., Equilibrium with investors using a diversity ..., J. Bank Finance (2007), doi:10.1016/j.jbankfin.2007.04.002

problems $\mathscr{P}_i(\Delta_i)$, but in the end it does not matter whether results are stated in terms of basic funds or master funds.

An important consequence of Theorem 2.1 concerning the basic funds/master funds associated with the deviation measures $\mathscr{D}_i$ for $i = 1, \ldots, m$, comes out of our assumption on the "relevance" of the risky assets.

Proposition 2.1. *Basic fund vectors $\bar{x}_i \in \overline{X}_i$ can be selected for $i = 1, \ldots, m$ in such a way that*

$$\text{for each risky asset } j, \text{ at least one of the vectors } \bar{x}_i \text{ has } \bar{x}_{ij} > 0. \tag{6}$$

Equivalently, master fund vectors $\tilde{x}_i$ exist with this property.

Proof. Our relevance assumption can be restated through Theorem 2.1 as the condition that no risky asset is rejected by every basic fund associated with the various deviation measures $\mathscr{D}_i$. This guarantees that we can choose for each i a collection of vectors $\bar{x}_i^k \in \overline{X}_i$ for $k = 1, \ldots, n_i$ (maybe just with $n_i = 1$), such that for each $j \in \{1, \ldots, n\}$ there will be some $\bar{x}_i^k$ having $\bar{x}_{ij}^k > 0$. The vectors $\bar{x}_i = (\bar{x}_i^1 + \cdots + \bar{x}_i^{n})/n_i$ then belong to $\overline{X}_i$ as well, by the convexity in (3), and they satisfy (6). □

3. Equilibrium model

Market equilibrium will be concerned with the effects of prices on the trading (buying and selling) of assets by the investors $i = 1, \ldots, m$.

With $p_j > 0$ denoting the (yet-to-be-determined) market price per share of asset j, the future value per share of asset j is $(1 + r_j)p_j$. The dependence in this way of future values on current prices is fundamental to the approach we are taking.

Let $Z_j > 0$ be the total number of shares of asset j for $j = 1, \ldots, n$. Let z_{ij}^0 be the (maybe fractional) number of shares of asset j held initially by investor i, with

$$z_{ij}^0 \geqslant 0 \quad \text{for } j = 1, \ldots, n, \tag{7}$$

as well as

$$\sum_{i=1}^{m} z_{ij}^0 = Z_j \quad \text{for } j = 1, \ldots, n. \tag{8}$$

Let z_{i0}^0 be the initial holding of investor i in the riskless asset, the units being the same as those of the prices (i.e., "money"). Under the prices p_j, the initial wealth of investor i is then

$$w_i(p) = z_{i0}^0 + \sum_{j=1}^{n} z_{ij}^0 p_j. \tag{9}$$

We do not require $z_{i0}^0 \geqslant 0$, so this initial wealth might conceivably be negative; investor i might be in debt. That causes no difficulties in what follows, but of course it could also be excluded by adding the nonnegativity of z_{i0}^0 to the assumptions in (7).

In the market, investor i trades the shares z_{ij}^0 for shares z_{ij} at the prices p_j subject to

$$z_{ij} \geqslant 0 \quad \text{for } j = 1, \ldots, n, \tag{10}$$

(no short positions in the risky assets) and the budget constraint that

$$z_{i0} + \sum_{j=1}^{n} z_{ij} p_j = w_i(p). \tag{11}$$

The random variable giving the future value of the resulting portfolio for investor i is

$$R_i = z_{i0}(1 + r_0) + \sum_{j=1}^{n} z_{ij} p_j (1 + r_j). \tag{12}$$

We adopt the modeling premise that investor i is only interested in two aspects of this random variable, namely its expected value ER_i and its deviation value $\mathcal{D}_i(R_i)$. Investor i balances these by means of a utility function $U_i(\mu_i,\delta_i)$ of parameters $\mu_i \in (-\infty, \infty)$ and $\delta_i \in [0, \infty)$ standing for mean and deviation. In response to a price vector $p = (p_1, \ldots, p_m)$, investor i solves the following optimization problem in order to determine the new portfolio:

$$\overline{\mathcal{P}}_i(p) \text{ choose } z_i = (z_{i0}, z_{i1}, \ldots, z_{in}) \text{ to maximize } U_i(ER_i, \mathcal{D}_i(R_i)) \text{ under (10) and (11).}$$

Despite appearances, this problem statement will be shown to allow investor i to impose a constraint in the form of a deviation cap, $\mathcal{D}_i(R_i) \leqslant \delta_i^*$, if desired. The generality of our assumptions about U_i will make this possible.

Definition 3.1 (*Market equilibrium*). A price vector p together with vectors $z_i = (z_{i0}, z_{i1}, \ldots, z_{in})$ for $i = 1, \ldots, m$ furnishes a market *equilibrium* if $p > 0$ (i.e., $p_j > 0$ for $j = 1, \ldots, n$), each z_i solves the corresponding optimization problem $\overline{\mathcal{P}}_i(p)$, and

$$\sum_{i=1}^{m} z_{ij} = Z_j \quad \text{for } j = 1, \ldots, n. \tag{13}$$

These equations mean that after the trading, as before, all the shares of all the risky assets $j = 1, \ldots, n$ are held among the investors $i = 1, \ldots, m$. The supply of shares has exactly met the demand for shares arising from the investors' desire to optimize utility. The corresponding equation

$$\sum_{i=1}^{m} z_{i0} = \sum_{i=1}^{m} z_{i0}^{0}$$

for the riskless asset will then be satisfied automatically through (8), (9) and (11). Since the terms in this equation are allowed to be negative, it has the interpretation that the *net* holdings in this asset are the same after trading as before.

Our goal is to provide a criterion for the existence of an equilibrium, and for this we need to put some restrictions on the utility functions.

Assumptions 3.1 (*Utility function*). For each investor i, the function U_i on $(-\infty,\infty) \times [0, \infty)$, which is allowed to take on $-\infty$ but not $+\infty$, has the following properties:

$$U_i(\mu_i,\delta_i) \text{ is} \begin{cases} \text{concave with respect to } \mu_i \text{ and } \delta_i \text{ jointly,} \\ \text{increasing with respect to } \mu_i, \\ \text{nonincreasing with respect to } \delta_i, \\ \text{upper semicontinuous with respect to } \mu_i \text{ and } \delta_i \text{ jointly,} \end{cases} \tag{14}$$

and, in relation to the basic deviation value $\bar{\delta}_i$ associated with $\mathscr{D}_i$, it has the property that

$$\text{for all } \mu_i,\ \textit{all } \delta_i \in (0,\bar{\delta}_i] \begin{cases} \text{there is a } \tau_1 > 0 \text{ with } U_i(\mu_i+\tau_1,\tau_1\delta_i) > U_i(\mu_i,0), \\ \text{there is a } \tau_2 > 0 \text{ with } U_i(\mu_i+\tau_2,\tau_2\delta_i) < U_i(\mu_i,0). \end{cases} \tag{15}$$

In allowing $U_i(\mu_i,\delta_i)$ to be $-\infty$ for some instances of μ_i and δ_i, we effectively force the random variable R_i in problem $\overline{\mathscr{P}}_i(p)$ not to have such a combination of mean μ_i and deviation δ_i (cf. Example 3.2 below). This is where a deviation cap can come in.

The requirement that $U_i(\mu_i,\delta_i)$ be *increasing* in μ_i means that $U_i(\mu_i',\delta_i) > U_i(\mu_i,\delta_i)$ when $\mu_i' > \mu_i$, whereas the requirement that $U_i(\mu_i,\delta_i)$ be *nonincreasing* with respect to δ_i means that $U_i(\mu_i,\delta_i') \leqslant U_i(\mu_i,\delta_i)$ when $\delta_i' > \delta_i$. Of course such monotonicity conditions could be captured by sign conditions on partial derivatives of U_i, but the assumption that partial derivatives exist would be a pointless limitation, potentially leading to conflicts with our tactic of allowing U_i to take on $-\infty$. The *upper semicontinuity* in (14) means that for every $c \in \mathbb{R}$ the set of $(\mu_i,\delta_i) \in (-\infty,\infty)\times[0,\infty)$ satisfying $U_i(\mu_i,\delta_i) \geqslant c$ is closed. This is a minor technical requirement, chiefly for coping with the possibility of U_i taking on $-\infty$.

These assumptions, so far, follow a pattern going all the way back to Mossin (1966), except that utility functions have elsewhere been assumed to be finite, continuous and differentiable. The properties in (15), however, distinguish our model from others. They will pave the way toward an application of convex analysis which will validate a crucial fixed point argument.

The significance of (15) lies in how investor i will react to adding to a riskless portfolio with future value μ_i some multiple τ of a risky portfolio with expected future value 1 and deviation δ_i, thus getting a portfolio with expected future value $\mu_i+\tau$ and deviation $\tau\delta_i$. In light of $U_i(\mu_i+\tau,\tau\delta_i)$ being concave as a function of τ by (14), the requirements in (15) are equivalent to insisting that the utility must increase at first, as τ rises above 0, but later decrease and actually tend to $-\infty$. The addition of some amount of the risky portfolio to the riskless portfolio should be attractive at least if the amount is small enough, but should be shunned if the amount is too large. But this is stipulated only for risky portfolios with deviation $\delta_i \leqslant \bar{\delta}_i$. That makes sense from (1)–(3): investor i should not be coerced to react favorably, even for small τ, to a higher rate of increase in deviation than $\bar{\delta}_i$. Clearly, if U_i satisfies (15) for all $\delta_i \in (0,\infty)$, not just for $\delta_i \in (0,\bar{\delta}_i]$, then all the better – there is no need to get involved with the basic deviation value $\bar{\delta}_i$.

Example 3.1. For a nonlinear utility function having the form

$$U_i(\mu_i,\delta_i) = \mu_i - \lambda_i\delta_i^{q_i} \quad \text{with} \quad \lambda_i \geqslant 0 \text{ and } q_i > 1, \tag{16}$$

all the assumptions are satisfied, regardless of the magnitude of $\bar{\delta}_i$. This corresponds to investor i maximizing $ER_i - \lambda_i\mathscr{D}_i(R_i)^{q_i}$ in problem $\overline{\mathscr{P}}_i(p)$. In the linear case where $q_i = 1$, the assumptions are unfulfilled because the conditions in (15) conflict with each other.

A classical choice for Example 3.1 would $q_i = 2$ with $\mathscr{D}_i$ taken to be standard deviation. Then investor i in problem $\overline{\mathscr{P}}_i(p)$ would be maximizing $ER_i - \lambda_i\sigma^2(R_i)$.

Example 3.2. For a specified upper bound $\delta_i^* > 0$ on deviation, let

$$U_i(\mu_i, \delta_i) = \begin{cases} \mu_i & \text{if } \delta_i \leqslant \delta_i^*, \\ -\infty & \text{if } \delta_i > \delta_i^*. \end{cases} \tag{17}$$

Then all the utility assumptions are satisfied, regardless of the magnitude of $\bar{\delta}_i$. In this case investor i, in solving problem $\overline{\mathcal{P}}_i(p)$, is concerned simply with choosing z_i to maximize ER_i subject to the budget constraint (11) and the deviation cap $\mathcal{D}_i(R_i) \leqslant \delta_i^*$.

As a matter of fact, the kind of implicit constraint on deviation incorporated in (17) is essentially the only kind that our assumptions can accommodate. The reason is that the set of (μ_i, δ_i) having $U_i(\mu_i, \delta_i) > -\infty$ is convex, owing to the concavity of U_i in (14), and through (15) it must include the μ_i-axis, and more besides. If not equal to all of $(-\infty,\infty) \times [0,\infty)$, this set must be a sort of "strip" that puts a cap δ_i^* on δ_i. (It could happen that $U_i(\mu_i,\delta_i)$ tends to $-\infty$ as δ_i rises to δ_i^*; this does not conflict with the upper semicontinuity in (14).)

The failings of the linear utility case in Example 3.1 with $q_i = 1$ could be countered by imposing a deviation cap as in Example 3.2, as long as $\lambda_i < 1/\bar{\delta}_i$. This demonstrates that the magnitude of $\bar{\delta}_i$ could make a difference in some situations.

More will be said about Examples 3.1 and 3.2 at the beginning of Section 5.

Theorem 3.1. *An equilibrium is certain to exist under the assumptions placed on the utility functions U_i and the risky assets $j = 1,\ldots,n$ with respect to the deviation measures $\mathcal{D}_i$.*

Theorem 3.2

(a) *In any equilibrium the vector $(x_{i1},\ldots,x_{in}) = (z_{i1}p_1,\ldots,z_{in}p_n)$, giving the amounts invested in the risky assets $j = 1,\ldots,n$ by an investor i, will be a positive multiple of a basic fund vector $\bar{x}_i = (\bar{x}_{i1},\ldots,\bar{x}_{in})$ associated with $\mathcal{D}_i$ (or the corresponding master fund vector $\tilde{x}_i$). In particular, it will be nonnegative but not the zero vector.*

(b) *In the case of nonuniqueness of basic funds, an arbitrary choice of $\bar{x}_i \in \overline{X}_i$ can be made for each deviation measure $\mathcal{D}_i$, subject only to (6), and then an equilibrium will exist in which (a) holds for those particular basic funds (or the corresponding master funds).*

Theorems 3.1 and 3.2 will be proved in Section 5 after some work in the Section 4 which connects the optimization problems in our model with the fundamental portfolio problems in Section 2.

4. Characterization in terms of masterfunds

As a step toward understanding the model better and working up to the proof of Theorems 3.1 and 3.2 in stages, let us view the budget equation in (11) as a prescription for obtaining z_{i0} from the other variables:

$$z_{i0} = w_i(p) - \sum_{j=1}^{n} z_{ij} p_j. \tag{18}$$

Substituting this into the formula for R_i in (12), we get

$$R_i = w_i(p)(1+r_0) + \sum_{j=1}^{n} z_{ij}p_j(r_j - r_0), \tag{19}$$

so that

$$ER_i = w_i(p)(1+r_0) + \sum_{j=1}^{n} z_{ij}p_j(Er_j - r_0) \tag{20}$$

and

$$\mathscr{D}_i(R_i) = \mathscr{D}_i\left(\sum_{j=1}^{n} z_{ij}p_j(r_j - r_0)\right) = \mathscr{D}_i\left(\sum_{j=1}^{n} z_{ij}p_j r_j\right). \tag{21}$$

In these terms, problem $\overline{\mathscr{P}}_i(p)$ comes down to the following:

$$\begin{cases} \text{choose } (z_{i1},\ldots,z_{in}) \geqslant (0,\ldots,0) \text{ to maximize} \\ U_i\left(w_i(p)(1+r_0) + \sum_{j=1}^{n} z_{ij}p_j(Er_j - r_0), \mathscr{D}_i\left(\sum_{j=1}^{n} z_{ij}p_j r_j\right)\right) \\ \text{and then get the corresponding } z_{i0} \text{ from(18).} \end{cases} \tag{22}$$

This formulation leads us to consider, alongside of $\overline{\mathscr{P}}_i(p)$, the auxiliary problem

$$\mathscr{P}'_i(p) \quad \begin{cases} \text{choose } x_i = (x_{i1},\ldots,x_{in}) \geqslant (0,\ldots,0) \text{ to maximize} \\ U_i\left(w_i(p)(1+r_0) + \sum_{j=1}^{n} x_{ij}(Er_j - r_0),\ \mathscr{D}_i\left(\sum_{j=1}^{n} x_{ij}r_j\right)\right). \end{cases}$$

Note that there are no constraints on the x_{ij}'s in $\overline{\mathscr{P}}'_i(p)$ beyond nonnegativity, other than those that may be implicit in ensuring that the value of U_i is not $-\infty$ (as for instance in the case of Example 3.2).

Proposition 4.1. *A price vector $p > 0$ furnishes an equilibrium with shares z_{ij} for $i = 1,\ldots,m$ and $j = 0,1,\ldots,n$, if and only if vectors $x_i = (x_{i1},\ldots,x_{in})$ exist such that*

(a) *x_i solves problem $\overline{\mathscr{P}}'_i(p)$ for $i = 1,\ldots,m$,*
(b) *$p_j = \sum_{i=1}^{m} x_{ij}/Z_j$ for $j = 1,\ldots,n$,*
(c) *$z_{ij} = x_{ij}/p_j$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$,*
(d) *$z_{i0} = w_i(p) - \sum_{j=1}^{n} z_{ij}p_j$ for $i = 1,\ldots,m$.*

Proof. Under the assumption that $p_j > 0$ for $j = 1,\ldots,n$, the change of variables $x_{ij} = z_{ij}p_j$ is reversible through (c). Having a solution to problem $\overline{\mathscr{P}}_i(p)$, which is characterized by the prescription in (22), is equivalent to having (a) and then invoking (c) and (d). The equations in (b) mean then that (13) holds, so that the conditions in the definition of an equilibrium are fulfilled. □

Further insights can now be gained by expressing the optimization problem $\overline{\mathscr{P}}'_i(p)$ in an equivalent but more extensive form in which another decision variable Δ_i appears:

$$\overline{\mathscr{P}}''_i(p) \quad \begin{cases} \text{choose } x_i = (x_{i1},\ldots,x_{in}) \text{ and } \Delta_i \text{ to maximize} \\ U_i\left(w_i(p)(1+r_0) + \Delta_i, \mathscr{D}_i\left(\sum_{j=1}^{n} x_{ij}r_j\right)\right) \\ \text{subject to } \quad x_{ij} \geqslant 0,\ \sum_{j=1}^{n} x_{ij}(Er_j - r_0) \geqslant \Delta_i. \end{cases}$$

This relies on our assumption that $U_i(\mu_i, \delta_i)$ increases as μ_i increases, which implies further that the expectation inequality must hold as an equation at optimality. The advantage of the seemingly redundant reformulation is that it allows us to apply the portfolio results from Rockafellar et al. (2005) which were summarized in Section 2.

Consider the subproblem of $\overline{\mathscr{P}}''_i(p)$ in which Δ_i is fixed and the minimization is carried out in x_i alone. Because $U_i(\mu_i, \delta_i)$ is nonincreasing with respect to δ_i, this subproblem can be reduced to the earlier problem $\mathscr{P}_i(\Delta_i)$; it does not depend on p. We arrive then at an illuminating prescription for solving the optimization problem $\overline{\mathscr{P}}_i(p)$ for investor i, when posed as $\overline{\mathscr{P}}''_i(p)$. In presenting it, we recall from (2) the notation $\bar{X}_i$ for the (nonempty) set of basic fund vectors $\bar{x}_i$ associated with the deviation measure $\mathscr{D}_i$ and introduce

$$M_i(p) = \{\text{set of all } \Delta_i \text{ maximizing } U_i(w_i(p)(1+r_0) + \Delta_i, \Delta_i\bar{\delta}_i) \quad \text{subject to } \Delta_i \geqslant 0\}. \tag{23}$$

Proposition 4.2. *A vector x_i solves problem $\overline{\mathscr{P}}'_i(p)$ if and only if there is a Δ_i such that x_i and Δ_i solve problem $\overline{\mathscr{P}}''_i(p)$. Moreover, determining x_i and Δ_i to solve $\overline{\mathscr{P}}''_i(p)$ comes down to the following: take $\Delta_i \in M_i(p)$ and then take $x_i = \Delta_i\bar{x}_i$ for any $\bar{x}_i \in \overline{X}_i$. Here $M_i(p)$ is sure to be a nonempty, closed interval of values $\Delta_i > 0$ (maybe just one value).*

Thus, x_i solves $\overline{\mathscr{P}}'_i(p)$ if and only if $x_i = \Delta_i\bar{x}_i$ for some $\bar{x}_i \in \overline{X}_i$ and $\Delta_i \in M_i(p)$.

Proof. The initial assertion is correct from the formulation of $\overline{\mathscr{P}}''_i(p)$ and the properties of U_i. Because $U_i(\mu_i, \delta_i)$ is nonincreasing in δ_i, the maximization of the utility expression in $\overline{\mathscr{P}}''_i(p)$ with respect to x_i for a fixed $\Delta_i > 0$ yields the value that corresponds to choosing x_i to minimize $\mathscr{D}_i\left(\sum_{j=1}^n x_{ij}r_j\right)$ subject to $\sum_{j=1}^n x_{ij}(Er_j - r_0) \geqslant \Delta_i$, as in the portfolio problem $\mathscr{P}_i(\Delta_i)$. According to (4) in Theorem 2.1, the maximum utility obtained this way for a fixed $\Delta_i > 0$ is $U_i(w_i(p)(1+r_0) + \Delta_i, \Delta_i\bar{\delta}_i)$. For fixed $\Delta_i \leqslant 0$, of course, the maximum with respect to x_i would be achieved just by taking $x_i = 0$ and getting deviation 0; but $U_i(w_i(p)(1+r_0) + \Delta_i, 0) \leqslant U_i(w_i(p)(1+r_0), 0)$ when $\Delta_i \leqslant 0$ by (15). Therefore, the residual problem of optimization in $\overline{\mathscr{P}}''_i(p)$ with respect to Δ_i, after the maximization with respect to x_i, is the one having $M_i(p)$ as the set of optimal values of Δ_i.

In that problem, the function $\varphi(\Delta_i) = U_i(w_i(p)(1+r_0) + \Delta_i, \Delta_i\bar{\delta}_i)$ is upper semicontinuous and concave with $\varphi(0)$ finite. Through (15), there exists $\Delta_i > 0$ such that $\varphi(\Delta_i) > \varphi(0)$, but also $\Delta_i > 0$ such that $\varphi(\Delta_i) < \varphi(0)$. The set $\{\Delta_i \geqslant 0 | \varphi(\Delta_i) \geqslant \varphi(0)\}$ is thus a closed, bounded interval on which the maximum of φ must somewhere be attained, but not at 0. Hence the set of Δ_i achieving the maximum is a nonempty, closed, bounded interval in $(0, \infty)$.

The pairs (Δ_i, x_i) solving $\mathscr{P}''_i(p)$ are thus the ones such that $\Delta_i \in M_i(p)$ and x_i solves the portfolio subproblem in $\mathscr{P}''_i(p)$ for that Δ_i. We know from (4) that the vectors $\Delta_i\bar{x}_i$ $\bar{x}_i \in \overline{X}_i$ solve that subproblem, for which the maximum is $U_i(w_i(p)(1+r_0) + \Delta_i, \Delta_i\bar{\delta}_i)$, and that if there somehow were others, they would have to achieve this same utility value with $\mathscr{D}\left(\sum_{j=1}^n x_{ij}\right) = \Delta'_i\bar{\delta}_i$ for $\Delta'_i > \Delta_i$. The monotonicity properties of U_i exclude this, however, because they imply for $\Delta'_i > \Delta_i$ that $U_i(w_i(p)(1+r_0) + \Delta_i, \Delta'_i\bar{\delta}_i) < U_i(w_i(p)(1+r_0)+$

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$\Delta_i', \Delta_i'\bar{\delta}_i) \leqslant U_i(w_i(p)(1+r_0) + \Delta_i, \Delta_i\bar{\delta}_i)$. Hence the prescription for solving $\mathscr{P}_i''(p)$ is complete. □

Propositions 4.1 and 4.2 allow us now to confirm the claims in part (a) of Theorem 3.2, even if the proof of part (b) has to await further developments.

Proof. Theorem 3.2(a). We merely have to apply the knowledge in Proposition 4.2 about solutions x_i to $\overline{\mathscr{P}}'(p)$ to the prescription in Proposition 4.1 for determining the shares z_{ij} in an equilibrium. Under our assumption about master funds, the portfolio vectors giving them are obtained by a simple rescaling of the vectors giving basic funds; therefore, saying that the vector in (4) is a positive multiple of a master fund vector is the same as saying it is a positive multiple of some $\bar{x}_i \in \overline{X}_i$. Such vectors are known to be nonzero but not the zero vector. □

From now on, our efforts will mainly be concentrated on additional background needed for the verification of the existence of equilibrium claimed in Theorem 3.1. The next result gives the key to the approach we will take.

Proposition 4.3. *A price vector p furnishes an equilibrium if and only if the equations*

$$\sum_{i=1}^{m} \bar{\Delta}_i \bar{x}_{ij} = p_j Z_j \quad \textit{for } j = 1, \ldots, n, \tag{24}$$

can be satisfied by some choice of vectors $\bar{x}_i \in \overline{X}_i$ satisfying (6) and values $\bar{\Delta}_i \in M_i(p)$. Thus, an equilibrium exists if and only if there exist values $\bar{\Delta}_i$ with the property that, for some choice of vectors $\bar{x}_i \in \overline{X}_i$, the prices defined by

$$p_j = \sum_{i=1}^{m} \bar{\Delta}_i \bar{x}_{ij} / Z_j \quad \textit{for } j = 1, \ldots, n, \tag{25}$$

will be such that $\bar{\Delta}_i \in M_i(p)$ for $i = 1, \ldots, m$. The equilibrium holdings of the investors are given then for the risky assets by

$$z_{ij} = f_{ij} Z_j \quad \textit{with } f_{ij} = \frac{\bar{\Delta}_i \bar{x}_{ij}}{\bar{\Delta}_1 \bar{x}_{1j} + \cdots + \bar{\Delta}_m \bar{x}_{mj}} \geqslant 0, \quad \textit{for } j = 1, \ldots, n, \tag{26}$$

and for the riskless asset by

$$z_{i0} = w_i(p) - \sum_{j=1}^{n} \bar{\Delta}_i \bar{x}_{ij}. \tag{27}$$

Proof. This combines Proposition 4.2 with Proposition 4.1. By Proposition 4.2, the vectors x_i in (a) of Proposition 4.1 must be of the form $\Delta_i \bar{x}_i$, and the values Δ_i in question have to be positive. Because of (6), our prescription for prices in (b) of Proposition 4.1 yields $p_j > 0$. The substitution of these prices into the formulas in (c) and (d) of Proposition 4.1, where $z_{ij} p_j = x_{ij} = \bar{\Delta}_i \bar{x}_{ij}$, then completes the specification of the equilibrium. □

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5. Existence arguments

The characterization of the existence of equilibrium in Proposition 4.3 will lead us to a fixed point argument which will furnish a proof of Theorem 3.1. In an important case, however, the existence of equilibrium can immediately be verified at this stage without resorting to a fixed point argument. It is worthwhile to record this fact directly, even though it could also just be written down later as a corollary of our broader result.

Theorem 5.1. *Suppose that the sets $M_i(p)$ in (23) do not actually depend on p and thus can be denoted simply by M_i, as is true for the utility functions in Examples 3.1 and 3.2. Then for any choice of $\bar{\Delta}_i \in M_i$ and $\bar{x}_i \in \bar{X}_i$ for $i = 1, \dots, m$ satisfying (6), the price vector $p = \sum_{i=1}^{m} \bar{\Delta}_i \bar{x}_i / Z_j$ will furnish an equilibrium.*

Proof. This follows at once from the characterization of equilibrium in Proposition 4.3. □

Example 5.1. If every investor i has a utility function U_i of the nonlinear form in (16) of Example 3.1, then an equilibrium exists and is described for any choice of vectors $\bar{x}_i \in \bar{X}_i$ satisfying (6) by (25)–(27) with

$$\bar{\Delta}_i = 1/(q_i \lambda_i \bar{\delta}_i^{q_i})^{1/(q_i - 1)} \quad \text{for } i = 1, \dots, m. \tag{28}$$

Example 5.2. If every investor i has a utility function U_i of the form (17) in Example 3.2, capping the deviation at level δ_i^*, then an equilibrium exists and is described for any choice of vectors $\bar{x}_i \in \overline{X}_i$ satisfying (6) by (25)–(27) with

$$\bar{\Delta}_i = \delta_i^* / \bar{\delta}_i \quad \text{for } i = 1, \dots, m. \tag{29}$$

Turning now to the task of setting up a fixed point argument for the existence of equilibrium in the general case of Theorem 3.1, we carry the ideas in Proposition 4.3 further by taking p from the formula in (25) and considering what this requires of $\bar{\Delta}_i$ and $\bar{x}_i$. In terms of the fractions

$$f_{ij}^0 = z_{ij}^0 / Z_j, \tag{30}$$

which describe the initial holdings of the investors, the initial wealth of investor i with respect to the price vector p specified by (25) has the expression

$$w_i(p) = z_{i0}^0 + \sum_{j=1}^{n} f_{ij}^0 \sum_{k=1}^{m} \bar{\Delta}_k \bar{x}_{kj}. \tag{31}$$

To say that $\Delta_i \in M_i(p)$ when p is given by (25) is to say that Δ_i maximizes

$$U_i\left(\left(z_{i0}^0 + \sum_{j=1}^{n} f_{ij}^0 \sum_{k=1}^{m} \bar{\Delta}_k \bar{x}_{kj}\right)(1 + r_0) + \Delta_i, \Delta_i \bar{\delta}_i\right),$$

subject to $\Delta_i \geqslant 0$. This leads us to introduce the shorthand notation

$$\Delta = (\Delta_1, \dots, \Delta_m), \quad \bar{\Delta} = (\bar{\Delta}_1, \dots, \bar{\Delta}_m), \quad \bar{x} = (\bar{x}_1, \dots, \bar{x}_m), \quad \overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_m,$$

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along with the vectors

$$a_{i,\bar{x}} = \left(\sum_{j=1}^{n} f_{ij}^{0}\bar{x}_{1j}, \ldots, \sum_{j=1}^{n} f_{ij}^{0}\bar{x}_{mj}\right), \tag{32}$$

in terms of which we can focus on the functions

$$\overline{U}_{i,\bar{x}}(\bar{\Delta}, \Delta_i) = U_i((z_{i0} + \bar{\Delta} \cdot a_{i,\bar{x}})(1 + r_0) + \Delta_i, \Delta_i\bar{\delta}_i) \tag{33}$$

and sets

$$\overline{M}_{i,\bar{x}}(\bar{\Delta}) = \{\text{set of all } \Delta_i \text{ maximizing } \overline{U}_{i,\bar{x}}(\bar{\Delta}, \Delta_i) \text{ subject to } \Delta_i \geqslant 0\}. \tag{34}$$

We consider then for any $\bar{x} \in \overline{X}$ satisfying (6) the set-valued mapping $S_{\bar{x}}$ defined by

$$S_{\bar{x}}(\bar{\Delta}) = \{\text{set of all } \Delta \text{ such that } \Delta_i \in \overline{M}_{i,\bar{x}}(\bar{\Delta}) \quad \text{for } i = 1, \ldots, m\}. \tag{35}$$

Proposition 5.1. *If for some $\bar{x} \in \overline{X}$ satisfying (6) the set-valued mapping $S_{\bar{x}}$ has a fixed point, i.e., a vector $\bar{\Delta}$ satisfying $\bar{\Delta} \in S_{\bar{x}}(\bar{\Delta})$, then an equilibrium exists, moreover one in which the master funds are positive multiples of the vectors $\bar{x}_i$ that comprise $\bar{x}$.*

Proof. This is hardly more than a restatement of the existence criterion of Proposition 4.3 in which the sets $M_i(p)$ are replaced by the sets $\overline{M}_{i,\bar{x}}(\bar{\Delta})$ which have been obtained by taking p from $\bar{\Delta}$ and $\bar{x}$ as in (25). □

Proposition 5.2. *For any $\bar{x} \in \overline{X}$ satisfying (6), $S_{\bar{x}}$ has the following properties:*

(a) *The set $S_{\bar{x}}(\bar{\Delta})$ is nonempty, convex and compact in $\mathbb{R}^m_+$ for every $\bar{\Delta} \in \mathbb{R}^m$.*
(b) *The graph of $S_{\bar{x}}$, consisting of all $(\bar{\Delta}, \Delta)$ such that $\Delta \in S_{\bar{x}}(\bar{\Delta})$, is a closed set.*
(c) *There is a $\rho > 0$ such that*

$$\Delta \in S_{\bar{x}}(\bar{\Delta}) \text{ with } \bar{\Delta} \geqslant 0 \quad \text{and} \quad \|\bar{\Delta}\|_\infty \leqslant \rho \Rightarrow \|\Delta\|_\infty < \rho. \tag{36}$$

Proof. For each i, the set $\overline{M}_{i,\bar{x}}(\bar{\Delta})$ is a nonempty, closed interval in $(0,\infty)$; we already know this from Proposition 4.2 – the only thing that has changed is the notation for the initial wealth of investor i. Because $S_{\bar{x}}(\bar{\Delta})$ is the product of the intervals $\overline{M}_{i,\bar{x}}(\bar{\Delta})$, it is a compact, convex subset of $\mathbb{R}^m_+$. This shows (a).

Much of what comes next will stem from the observation that the functions $\overline{U}_{i,\bar{x}}$ are upper semicontinuous and concave on $\mathbb{R}^m \times \mathbb{R}^m$, owing to our assumptions on the functions U_i and the linearity in the dependence on $\bar{\Delta}$ of the initial wealth expressions in (31) and (33), with coefficient vectors coming from (32). This implies in particular that the optimal value $\theta_{i,\bar{x}}(\bar{\Delta})$ in the problem of maximizing $\bar{U}_{i,\bar{x}}(\bar{\Delta}, \Delta_i)$ with respect to $\Delta_i \geqslant 0$ is a concave function of $\bar{\Delta} \in \mathbb{R}^m$ (Rockafellar and Wets, 1998, 2.22), which moreover is everywhere finite. Such a function is necessarily continuous (Rockafellar and Wets, 1998, 2.36). Having $\Delta_i \in \bar{M}_{i,\bar{x}}(\bar{\Delta})$ is the same as having $\overline{U}_{i,\bar{x}}(\bar{\Delta}, \Delta_i) = \theta_{i,\bar{x}}(\bar{\Delta})$, or for that matter, $\overline{U}_{i,\bar{x}}(\bar{\Delta}, \Delta_i) \geqslant \theta_{i,\bar{x}}(\bar{\Delta})$, and therefore

$$\Delta \in S_{\bar{x}}(\bar{\Delta}) \Longleftrightarrow \overline{U}_{i,\bar{x}}(\bar{\Delta}, \Delta) - \theta_{i,\bar{x}}(\bar{\Delta}) \geqslant 0 \quad \text{for } i = 1, \ldots, m.$$

The upper semicontinuity of $\overline{U}_{i,\bar{x}}$ and continuity of $\theta_{i,\bar{x}}$ ensure the upper semicontinuity of $\overline{U}_{i,\bar{x}} - \theta_{i,\bar{x}}$ and consequently the closedness of the set specified by this system of inequalities. Hence (b) is accurate.

In proving (c), a key ingredient will be the convexity and closedness of the sets

$$C_i = \{(\mu_i, \delta_i) | U_i(\mu_i, \delta_i) \geqslant c_i\} \quad \text{for } c_i = U_i(z_{i0}^0(1 + r_0), 0), \tag{37}$$

which follows from the concavity and upper semicontinuity assumed for the utility functions U_i, along with the observation that

$$\Delta_i \in \overline{M}_{i,\bar{x}}(\bar{\Delta}), \quad \bar{\Delta} \geqslant 0 \Rightarrow ((z_{i0} + \bar{\Delta} \cdot a_{i,\bar{x}})(1 + r_0) + \Delta_i, \Delta_i \bar{\delta}_i) \in C_i. \tag{38}$$

That holds because $\Delta_i \in \overline{M}_{i,\bar{x}}(\bar{\Delta})$ implies

$$U_i((z_{i0} + \bar{\Delta} \cdot a_{i,\bar{x}})(1 + r_0) + \Delta_i, \Delta_i \bar{\delta}_i) \geqslant U_i((z_{i0} + \bar{\Delta} \cdot a_{i,\bar{x}})(1 + r_0), 0),$$

(since $\Delta_i = 0$ is one of the candidates in the maximization), and on the other hand

$$U_i((z_{i0} + \bar{\Delta} \cdot a_{i,\bar{x}})(1 + r_0), 0) \geqslant U_i(z_{i0}(1 + r_0), 0),$$

because $U_i(\mu_i, \delta_i)$ increases when μ_i increases, while

$$\bar{\Delta} \cdot a_{i,\bar{x}}(1 + r_0) \geqslant 0 \quad \text{when } \bar{\Delta} \geqslant 0 \tag{39}$$

through the nonnegativity of the vector $a_{i,\bar{x}}$ in (32).

If (c) were false, there would have to exist sequences $\{\bar{\Delta}^k\}_{k=1}^{\infty}$ and $\{\Delta^k\}_{k=1}^{\infty}$ in $\mathbb{R}_+^m$ such that

$$\Delta^k \in S_{\bar{x}}(\bar{\Delta}^k) \quad \text{and} \quad \|\Delta^k\|_\infty \geqslant \|\bar{\Delta}^k\|_\infty \to \infty. \tag{40}$$

We will argue this to a contradiction.

The inequality in (40) means that for each k there is some i such that $\Delta_i^k \geqslant \|\bar{\Delta}^k\|_\infty$. Since there are only finitely many investors i, we can suppose, by passing to subsequences if necessary, that the i in this inequality is always the same, say $i = 1$. Then the vectors $\tilde{\Delta}^k = \bar{\Delta}^k / \Delta_1^k$ have $\|\tilde{\Delta}^k\|_\infty \leqslant 1$, and by passing once again to subsequences if necessary, we can arrange that

$$\lim_{k\to\infty} \tilde{\Delta}^k = \tilde{\Delta} \in \mathbb{R}_+^m \quad \text{for } \tilde{\Delta}^k = \bar{\Delta}^k / \Delta_1^k. \tag{41}$$

We return now to the observation in (38), fixing on the case of $i = 1$:

$$((z_{10} + \bar{\Delta}^k \cdot a_{1,\bar{x}})(1 + r_0) + \Delta_1^k, \Delta_1^k \bar{\delta}_1) \in C_1,$$

which can be written as

$$(((1/\Delta_1^k) z_{10} + \tilde{\Delta}^k \cdot a_{1,\bar{x}})(1 + r_0) + 1, \bar{\delta}_1) \in (1/\Delta_1^k) C_1. \tag{42}$$

Since $1/\Delta_1^k \to 0$, this implies by way of (41) that

$$(\tilde{\Delta} \cdot a_{1,\bar{x}}(1 + r_0) + 1, \bar{\delta}_i) \in C_1^\infty,$$

where C_1^∞ denotes the horizon cone of the set C_1 (Rockafellar and Wets, 1998, p. 81), which by the convexity of C_1 is the same as the recession cone 0^+C_1 of convex analysis (Rockafellar, 1970) (cf. (Rockafellar and Wets, 1998, p. 82)) and is a closed, convex cone in $\mathbb{R}^2$. Then likewise

$$(1, \tilde{\delta}_1) \in C_1^\infty \quad \text{for } \tilde{\delta}_1 = \bar{\delta}_1 / (\tilde{\Delta} \cdot a_{1,\bar{x}}(1 + r_0) + 1) > 0. \tag{43}$$

Since C_1 contains points of the form $(\mu_1, 0)$, namely for $\mu_1 \geqslant z^0_{10}(1 + r_0)$, the criterion in Rockafellar and Wets (1998, Theorem 3.6) for membership in C_1^∞ can be expressed, in the case of a vector of form $(1, \delta_1)$ with $\delta_1 \geqslant 0$, as the property that

$$(\mu_1, 0) + \tau(1, \delta_1) \in C_1 \quad \text{for all } (\mu_1, 0) \in C_1 \text{ and } \tau > 0,$$

which is equivalent to having

$$U_1(\mu_1 + \tau, \tau\delta_1) \geqslant U_1(\mu_1, 0) \quad \text{for all } \tau > 0 \textit{ when } \mu_1 \geqslant z^0_{10}. \tag{44}$$

We are told by (43) that this holds for $\delta_1 = \tilde{\delta}_1$. On the other hand, we know from assumption (14) that (44) holds for $\delta_1 = 0$. The convexity of C_1^∞ implies then that it holds for all $\delta_1 \in [0, \tilde{\delta}_1]$. But we also know from assumption (15) that (44) does not hold for any $\delta_1 \in (0, \bar{\delta}_1]$. These two properties are incompatible with each other, inasmuch as both $\tilde{\delta}_1$ and $\bar{\delta}_1$ are positive. This conflict forces us to conclude that (c) cannot be false. □

With all these pieces in place, we can move on to the final verification of the main results in this paper.

Proof. Theorem 3.1. Take any $\bar{x} \in \bar{X}$ satisfying (6). Take ρ as in (c) of Proposition 5.2, and let D be the compact, convex set consisting of all $\bar{\Delta} \in \mathbb{R}^m_+$ with $\|\bar{\Delta}\|_\infty \leqslant \rho$. We have $S_{\bar{x}}(\bar{\Delta}) \subset D$ for all $\bar{\Delta} \in D$. The graph of $S_{\bar{x}}$ relative to D, which is the intersection of $D \times D$ with the overall graph in (b) of Proposition 5.2, is closed, and the sets $S_{\bar{x}}(\bar{\Delta})$ are nonempty and convex by (a) of Proposition 5.2. The Kakutani fixed point theorem (Border, 1985) is therefore applicable and yields a $\bar{\Delta} \in D$ such that $\bar{\Delta} \in S_{\bar{x}}(\bar{\Delta})$. This assures us through Proposition 5.1 that an equilibrium exists. □

Proof. Theorem 3.2(b). This is immediate now from the fact that the preceding fixed point argument was carried out for an arbitrary choice of $\bar{x} \in \bar{X}$ satisfying (6). □

6. Conclusions

The existence of equilibrium in a financial market has been proved in a setting in which investors optimize portfolios according to their individual preferences for combinations of expected future values and the extent to which those future values may deviate from expectations. The principle new feature is that nonstandard measures of deviation, fitting with axiomatic developments elsewhere, are allowed to be used in place of standard deviation. Different investors can even select different measures of deviation. Moreover they can introduce hard upper bounds on portfolio deviation values, if they want such constraints as part of their expressions of preferences. Despite all these new possibilities, the portfolios at which the various investors arrive will necessarily be scaled from generalized master funds associated with the chosen deviation measure, or measures.

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Please cite this article in press as: Rockafellar, R.T. et al., Equilibrium with investors using a diversity ..., J. Bank Finance (2007), doi:10.1016/j.jbankfin.2007.04.002

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