Numerical Comparison of CVaR and CDaR Approaches: Application to Hedge Funds[1]

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This paper applies risk management methodologies to optimization of a portfolio of hedge funds (fund of funds). We compare two recently developed risk management methodologies: Conditional Value-at-Risk and Conditional Drawdown-at-Risk. The common property of the risk management techniques is that they admit the formulation of a portfolio optimization model as a linear programming (LP) problem. LP formulations allow for implementing efficient and robust portfolio allocation algorithms, which can successfully handle optimization problems with thousands of instruments and scenarios. The performance of various risk constraints is investigated and discussed for in-sample and out-of-sample testing of the algorithm. The numerical experiments show that imposing risk constraints may improve the "real" performance of a portfolio rebalancing strategy in out-of-sample runs. It is beneficial to combine several types of risk constraints that control different sources of risk.

1 Introduction

This paper applies risk management methodologies to the optimization of a portfolio of hedge funds (fund of funds). We compare risk management techniques based on two recently developed risk measures, Conditional Value-at-Risk and Conditional Drawdown-at-Risk (Rockafellar and Uryasev 2000, 2002, Chekhlov et al. 2000). Both risk management techniques utilize stochastic programming approaches and allow for construction of *linear portfolio rebalancing strategies,* and, as a result, have proven their high efficiency in various portfolio management applications (Andersson et al. 2001, Chekhlov et al. 2000, Krokhmal et al. 2002, Rockafellar and Uryasev 2000, 2002). The choice of hedge funds, as a subject for the portfolio optimization strategy, was stimulated by a strong interest to this class of assets by both practitioners and scholars, as well as by challenges related to relatively small datasets available for hedge funds.

Recent studies[2] of the hedge funds industry are mostly concentrated on the classification of hedge funds and the relevant investigation of their activity. However, this paper is focused on possible realization of investment opportunities existing in this market from the viewpoint of portfolio rebalancing strategies (for an extensive discussion of stochastic programming approaches to hedge fund management, see Ziemba 2002).

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[2] See, for example, papers by Ackermann et al. 1999, Amin and Kat 2001, Brown and Goetzmann 2000, Fung and Hsieh 1997, 2000, 2001, and Lhabitant 2001.

Hedge funds are investment pools employing sophisticated trading and arbitrage techniques including leverage and short selling, wide usage of derivative securities etc. Generally, hedge funds restrict share ownership to high net worth individuals and institutions, and are not allowed to offer their securities to the general public. Many hedge funds are limited to 99 investors. This private nature of hedge funds has resulted in few regulations and disclosure requirements, compared for example, with mutual funds (however, stricter regulations exist for hedge funds trading futures). Also, the hedge funds may take advantage of specialized, risk-seeking investment and trading strategies, which other investment vehicles are not allowed to use.

The first official[3] hedge fund was established in the United States by A. W. Jones in 1949, and its activity was characterized by the use of short selling and leverage, which were separately considered risky trading techniques, but in combination could limit market risk. The term "hedge fund" attributes to the structure of Jones fund's portfolio, which was split between long positions in stocks that would gain in value if market went up, and short positions in stocks that would protect against market drop. Also, Jones has introduced another two initiatives, which became a common practice in hedge fund industry, and with more or less variations survived to this day: he made the manager's incentive fee a function of fund's profits, and kept his own capital in the fund, in this way making the incentives of fund's clients and of his own coherent.

Nowadays, hedge funds become a rapidly growing part of the financial industry. According to Van Hedge Fund Advisors, the number of hedge funds at the end of 1998 was 5830, they managed 311 billion USD in capital, with between $800 billion and $1 trillion in total assets. Nearly 80% of hedge funds have market capitalization less than 100 million, and around 50% are smaller than $25 million, which indicates high number of new entries. More than 90% of hedge funds are located in the U.S.

Hedge funds are subject to far fewer regulations than other pooled investment vehicles, especially to regulations designed to protect investors. This applies to such regulations as regulations on liquidity, requirements that fund's shares must be redeemable an any time, protecting conflicts of interests, assuring fairness of pricing of fund shares, disclosure requirements, limiting usage of leverage, short selling etc. This is a consequence of the fact that hedge funds' investors qualify as sophisticated high-income individuals and institutions, which can stand for themselves. Hedge funds offer their securities as private placements, on individual basis, rather than through public advertisement, which allows them to avoid disclosing publicly their financial performance or asset positions. However, hedge funds must provide to investors some information about their activity, and of course, they are subject to statutes governing fraud and other criminal activities.

As market's subjects, hedge funds do subordinate to regulations protecting the market integrity that detect attempts of manipulating or dominating in

[3] Ziemba (2002) traces early unofficial hedge funds, such as Keynes Chest Fund etc., that existed in the 1920's to 1940's.

markets by individual participants. For example, in the United States hedge funds and other investors active on currency futures markets, must regularly report large positions in certain currencies. Also, many option exchanges have developed Large Option Position Reporting System to track changes in large positions and identify outsized short uncovered positions.

In this paper, we consider problem of managing fund of funds, i.e., constructing optimal portfolios from sets of hedge funds, subject to various risk constraints, which control different types of risks. However, the practical use of the strategies is limited by restrictive assumptions[4] imposed in this case study: 1) liquidity considerations are not taken into account, 2) no transaction costs, 3) considered funds may be closed for new investors, 4) credit and other risks which directly are not reflected in the historical return data are not taken into account, and 5) survivorship bias is not considered. The obtained results cannot be treated as direct recommendations for investing in hedge funds market, but rather as a description of the risk management methodologies and portfolio optimization techniques in a realistic environment. For an overview of the potential problems related to the data analysis and portfolio optimization of hedge funds, see Lo (2001).

Section 2 presents an overview of linear portfolio optimization algorithms and the related risk measures, which were explored in this paper. Section 3 contains description of our case study, results of in-sample and out-of-sample experiments and their detailed discussion. Section 4 presents the concluding remarks.

2 Risk management using Conditional Value-at-Risk and Conditional Drawdown-at-Risk

Formal portfolio management methodologies assume some measure of risk that impacts allocation of instruments in the portfolio. The classical Markowitz theory, for example, identifies risk with the volatility (standard deviation) of a portfolio. In this study we investigate a portfolio optimization problem with three different constraints on risk: Conditional Value-at-Risk (Rockafellar and Uryasev 2000, 2002), Conditional Drawdown-at-Risk (Chekhlov et al. 2000), and the market-neutrality ("beta" of the portfolio equals zero)[5]. CVaR and CDaR risk measures represent relatively new developments in the risk management field. Application of these risk measures to portfolio allocation problems relies on the scenario representation of uncertainties and stochastic programming approaches.

A *linear portfolio rebalancing algorithm* is a trading (investment) strategy with mathematical model that can be formulated as a linear programming (LP) problem. The focus on LP techniques in application to portfolio rebalancing and

[4]These assumptions can be relaxed and incorporated in the model as linear constraints. Here we focus on comparison of risk constraints and have not included other constraints.

[5]There are different interpretations for the term "market-neutral" (see, for instance, BARRA RogersCasey 2002). In this paper market neutrality means zero beta.

trading problems is explained by the exceptional effectiveness and robustness of LP algorithms, which becomes especially important in finance applications. Recent developments (see, for example, Andersson et al. 2001, Cariño and Ziemba 1998, Cariño et al. 1998, Chekhlov et al. 2000, Consigli and Dempster 1997, 1998, Dembo and King 1992, Duarte 1999, Krokhmal et al. 2002, Rockafellar and Uryasev 2000, 2002, Turner et al. 1994, Zenios 1999, Ziemba and Mulvey 1998, Young 1998) show that LP-based algorithms can successfully handle portfolio allocation problems with thousands and even millions of decision variables and scenarios, which makes those algorithms attractive to institutional investors.

In the cited papers, along with Conditional Value-at-Risk and Conditional Drawdown-at-Risk, other, much earlier established measures of risk, such as Maximum Loss, Mean-Absolute Deviation, Low Partial Moment with power one and Expected Regret[6], have been employed in the framework of linear portfolio rebalancing algorithms (see, for example, Ziemba and Vickson 1975). Some of these risk measures are quite closely related to CVaR concept[7]. We restricted ourselves to considering CVaR- and CDaR-based risk management techniques.

However, the class of linear trading or portfolio optimization techniques is far from encompassing the entire universe of portfolio management techniques. For example, the famous portfolio optimization model by Markowitz (1952, 1991), which utilizes the mean-variance approach, belongs to the class of quadratic programming (QP) problems; the well-known constant-proportion rule leads to nonconvex multiextremum problems, etc.

2.1 Conditional Value-at-Risk

The Conditional Value-at-Risk (CVaR) measure (see Rockafellar and Uryasev 2000, 2002) develops and enhances the ideas of risk management, which have been put in the framework of Value-at-Risk (VaR) (see, for example, Duffie and Pan 1997, Jorion 1997, Pritzker 1997, Staumbaugh 1996). Incorporating such merits as easy-to-understand concept, simple and convenient representation of risks (one number), applicability to a wide range of instruments, VaR has evolved into a current industry standard for estimating risks of financial losses. Basically, VaR answers the question "what is the maximum loss, which is expected to be exceeded, say, only in 5% of the cases within the given time horizon?" For example, if daily VaR for the portfolio of some fund XYZ is equal

[6]Low partial moment with power one is defined as the expectation of losses exceeding some fixed threshold, see Harlow (1991). Expected regret (see, for example, Dembo and King 1992) is a concept similar to the lower partial moment. However, the expected regret may be calculated with respect to a random benchmark, while the low partial moment is calculated with respect to a fixed threshold.

[7]Maximum Loss is a limiting case of CVaR risk measure (see below). Also, Testuri and Uryasev (2000) showed that the CVaR constraint and the low partial moment constraint with power one are equivalent in the sense that the efficient frontier for portfolio with CVaR constraint can be generated by the low partial moment approach. Therefore, the risk management with CVaR and with low partial moment leads to similar results. However, the CVaR approach allows for direct controlling of percentiles, while the low partial moment penalizes losses exceeding some fixed thresholds.

to 10 millions USD at the confidence level 0.95, it means that there is only a 5% chance of losses exceeding 10 millions during a trading day.

The formal definition of VaR is as follows. Consider a loss function $f(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is a decision vector (e.g., portfolio positions), and $\mathbf{y}$ is a stochastic vector standing for market uncertainties (in this paper, $\mathbf{y}$ is the vector of returns of instruments in the portfolio). Let $\Psi(\mathbf{x},\zeta)$ be the cumulative distribution function of $f(\mathbf{x},\mathbf{y})$,

$$\Psi(\mathbf{x},\zeta) = P\left[f(\mathbf{x},\mathbf{y}) \leq \zeta\right].$$

Then, the Value-at-Risk function $\zeta_\alpha(\mathbf{x})$ with the confidence level α is the α-quantile of $f(\mathbf{x},\mathbf{y})$ (see Figure 1):

$$\zeta_\alpha(\mathbf{x}) = \min_{\zeta\in \mathrm{R}} \left\{\Psi(\mathbf{x},\zeta) \geq \alpha\right\}.$$

Using VaR as a risk measure in portfolio optimization is, however, a very difficult problem, if the return distributions of a portfolio's instruments are not normal or log-normal. The optimization difficulties with VaR are caused by its non-convex and non-subadditive nature (Artzner et al. 1997, 1999, Mausser and Rosen 1998). Non-convexity of VaR means that as a function of portfolio positions, it has multiple local extrema, which precludes using efficient optimization techniques.

The difficulties with controlling and optimizing VaR in non-normal portfolios have forced the search for similar percentile risk measures, which would also quantify downside risks and at the same time could be efficiently controlled and optimized. From this viewpoint, CVaR is a perfect candidate for conducting a "VaR"-style portfolio management.

For continuous distributions, CVaR is defined as an average (expectation) of high losses residing in the α-tail of the loss distribution, or, equivalently, as a conditional expectation of losses exceeding the α-VaR level (Fig. 1). From this follows that CVaR incorporates information on VaR and on the losses exceeding VaR.

For general (non-continuous) distributions, Rockafellar and Uryasev (2002) defined α-CVaR function $\phi_\alpha(\mathbf{x})$ as the α-tail expectation of a random variable z,

$$\phi_\alpha(\mathbf{x}) = E_{\alpha-\mathrm{tail}}\left[z\right],$$

where the α-tail cumulative distribution functions of z has the form

$$\Psi_\alpha(\mathbf{x},\zeta) = P\left[z \leq \zeta\right] = \begin{cases} 0, & \zeta < \zeta_\alpha(\mathbf{x}), \\ \left[\Psi(\mathbf{x},\zeta)-\alpha\right]/\left[1-\alpha\right], & \zeta \geq \zeta_\alpha(\mathbf{x}). \end{cases}$$

Also, Acerbi et al. (2001), Acerbi and Tasche (2001) redefined expected shortfall similar to the CVaR definition presented above.

Along with α - CVaR function $\phi_\alpha(\mathbf{x})$, the following functions called "upper" and "lower" CVaR (α-CVaR$^+$ and α-CVaR$^-$), are considered:

$$\phi^+_\alpha(\mathbf{x}) = E\left[f(\mathbf{x},\mathbf{y}) | f(\mathbf{x},\mathbf{y}) > \zeta_\alpha(\mathbf{x})\right],$$

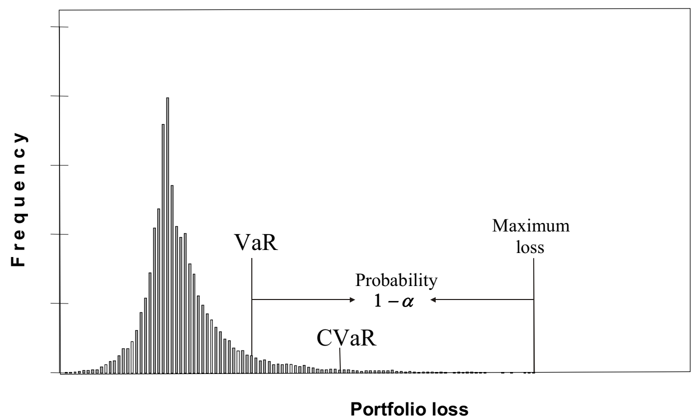

Figure 1: Loss distribution, VaR, CVaR, and Maximum Loss.

$$\phi_\alpha^-(\mathbf{x}) = E\,[f(\mathbf{x},\mathbf{y})|f(\mathbf{x},\mathbf{y}) \geq \zeta_\alpha(\mathbf{x})].$$

The CVaR functions satisfy the following inequality:

$$\phi_\alpha^-(\mathbf{x}) \leq \phi_\alpha(\mathbf{x}) \leq \phi_\alpha^+(\mathbf{x}).$$

Rockafellar and Uryasev (2002) showed that α-CVaR can be presented as a convex combination of α-VaR and α-CVaR$^+$,

$$\phi_\alpha(\mathbf{x}) = \lambda_\alpha(\mathbf{x})\,\zeta_\alpha(\mathbf{x}) + [1-\lambda_\alpha(\mathbf{x})]\,\phi_\alpha^+(\mathbf{x}),$$

where

$$\lambda_\alpha(\mathbf{x}) = [\Psi(\mathbf{x},\zeta_\alpha(\mathbf{x})) - \alpha]/[1-\alpha], \quad 0 \leq \lambda_\alpha(\mathbf{x}) \leq 1.$$

For a discrete loss distribution, where the stochastic parameter $\mathbf{y}$ may take values $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_J$ with probabilities $\theta_j, \quad j = 1, ..., J$, the α-VaR and α-CVaR functions respectively are[8]

$$\zeta_\alpha(\mathbf{x}) = f(\mathbf{x},\, \mathbf{y}_{j_\alpha}),$$

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha}\left[\left(\sum_{j=1}^{j_\alpha}\theta_j - \alpha\right) f(\mathbf{x},\mathbf{y}_{j_\alpha}) + \sum_{j=j_\alpha+1}^{J}\theta_j f(\mathbf{x},\mathbf{y}_j)\right],$$

where j_α satisfies

$$\sum_{j=1}^{j_\alpha-1}\theta_j < \alpha \leq \sum_{j=1}^{j_\alpha}\theta_j.$$

For values of confidence level α close to 1, Conditional Value-at-Risk coincides with the Maximum Loss (see Figure 1).

[8]This proposition has been derived in assumption that, without loss of the generality, scenarios $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_J$ satisfy inequalities $f(\mathbf{x},\mathbf{y}_1) \leq ... \leq f(\mathbf{x},\mathbf{y}_J)$.

While inheriting some of the nice properties of VaR, such as measuring downside risks and representing them by a single number, applicability to instruments with non-normal distributions etc., CVaR has substantial advantages over VaR from the risk management standpoint. First of all, CVaR is a convex function[9] of portfolio positions. Hence, it has a convex set of minimum points on a convex set, which greatly simplifies control and optimization of CVaR. Calculation of CVaR, as well as its optimization, can be performed by means of a convex programming shortcut (Rockafellar and Uryasev 2000, 2002), where the optimal value of CVaR is calculated simultaneously with the corresponding VaR; for linear or piecewise-linear loss functions these procedures can be reduced to linear programming problems. Also, unlike α-VaR, α-CVaR is continuous with respect to confidence level α. A comprehensive description of the CVaR risk measure and CVaR-related optimization methodologies can be found in Rockafellar and Uryasev (2000, 2002). Also, Rockafellar and Uryasev (2000) showed that for normal loss distributions, the CVaR methodology is equivalent to the standard Mean-Variance approach. Similar result also was independently proved for elliptic distributions by Embrechts et al. (2002).

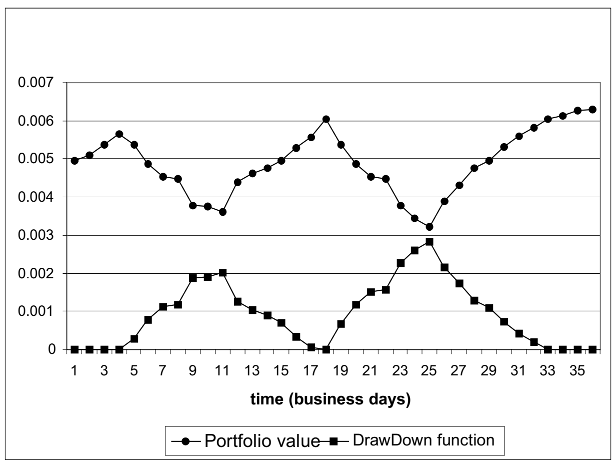

Figure 2: Portfolio value and drawdown.

According to Rockafellar and Uryasev (2000, 2002), the optimization problem with multiple CVaR constraints

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in X} & g(\mathbf{x}) \\ \text{subject to} & \phi_{\alpha_i}(\mathbf{x}) \le \omega_i, \;\; i = 1, ..., I, \end{array}$$

is equivalent to the following problem:

$$\min_{\mathbf{x}\in X,\ \zeta_k \in \mathrm{R},\ \forall k} \quad g(\mathbf{x})$$

[9] For a background on convex functions and sets see Rockafellar (1970).

$$\text{subject to} \quad \zeta_k + \frac{1}{1-\alpha_k}\sum_{j=1}^{J}\theta_j \max\{0,\ f(\mathbf{x},\mathbf{y}_j)-\zeta_k\} \le \omega_k,\ \ k=1,...,K,$$

provided that the objective function $g(\mathbf{x})$ and the loss function $f(\mathbf{x},\mathbf{y})$ are convex in $\mathbf{x}\in X$. When the objective and loss functions are linear in $\mathbf{x}$ and constraints $\mathbf{x}\in X$ are given by linear inequalities, the last optimization problem can be reduced to LP, see Rockafellar and Uryasev (2000, 2002).

Except for the fact that CVaR can be easily controlled and optimized, CVaR is a more adequate measure of risk as compared to VaR because it accounts for losses beyond the VaR level. The fundamental difference between VaR and CVaR as risk measures are: VaR is the "optimistic" low bound of the losses in the tail, while CVaR gives the value of the *expected* losses in the tail. In risk management, we may prefer to be neutral or conservative rather than optimistic. Moreover, CVaR satisfies several nice mathematical properties and is coherent in the sense of Artzner et al. (1997, 1999).

2.2 Conditional Drawdown-at-Risk

Conditional Drawdown-at-Risk (CDaR) is a portfolio performance measure (Chekhlov et al., 2000) closely related to CVaR. By definition, a portfolio's *drawdown* on a sample-path is the drop of the uncompounded[10] portfolio value as compared to the maximal value attained in the previous moments on the sample-path. Suppose, for instance, that we start observing a portfolio in January 2001, and record its uncompounded value every month[11]. If the initial portfolio value was \$100,000,000 and in February it reached \$130,000,000, then, the portfolio drawdown as of February 2001 is \$0. If, in March 2001, the portfolio value drops to \$90,000,000, then the current drawdown equals \$40,000,000 (in absolute terms), or 30.77%. Mathematically, the drawdown function for a portfolio is

$$\tilde{f}(\mathbf{x},t) = \max_{0\le\tau\le t}\{v_\tau(\mathbf{x})\} - v_t(\mathbf{x}), \tag{1}$$

where $\mathbf{x}$ is the vector of portfolio positions, and $v_t(\mathbf{x})$ is the *uncompounded* portfolio value at time t. We assume that the initial portfolio value is equal to 1; therefore, the drawdown is the uncompounded portfolio return starting from the previous maximum point. Figure 2 illustrates the relation between the portfolio value and the drawdown.

The drawdown quantifies the financial losses in a conservative way: it calculates losses for the most "unfavorable" investment moment in the past as

[10] Drawdowns are calculated with uncompounded portfolio returns. This is related to the fact that risk measures based on drawdowns of uncompounded portfolios have nice mathematical properties. In particular, these measures are convex in portfolio positions. Suppose that at the initial moment $t=0$ the portfolio value equals v and portfolio returns in the moments$t = 1,...,T$ equal $r_1,..., r_T$. By definition, the uncompounded portfolio value v_τ at time moment τ equals $v_\tau = v\sum_{t=1}^{\tau} r_t$. We assume that the initial portfolio value $v=1$.

[11] Usually, portfolio value is observed much more frequently. However, for the hedge funds considered in this paper, data are available on monthly basis.

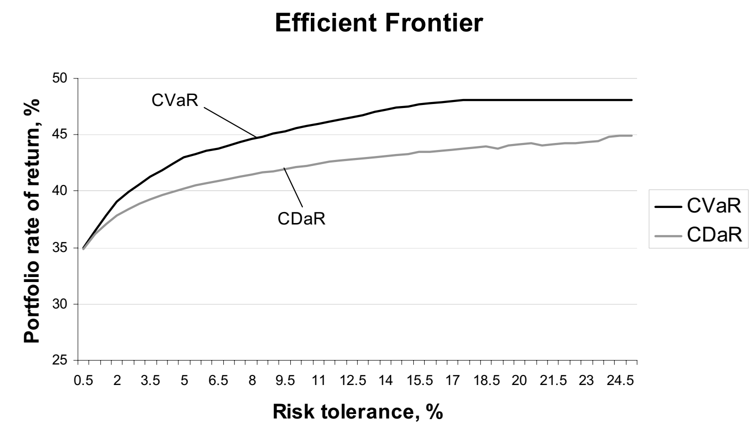

Figure 3: Efficient frontiers for portfolios with various risk constraints. The market-neutrality constraint is inactive.

compared to the current (discrete) moment. This approach reflects quite well the preferences of investors who define their allowed losses in percentages of their initial investments (e.g., an investor may consider it unacceptable to lose more than 10% of his investment). While an investor may accept small drawdowns in his account, he would definitely start worrying about his capital in the case of a large drawdown. Such drawdown may indicate that something is wrong with that fund, and maybe it is time to move the money to a more successful investment pool. The mutual and hedge fund concerns are focused on keeping existing accounts and attracting new ones; therefore, they should ensure that clients' accounts do not have large drawdowns.

One can conclude that *drawdown* accounts not only for the amount of losses over some period, but also for the *sequence* of these losses. This highlights the unique feature of the drawdown concept: it is a loss measure "with memory" taking into account the time sequence of losses.

For a specified sample-path, the drawdown function is defined for each time moment. However, in order to evaluate performance of a portfolio on the whole sample-path, we would like to have a function, which aggregates all drawdown information over a given time period into one measure. As this function one can pick, for example, the Maximum Drawdown,

$$\mathrm{MaxDD} = \max_{0 \le t \le T} \left\{ \tilde{f}(\mathbf{x}, t) \right\},$$

or the Average Drawdown,

$$\mathrm{AverDD} = \frac{1}{T} \int_0^T \tilde{f}(\mathbf{x}, t)\, dt.$$

However, both these functions may inadequately measure losses. The Maximum Drawdown is based on one “worst case” event in the sample-path. This event may represent some very specific circumstances, which may not appear in the future. The risk management decisions based only on this event may be too conservative.

On the other hand, the Average Drawdown takes into account all drawdowns in the sample-path. However, small drawdowns are acceptable (e.g., 1-2% drawdowns) and averaging may mask large drawdowns.

Chekhlov et al. (2000) suggested a new drawdown measure, Conditional Drawdown-at-Risk, that combines both the drawdown concept and the CVaR approach. For instance, 0.95-CDaR can be thought of as an average of 5% of the highest drawdowns. Formally, α-CDaR is α-CVaR with drawdown loss function $\tilde{f}(\mathbf{x}, t)$ given by (1). Namely, assume that possible realizations of the random vectors describing uncertainties in the loss function is represented by a sample-path (time-dependent scenario), which may be obtained from historical or simulated data. In this paper, it is assumed that we know one sample-path of returns of instruments included in the portfolio. Let r_{ij} be the rate of return of i-th instrument in j-th trading period (that corresponds to$j$-th month in the case study, see below), $j = 1, ..., J$. Suppose that the initial portfolio value equals 1. Let x_i, $i = 1, ..., n$ be weights of instruments in the portfolio. The uncompounded portfolio value at time j equals

$$v_j(\mathbf{x}) = \sum_{i=1}^{n} \left(1 + \sum_{s=1}^{j} r_{is} \right) x_i.$$

The drawdown function $\tilde{f}(\mathbf{x}, \mathbf{r}_j)$ at the time j is defined as the drop in the portfolio value compared to the maximum value achieved before the time moment j,

$$\tilde{f}(\mathbf{x}, j) = \max_{1 \le k \le j} \left\{ \sum_{i=1}^{n} \left(\sum_{s=1}^{k} r_{is} \right) x_i \right\} - \sum_{i=1}^{n} \left(\sum_{s=1}^{j} r_{is} \right) x_i.$$

Then, the Conditional Drawdown-at-Risk function $\Delta_\alpha(\mathbf{x})$ is defined as follows. If the parameter α and number of scenarios J are such that their product $(1 - \alpha)\, J$ is an integer number, then $\Delta_\alpha(\mathbf{x})$ is defined as

$$\Delta_\alpha(\mathbf{x}) = \eta_\alpha + \frac{1}{(1-\alpha)J} \sum_{j=1}^{J} \max \left\{ 0,\ \max_{1 \le k \le j} \left[\sum_{i=1}^{n} \left(\sum_{s=1}^{k} r_{is} \right) x_i \right] \right.$$
$$\left. - \sum_{i=1}^{n} \left(\sum_{s=1}^{j} r_{is} \right) x_i - \eta_\alpha \right\},$$

where $\eta_\alpha = \eta_\alpha(\mathbf{x})$ is the threshold that is exceeded by $(1 - \alpha)\, J$ drawdowns. In this case the drawdown functions $\Delta_\alpha(\mathbf{x})$ is the average of the worst case $(1 - \alpha)\, J$ drawdowns observed in the considered sample-path. If $(1 - \alpha)\, J$ is not integer, then the CDaR function, $\Delta_\alpha(\mathbf{x})$, is the solution of

$$\Delta_\alpha(\mathbf{x}) = \min_{\eta} \left\{ \eta + \frac{1}{1-\alpha}\frac{1}{J} \right.$$
$$\left. \times \sum_{j=1}^{J} \max\left[0,\ \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - \eta\right]\right\}.$$

The CDaR risk measure holds nice properties of CVaR such as convexity with respect to portfolio positions. Also CDaR can be efficiently treated with linear optimization algorithms (Chekhlov et al. 2000).

2.3 Market-neutrality

The market itself constitutes a risk factor. If the instruments in the portfolio are positively correlated with the market, then the portfolio would follow not only market growth, but also market drops. Naturally, portfolio managers are willing to avoid situations of the second type, by constructing portfolios, which are uncorrelated with market, or *market-neutral*. To be market-uncorrelated, the portfolio must have zero *beta,*

$$\beta_p = \sum_{i=1}^{n} \beta_i\, x_i = 0,$$

where $x_1, ..., x_n$ denote the proportions in which the total portfolio capital is distributed among n assets, and β_i are betas of individual assets,

$$\beta_i = \frac{\mathrm{Cov}\,(r_i, r_M)}{\mathrm{Var}\,(r_M)},$$

where r_M stands for market rate of return. Instruments' betas, β_i, can be estimated, for example, using historical data:

$$\beta_i = \left(\sum_{j=1}^{J} (r_{M,j} - \bar{r}_M)^2\right)^{-1} \sum_{j=1}^{J} (r_{i,j} - \bar{r}_i)\,(r_{M,j} - \bar{r}_M),$$

where J is the number of historical observations, and $\bar{r}$ denotes the sample average, $\bar{r} = J^{-1}\sum r_j$. As a proxy for market returns r_M, historical returns of the S&P500 index can be used.

In our case study, we investigate the effect of constructing a market-neutral (zero-beta) portfolio, by including a market-neutrality constraint in the portfolio optimization problem. We compare the performance of the optimal portfolios obtained with and without market-neutrality constraint.

2.4 Problem formulation

This section presents the "generic" problem formulation, which was used to construct an optimal portfolio. We suppose that some historical sample-path of returns of n instruments is available. Based on this sample-path, we calculate the expected return of the portfolio and the various risk measures for that portfolio. We maximize the expected return of the portfolio subject to different operating, trading, and risk constraints,

$$\max_{x} \; E\left[\sum_{i=1}^{n} r_i x_i\right] \tag{2}$$

subject to

$$0 \leq x_i \leq 1, \quad i = 1, ..., n, \tag{3}$$

$$\sum_{i=1}^{n} x_i \leq 1, \tag{4}$$

$$\Phi_{\text{Risk}}(x_1, ..., x_n) \leq \omega\,, \tag{5}$$

$$-k \leq \sum_{i=1}^{n} \beta_i x_i \leq k\,, \tag{6}$$

where x_i is the portfolio position (weight) of asset i, r_i is the (random) rate of return, and β_i is market beta of instrument i.

The objective function (2) represents the expected return of the portfolio. The first constraint (3) of the optimization problem imposes limitations on the amount of funds invested in a single instrument (we do not allow short positions). The second constraint (4) is the budget constraint. Constraints (5) and (6) control risks of financial losses. The key constraint in the presented approach is the risk constraint (5). Function $\Phi_{\text{Risk}}(x_1, ..., x_n)$ represents either a CVaR or a CDaR risk measure, and risk tolerance level ω is the fraction of the portfolio value that is allowed for risk exposure.

Constraint (6), with β_i representing market's *beta* for instrument i, forces the portfolio to be market-neutral in the "zero-beta" sense, i.e., the portfolio correlation with the market is bounded. The coefficient k in (6) is a small number that sets the portfolio's beta close to zero. To investigate the effects of imposing a "zero-beta" requirement on the portfolio-rebalancing algorithm, we solved the optimization problem with and without this constraint. Constraint (6) significantly improves the out-of-sample performance of the algorithm.

The risk measures considered in this paper allow for formulating the risk constraint (5) in terms of linear inequalities, which makes the optimization problem (2)–(6) linear, given the linearity of objective function and other constraints. Below we present the explicit form of the risk constraint (5) for CVaR and CDaR risk measures.

2.5 Conditional Value-at-Risk constraint

The loss function incorporated into CVaR constraint, is the negative portfolio's return,

$$f(\mathbf{x},\mathbf{y}) = -\sum_{i=1}^{n} r_i x_i, \tag{7}$$

where the vector of instruments' returns $\mathbf{y} = \mathbf{r} = (r_1, ..., r_n)$ is random. The risk constraint (5), $\phi_\alpha(\mathbf{x}) \le \omega$, where CVaR risk function replaces the function $\Phi_{\text{Risk}}(\mathbf{x})$, is

$$\zeta + \frac{1}{(1-\alpha)\,J}\sum_{j=1}^{J} \max\left\{0, -\sum_{i=1}^{n} r_{ij}x_i - \zeta\right\} \le \omega, \tag{8}$$

where r_{ij} is return of i-th instrument in scenario j, $j = 1, ..., J$. Since the loss function (7) is linear, the risk constraint (8) can be equivalently represented by the linear inequalities,

$$\begin{aligned} \zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j &\le \omega, \\ -\sum_{i=1}^{n} r_{ij}x_i - \zeta &\le w_j, \quad j = 1, ..., J, \\ \zeta \in \mathrm{R}, \quad w_j &\ge 0, \quad j = 1, ..., J. \end{aligned} \tag{9}$$

This representation allows for reducing the optimization problem (2)–(6) with the CVaR constraint to a linear programming problem.

2.6 Conditional Drawdown-at-Risk constraint

The CDaR risk constraint $\Delta_\alpha(\mathbf{x}) \le \omega$ has the form

$$\eta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J}\max\left[0, \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - \eta\right] \le \omega,$$

and it can be reduced to a set of linear constraints similarly to the CVaR constraint.

3 Case study: portfolio of hedge funds

The case study investigates investment opportunities and tests portfolio management strategies for a portfolio of hedge funds. Hedge funds are subject to less regulations as compared with mutual or pension funds. Hence, very little information on hedge funds' activities is publicly available (for example, many funds report their share prices only monthly). On the other hand, fewer regulations and weaker government control provide more room for aggressive, risk-seeking trading and investment strategies. As a consequence, the revenues in this industry are on average much higher than elsewhere, but the risk exposure is also higher (for example, the typical "life" of a hedge fund is about five years, and very few of them perform well in long run). Data availability and sizes of datasets impose challenging requirements on portfolio rebalancing algorithms. Also, the specific nature of hedge fund securities imposes some limitations on using them in trading or rebalancing algorithms. For example, hedge funds are far from being perfectly liquid: hedge funds may *not* be publicly traded or may be closed to new investors. From this point of view, our results contain a rather schematic representation of investment opportunities existing in the hedge fund market and do not give direct recommendations on investing in that market. The goal of this study is to compare the recently developed risk management approaches and to demonstrate their high numerical efficiency in a realistic setting.

The dataset for conducting the numerical experiments was provided to the authors by the Foundation for Managed Derivatives Research. It contained a monthly data for more than 5000 hedge funds, from which we selected those with significantly long history and some minimum level of capitalization. To pass the selection, a hedge fund should have 66 months of historical data from December 1995 to May 2001, and its capitalization should be at least 5 million U.S. dollars at the beginning of this period. The total number of funds, which satisfied these criteria and accordingly constituted the investment pool for our algorithm, was 301. In this dataset, the field with the names of hedge funds was unavailable; therefore, we identified the hedge funds with numbers, i.e., HF 1, HF 2, and so on. The historical returns from the dataset were used to generate scenarios for algorithm (2)–(6). Each scenario is a vector of monthly returns for all securities involved in the optimization, and all scenarios are assigned equal probabilities.

We performed separate runs of the optimization problem (2)–(5), with and without constraint (6) with CVaR and CDaR risk measure in constraint (5), varying such parameters as confidence levels, risk tolerance levels etc.

The case study consisted from two sets of numerical experiments. The first set of *in-sample* experiments included the calculation of efficient frontiers and the analysis of the optimal portfolio structure for each of the risk measures. The second set of experiments, *out-of-sample* testing, was designed to demonstrate the performance of our approach in a simulated historical environment.

3.1 In-sample results

Efficient frontier. For constructing the efficient frontier for the optimal portfolio with different risk constraints, we solved the optimization problem (2)–(5) with different risk tolerance levels ω in constraint (5), varied from $\omega = 0.005$ to $\omega = 0.25$. The parameter α in CVaR and CDaR risk constraints was set to $\alpha = 0.90$. The efficient frontier is presented in Figure 3, where the portfolio rate of return means expected yearly rate of return. In these runs, the market-neutrality constraint (6) is inactive. For optimal portfolios, in the sense

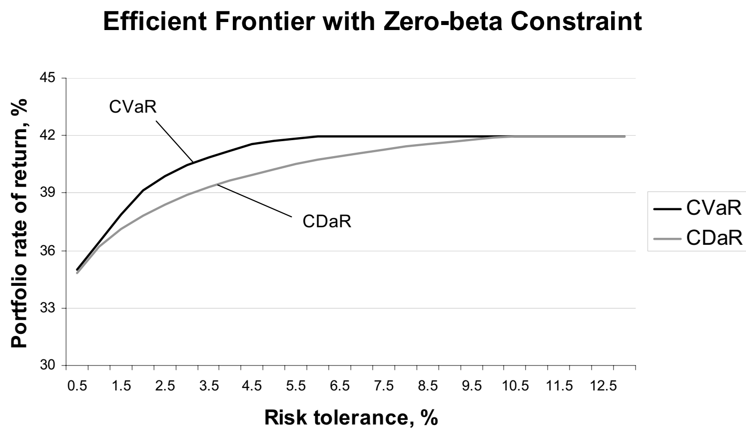

Figure 4: Efficient frontier for market-neutral portfolio with various risk constraints ($k = 0.01$).

of problem (2)–(5), there exists an upper bound (equal to 48.13%) for the portfolio's rate of return. Optimal portfolio with CVaR constraint reach this bound at about 18%-risk tolerance level, but the CDaR-constrained portfolio does not achieve the maximal expected return within the given range of ω values. CDaR is a relatively conservative constraint imposing requirements not only on the magnitude of loses, but also on the time sequence of losses (small consecutive losses may lead to large drawdown, without a significant increase of CVaR).

Figure 4 presents efficient frontiers of optimal portfolio (2)–(5) with active market-neutrality constraint (6), where coefficient k equals to 0.01. Imposing the extra constraint (6) causes a decrease in the in-sample optimal expected return. For example, the "saturation" level of the portfolio's expected return is now 41.94%, and both portfolios reach that level at much lower values of risk tolerance ω. However, the market-neutrality constraint almost does not affect the curves of efficient portfolios in the leftmost points of efficient frontiers, which correspond to the lowest values of risk tolerance ω.

Quite high rates of return for CVaR- and CDaR-efficient portfolios are explained by the fact that 301 funds, selected to form the optimal portfolios, constitute about 6% of the initial hedge fund pool, and already are "the best of the best" in our data sample.

Optimal portfolio configuration. We now discuss the structure of the optimal portfolio with various risk constraints. We selected those optimal portfolios on the efficient frontiers whose expected return is equal 35% (the market-neutrality constraint is not active).

Table 1 shows the configuration (portfolio weights) of the optimal portfolios with CVaR and CDaR constraints. Among the 301 available instruments, only

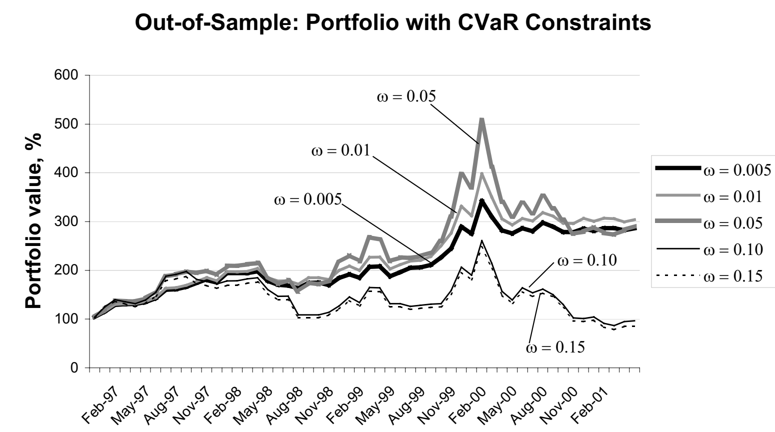

Figure 5: Historical trajectories of optimal portfolio with CVaR constraints.

few of them contribute to constructing the optimal portfolio. Moreover, a closer look at Table 1 shows that nearly two thirds of the portfolio value for both risk measures is formed by three hedge funds HF 209, HF 219 and HF 231. In general, CVaR- and CDaR-optimal portfolios have quite similar structure.

Table 1: Portfolio weights for optimal portfolio with CVaR and CDaR constraints.

	HF49	HF84	HF93	HF100	HF126	HF169	HF196	HF209	HF219	HF231	HF258	HF259	HF298
CVaR	4.39%	0.00%	8.75%	7.37%	0.87%	1.01%	1.56%	22.47%	25.93%	16.92%	1.42%	8.94%	0.38%
CDaR	11.02%	4.19%	8.14%	6.63%	0.00%	5.43%	0.00%	21.47%	13.72%	18.30%	3.41%	5.87%	1.83%

3.2 Out-of-sample calculations

The out-of-sample testing of the portfolio optimization algorithm (2)–(6) sheds light on the "*actual*" performance of the approaches. The question is how well the algorithms with different risk measures utilize the scenario information based on past history in producing a successful portfolio management strategy? An answer can be obtained, for instance, by interpreting the results of the preceding section as follows: suppose we were back in May 2001, and we would like to invest a certain amount of money in a portfolio of hedge funds to deliver the highest reward under a specified risk level. Then, according to in-sample results, the best portfolio would be the one on the efficient frontier of a particular rebalancing strategy. In fact, such a portfolio offers the best return-to-risk ratio *provided that the historical distribution of returns will repeat in the future.*

To estimate the "actual" performance of the optimization approach, we used part of the data for scenario generation, and the rest for evaluating the performance of the strategy.

We present the results of a "plain" out-of-sample test, where the older data is considered as the 'in-sample' data for the algorithm, and the newer data are treated as "to-be-realized" future. First, we took the 12 monthly returns from December 1995 to November 1996 as the initial historical data for constructing the first portfolio to invest in, and observed the portfolio's "realized" value by observing the historical prices for December 1996. Then, we added one more month, December 1996, to the data which were used for scenario generation (12 months of historical data in total) to generate an optimal portfolio and to allocate to investments in January, 1997, and so on. Note that we did not implement the "moving window" method for out-of-sample testing, where the same number of scenarios (i.e., the most recent historical points) is used for solving the portfolio-rebalancing problem. Instead, we accumulated the historical data for portfolio optimization.

First, we perform the out-of-sample runs for each risk measure in constraint (5) for different values of risk tolerance level ω (market-neutrality constraint, (6), is inactive). Figures 5 and 6 illustrate the historical trajectories of the optimal portfolio under different risk constraints (the portfolio values are given in % relatively to the initial portfolio value). Risk tolerance level ω was set to 0.005, 0.01, 0.03, 0.05, 0.10, 0.12, 0.15, 0.17 and 0.20, but for better reading of figures, we report only results with ω = 0.005, 0.01, 0.05, 0.10, and 0.15. The parameter α, which is risk confidence level in CDaR and CVaR constraints was set to $\alpha = 0.90$. Figures 7–8 shows that risk constraint (5) has a signifi-

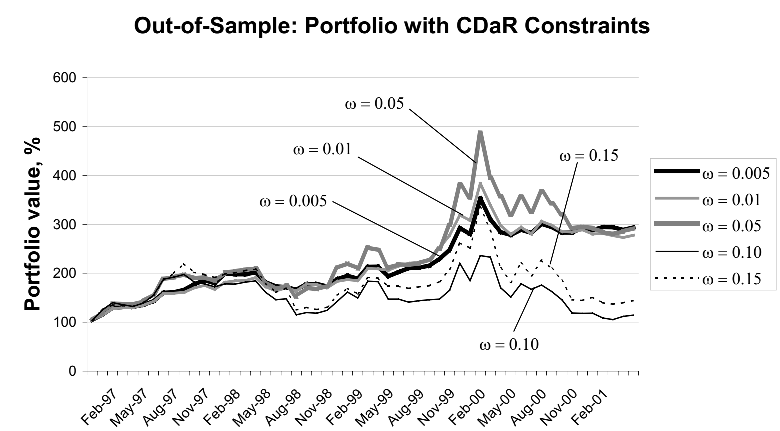

Figure 6: Historical trajectories of optimal portfolio with CDaR constraints.

cant impact on the algorithm's out-of-sample performance. Earlier, we had also

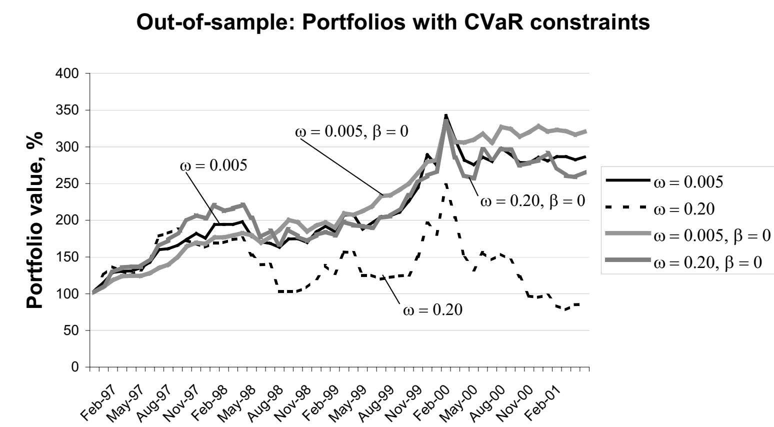

Figure 7: Historical trajectories of optimal portfolio with CVaR constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

observed that this constraint has significant impact on the in-sample performance. Constraining risk in the in-sample optimization decreases the optimal value of the objective function, and the results reported in the preceding subsection reflect this. The risk constraints force the algorithm to favor less profitable but safer decisions over more profitable but "dangerous" ones. Imposing extra constraints always reduces the feasibility set, and consequently leads to lower optimal objective values. However, the situation changes dramatically for an out-of-sample application of the optimization algorithm. The numerical experiments show that constraining risks *improves* the overall performance of the portfolio rebalancing strategy in out-of-sample runs; *tighter* in-sample risk constraint may lead to *both lower risks and higher out-of-sample returns.* For both risk measures, loosening the risk tolerance (i.e., increasing ω values) results in an increased volatility of the out-of-sample portfolio returns and, after exceeding some threshold value, in degradation of the algorithm's performance, especially during the last 13 months (March 2000 – May 2001). For all risk functions in constraint (5), the most attractive portfolio trajectories are obtained for risk tolerance level $\omega = 0.005$, which means that these portfolios have high returns (high final portfolio value), low volatility, and low drawdowns. Increasing ω to 0.01 leads to a slight increase of the final portfolio value, but it also increases portfolio volatility and drawdowns, especially for the second quarter of 2001. For larger values of ω the portfolio returns deteriorate, and for all risk measures portfolio curves with $\omega = 0.10$ show quite poor performance. Further increasing the risk tolerance to $\omega = 0.15$ in some cases allows for achieving higher returns at the end of 2000, but after this high peak the portfolio suffers severe drawdowns.

Figures 7 and 8 illustrate the effects of imposing market-neutrality constraint (6) in addition to risk constraint (5). The primary purpose of (6) is making the portfolio uncorrelated with market. The main idea of composing a market-neutral portfolio is protecting it from market drawdowns. Figures 7–8 compare the trajectories of market-neutral and without risk-neutrality optimal portfolios. Additional constraining resulted in most cases in a further improvement of the portfolio's out-of-sample performance. To clarify how the risk-neutrality condition (6) influences the portfolio's performance, we displayed only figures for lowest and highest values of the risk tolerance parameter, namely for $\omega = 0.005$ and $\omega = 0.20$. Coefficient k in (6) was set to $k = 0.01$, and instruments' betas β_i were calculated by correlating with the benchmark S&P 500 index. For portfolios with tight risk constraints ($\omega = 0.005$) imposing market-neutrality constraint (6) straightened their trajectories (reduced volatility and drawdowns), which made the historic curves almost monotone curves with a positive slope. On top of that, portfolios with market-neutrality constraint had a higher final portfolio value, compared to those without market-neutrality. Also, for portfolios with loose risk constraints ($\omega = 0.20$) imposing market-neutrality constraint had a positive effect on the form of their trajectories, dramatically reducing volatility and drawdowns.

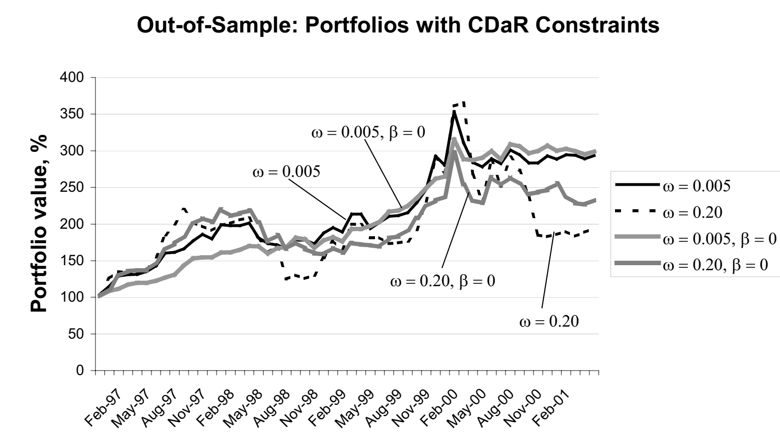

Figure 8: Historical trajectories of optimal portfolio with CDaR constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

Finally, Figures 9 and 10 demonstrate the performance of the optimal portfolios versus two benchmarks: 1) S&P500 index; 2) "Best20", representing the portfolio distributed equally among the "best" 20 hedge funds. These 20 hedge funds include funds with the highest expected monthly returns calculated with past historical information. Similarly to the optimal portfolios (2)–(6), the "Best20" portfolio was monthly rebalanced (without risk constraints).

According to Figures 9–10, CVaR and CDaR constrained portfolios, both without and with market-neutrality condition, outperform benchmarks, which

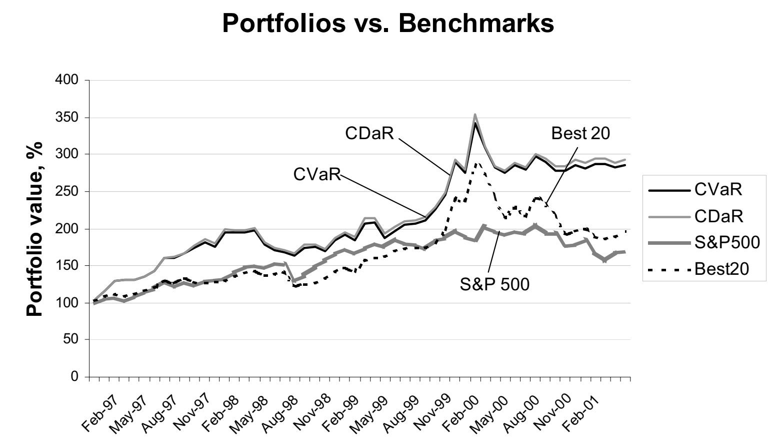

Figure 9: Performance of the optimal portfolios with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level $\omega = 0.005$, parameter $\alpha = 0.90$. Market-neutrality constraint is inactive.

provides an evidence of high efficiency of the risk-constrained portfolio management algorithm (2)–(6). Also, we would like to emphasize the behavior of market-neutral portfolios in "down" market conditions. Two marks on Figure 10 indicate the points when two risk-constrained portfolios gained positive returns, while the market was falling. Also, all risk-constrained portfolios seem to withstand the down market in 2000, when the market experienced significant drawdown. This demonstrates the efficiency and appropriateness of risk management approaches considered in this paper.

The "Best20" benchmark evidently lacks the solid performance of its competitors. It not only significantly underperforms all the portfolios constructed with algorithm (2)–(6), but also underperforms the market half of the time. Unlike portfolios (2)–(6), the "Best20" portfolio pronouncedly follows the market drop in the second half of 2000, and moreover, it suffers much more severe drawdowns than the market does. This indicates that the risk constraints in the algorithm (2)–(6) play an important role in selecting the funds.

Summarizing, we emphasize the general inference about the role of risk constraints in the out-of-sample and in-sample application of an optimization algorithm, which can be drawn from our experiments: risk constraints decrease the in-sample returns, while out-of-sample performance may be improved by adding risk constraints, and moreover, stronger risk constraints usually ensure better out-of-sample performance.

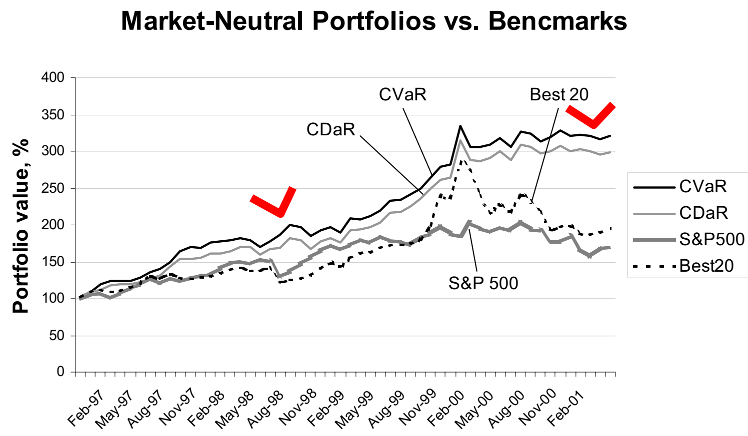

Figure 10: Figure 10. Performance of market-neutral optimal portfolio with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level is $\omega = 0.005$, parameter is $\alpha = 0.90$.

4 Conclusions

We tested the performance of a portfolio allocation algorithm with different types of risk constraints in an application for managing a portfolio of hedge funds. As the risk measure in the portfolio optimization problem, we used Conditional Value-at-Risk and Conditional Drawdown-at-Risk. We combined these risk constraints with the market-neutrality (zero-beta) constraint making the optimal portfolio uncorrelated with the market.

The numerical experiments consist of in-sample and out-of-sample testing. We generated efficient frontiers and compared algorithms with various constraints. The out-of-sample part of experiments was performed in two setups, which differed in constructing the scenario set for the optimization algorithm.

The results obtained are dataset-specific and we cannot make direct recommendations on portfolio allocations based on these results. However, we learned several lessons from this case study. Imposing risk constraints may significantly degrade in-sample expected returns while improving risk characteristics of the portfolio. In-sample experiments showed that for tight risk tolerance levels, all risk constraints produce relatively similar portfolio configurations. Imposing risk constraints may improve the out-of-sample performance of the portfolio-rebalancing algorithms in the sense of risk-return tradeoff. Especially promising results can be obtained by combining several types of risk constraints. In particular, we combined the market-neutrality (zero-beta) constraint with CVaR or CDaR constraints. We found that tightening of risk constraints greatly improves portfolio dynamic performance in out-of-sample tests, increasing the overall portfolio return and decreasing both losses and drawdowns. In addition, imposing the market-neutrality constraint adds to the stability of portfolio's return, and reduces portfolio drawdowns. Both CDaR and CVaR risk measures demonstrated a solid performance in out-of-sample tests.

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