Comparative Analysis of Linear Portfolio Rebalancing Strategies: An Application to Hedge Funds*

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This paper applies formal risk management methodologies to optimization of a portfolio of hedge funds (fund of funds). We compare recently developed risk management methodologies: Conditional Value-at-Risk and Conditional Drawdown-at-Risk with more established Mean-Absolute Deviation, Maximum Loss, and Market Neutrality approaches. The common property of the considered risk management techniques is that they admit the formulation of a portfolio optimization model as a linear programming (LP) problem. LP formulations allow for implementing efficient and robust portfolio allocation algorithms, which can successfully handle optimization problems with thousands of instruments and scenarios. The performance of various risk constraints is investigated and discussed in detail for in-sample and out-of-sample testing of the algorithm. The numerical experiments show that imposing risk constraints may improve the "real" performance of a portfolio rebalancing strategy in out-of-sample runs. It is beneficial to combine several types of risk constraints that control different sources of risk.

1. Introduction

This paper applies formal risk management methodologies to the optimization of a portfolio of hedge funds (fund of funds). We compare Conditional Value-at-Risk and Conditional Drawdown-at-Risk with more established Mean-Absolute Deviation, Maximum Loss, and Market-Neutrality approaches. These risk management techniques allow for the formulation of linear portfolio rebalancing strategies, and have proven their high efficiency in various portfolio management applications (Andersson et al. (2001), Chekhlov et al. (2000), Dembo and King (1992), Duarte (1999), Konno and Wijayanayake (1999), Konno and Yamazaki (1991), Krokhmal et al. (2002), Rockafellar and Uryasev (2000, 2001), Ziemba and Mulvey (1998), Zenios (1999), Young (1998)). The choice of hedge funds, as a subject for the portfolio optimization strategy, was stimulated by a strong interest to this class of assets by both practitioners and scholars, as well as by challenges related to relatively small datasets available for hedge funds.

Recent studies[1] of the hedge funds industry are mostly concentrated on the classification of hedge funds and the relevant investigation of their activity. However, this paper is

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[1] See, for example, papers by Ackermann et al. (1999), Amin and Kat (2001), Brown and Goetzmann (2000), Fung and Hsieh (1997, 2000, 2001), and Lhabitant (2001).

focused on possible realization of investment opportunities existing in this market from the viewpoint of portfolio rebalancing strategies.

Hedge funds are investment pools employing sophisticated trading and arbitrage techniques including leverage and short selling, wide usage of derivative securities etc. Generally, hedge funds restrict share ownership to high net worth individuals and institutions, and do not offer their securities to the general public. Some hedge funds are limited to 100 investors. This private nature of hedge funds has resulted in few regulations and disclosure requirements, compared for example, with mutual funds. Also, the hedge funds may take advantage of specialized, risk-seeking investment and trading strategies, which other investment vehicles are not allowed to use.

The first hedge fund was established in the United States by A. W. Jones in 1949, and its activity was characterized by the use of short selling and leverage, which were separately considered risky trading techniques, but in combination could limit the market risk. The term “hedge fund” attributes to the structure of Jones fund’s portfolio, which was split between long positions in stocks that would grow if market went up, and short positions in stocks that would protect against market drop. Also, Jones has introduced another two initiatives, which became a common practice in hedge fund industry, and with more or less variations survived to this day: he made the manager’s incentive fee a function of fund’s profits, and kept his own capital in the fund, in this way making the incentives of fund’s clients and of his own coherent.

Nowadays, hedge funds become a rapidly growing part of the financial industry. According to Van Hedge Fund Advisors, the number of hedge funds at the end of 1998 was 5830, they managed 311 billion USD in capital, with between $800 billion and $1 trillion in total assets. Nearly 80% of hedge funds have market capitalization less than 100 million, and around 50% are smaller than $25 million, which indicates high number of new entries. More than 90% of hedge funds are located in the U.S.

Hedge funds are subject to far fewer regulations than other pooled investment vehicles, especially to regulations designed to protect investors. This applies to such regulations as regulations on liquidity, requirements that fund’s shares must be redeemable an any time, protecting conflicts of interests, assuring fairness of pricing of fund shares, disclosure requirements, limiting usage of leverage, short selling etc. This is a consequence of the fact that hedge funds’ investors qualify as sophisticated high-income individuals and institutions, which can stand for themselves. Hedge funds offer their securities as private placements, on individual basis, rather than through public advertisement, which allows them to avoid disclosing publicly their financial performance or asset positions. However, hedge funds must provide to investors some information about their activity, and of course, they are subject to statutes governing fraud and other criminal activities.

As market’s subjects, hedge funds do subordinate to regulations protecting the market integrity that detect attempts of manipulating or dominating in markets by individual participants. For example, in the United States hedge funds and other investors active on currency futures markets, must regularly report large positions in certain currencies. Also, many option exchanges have developed Large Option Position Reporting System to track changes in large positions and identify outsized short uncovered positions.

In the presented paper, we consider problem of managing fund of funds, i.e., constructing optimal portfolios from sets of hedge funds, subject to various risk constraints, which control different types of risks. However, the practical use of considered strategies is limited by restrictive assumptions[2] imposed in this case study: 1) liquidity considerations are not taken into account, 2) no transaction costs, 3) considered funds may be closed for new investors, 4) credit and other risks which directly are not reflected in the historical return data are not taken into account, and 5) survivorship bias is not considered. The obtained results cannot be treated as direct recommendations for investing in hedge funds market, but rather as a description of the risk management methodologies and portfolio optimization techniques in a realistic environment. For an overview of the potential problems related to the data analysis and portfolio optimization of hedge funds, see Lo (2001).

The paper is organized as follows. The next section presents an overview of linear portfolio optimization algorithms and related risk measures, which were explored in this paper. Section 3 contains description of our case study, results of in-sample and out-of-sample experiments and their detailed discussion. Appendix A presents a rigorous mathematical description of the considered risk measures and formulations of risk constraints for portfolio optimization problems.

2. Linear Portfolio Rebalancing Algorithms

A linear portfolio rebalancing algorithm is a trading or investment strategy with mathematical model that can be formulated as a linear programming (LP) problem. Linear programming problems form a class of optimization problems, where the function to be optimized is linear, and the set of constraints is given by linear equalities and inequalities. The focus on LP techniques in application to portfolio rebalancing and trading problems is explained by the exceptional effectiveness and robustness of LP solving algorithms, which becomes especially important in finance applications. Recent developments (see, for example, Andersson et al. (2001), Chekhlov et al. (2000), Consigli and Dempster (1997, 1998), Duarte (1999), Rockafellar and Uryasev (2000, 2001), Ziemba and Mulvey (1998), Zenios (1999), Dembo and King (1992), Young (1998)) show that LP-based algorithms can successfully handle portfolio allocation problems with thousands of instruments and scenarios, which makes those algorithms attractive to institutional investors. However, the class of linear trading or portfolio optimization techniques is far from encompassing the entire universe of portfolio management techniques. For example, the famous portfolio optimization model by Markowitz (1952, 1959), which utilizes the mean-variance approach, belongs to the class of quadratic programming (QP) problems; the well-known constant-proportion rule leads to nonconvex multiextremum problems etc.

[2] These assumptions can be relaxed and incorporated in the model as linear constraints. However, here we focused on comparison of risk constraints and have not included other constraints.

Formal portfolio management methodologies assume some measure of risk that impacts allocation of instruments in the portfolio. The classical Markowitz theory identifies risk with the volatility (standard deviation) of a portfolio. In the present study we investigate a portfolio optimization problem with five different constraints on risk, including Conditional Value-at-Risk (Rockafellar and Uryasev, 2000, 2001), Conditional Drawdown-at-Risk (Chekhlov et al., 1999), Mean-Absolute Deviation (Konno and Yamazaki (1991), Konno and Shirakawa (1994), Konno and Wijayanayake (1999)), Maximum Loss (Young, 1998) and the market-neutrality ("beta" of the portfolio equals zero)[3]. The first two risk measures represent relatively new developments in the risk management field. The considered risk management constraints can be used in linear portfolio rebalancing algorithms without destroying their linear structure. We have not included in this study the low partial moment with power one[4], which can also be treated using LP approaches. Testuri and Uryasev (2000) showed that the CVaR constraint and the low partial moment constraint with power one are equivalent in the sense that the efficient frontier for portfolio with CVaR constraint can be generated by the low partial moment approach. Therefore, the risk management with CVaR and with low partial moment leads to similar results. However, the CVaR approach allows for direct controlling of percentiles, while the low partial moment penalizes losses exceeding some fixed thresholds.

2.1. Conditional Value-at-Risk

The Conditional Value-at-Risk (CVaR) measure (see Rockafellar and Uryasev, 2000, 2001) develops and enhances the ideas of risk management, which have been explored in the framework of Value-at-Risk (VaR) (see, for example, Duffie and Pan (1997), Jorion (1997), Pritzker (1997), Staumbaugh (1996)). Incorporating such merits as easy-to-understand concept, simple and convenient representation of risks (one number), applicability to a wide range of instruments, VaR has evolved into a current industry standard for estimating risks of financial losses. Basically, VaR answers the question "what is the maximum loss, which is expected to be exceeded, say, only in 5% of the cases within the given time horizon?" For example, if daily VaR for the portfolio of some fund XYZ is equal to 10 millions USD at the confidence level 0.95, it means that there is only a 5% chance of losses exceeding 10 millions during a trading day. Mathematically, VaR with confidence level α is α-quantile of the loss distribution, see Fig. 1, (formal definitions are included in Appendix A).

However, using VaR as a risk measure in portfolio optimization is a very difficult problem, if the return distributions of a portfolio's instruments are not normal or log-normal. The optimization difficulties with VaR are caused by its non-convex and non-

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[3] There are different interpretations for the term "market-neutral" (see, for instance, BARRA RogersCasey (2002)). In the present paper, the market neutrality always means zero beta.

[4] Low partial moment with power one is defined as the expectation of losses exceeding some fixed threshold, see Harlow (1991). Expected regret (see, for example, Dembo and King, 1992) is a concept similar to the lower partial moment. However, the expected regret may be calculated with respect to a random benchmark, while the low partial moment is calculated with respect to a fixed threshold.

subadditive nature (Artzner et al. (1997, 1999), Mausser and Rosen (1991)). Non-convexity of VaR means that as a function of portfolio positions, it has multiple local extrema, which precludes using efficient optimization techniques.

The difficulties with controlling and optimizing VaR in non-normal portfolios have forced the search for similar percentile risk measures, which would also quantify downside risks and at the same time could be efficiently controlled and optimized. From this viewpoint, CVaR is a perfect candidate for conducting a "VaR"-style portfolio management.

For continuous distributions, CVaR is defined as an average (expectation) of high losses residing in the a-tail of the loss distribution, or, equivalently, as a conditional expectation of losses exceeding the a-VaR level (Fig. 1). For continuous loss distributions the a-CVaR function $f_a(\mathbf{x})$, where $\mathbf{x}$ is the vector of portfolio positions, can be written in the following way:

$$f_a(\mathbf{x}) = \frac{1}{1-a} \int_{f(\mathbf{x},\mathbf{y}) \geq z_a(\mathbf{x})} f(\mathbf{x},\mathbf{y})\, p(\mathbf{y})\, d\mathbf{y} ,$$

where $f(\mathbf{x},\mathbf{y})$ is the loss function depending on $\mathbf{x}$, the stochastic vector $\mathbf{y}$ represents uncertainties and has distribution $p(\mathbf{y})$, and $z_a(\mathbf{x})$ is a-VaR of the portfolio. For general distributions, e.g., for discrete distributions, a-CVaR (see, Rockafellar and Uryasev, 2001) is a weighted average of a-VaR and the expected value of losses strictly exceeding a-VaR (see Appendix A for formal definition). From this follows that CVaR incorporates information on VaR and on the losses exceeding VaR.

While inheriting some of the nice properties of VaR, such as measuring downside risks and representing them by a single number, applicability to instruments with non-normal distributions etc., CVaR has substantial advantages over VaR from the risk management standpoint. First of all, CVaR is a convex function[5] of portfolio positions. Thus, it has a convex set of minimum points on a convex set, which greatly simplifies control and optimization of CVaR. Calculation of CVaR, as well as its optimization, can be performed by means of a convex programming shortcut (Rockafellar and Uryasev, 2000, 2001), where the optimal value of CVaR is calculated simultaneously with the corresponding VaR; for linear or piecewise-linear loss functions these procedures can be reduced to linear programming problems. Also, unlike a-VaR, a-CVaR is continuous with respect to confidence level a, and has relatively stable statistical estimates. A comprehensive description of the CVaR risk measure and CVaR-related optimization methodologies can be found in Rockafellar and Uryasev (2000, 2001). Also, Rockafellar and Uryasev, (2000) showed that for normal loss distributions, the CVaR methodology is equivalent to the standard Mean-Variance approach.

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[5] For a background on convex functions and sets see Rockafellar (1970).

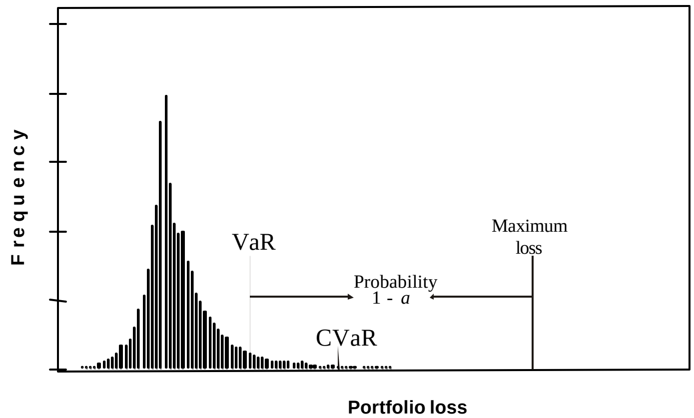

Figure 1. VaR, CVaR and Maximum Loss

Except for the fact that CVaR can be easily controlled and optimized, CVaR it is a more adequate measure of risk as compared to VaR because it accounts for losses beyond the VaR level. The fundamental difference between VaR and CVaR as risk measures are: VaR is the "optimistic" low bound of the losses in the tail, while CVaR gives the value of the *expected* losses in the tail. In risk management, we may prefer to be neutral or conservative rather than optimistic.

2.2. Conditional Drawdown-at-Risk

Conditional Drawdown-at-Risk (CDaR) is a portfolio performance measure (Chekhlov et al., 2000) closely related to CVaR. By definition, a portfolio's *drawdown* on a sample-path is the drop of the uncompounded[6] portfolio value as compared to the maximal value attained in the previous moments on the sample-path. Suppose, for instance, that we start observing a portfolio in January 2001, and record its uncompounded value every month[7]. If the initial portfolio value was \$100,000,000 and in February it reached \$130,000,000, then, the portfolio drawdown as of February 2001 is \$0. If, in March 2001, the portfolio value drops to \$90,000,000, then the current drawdown will equal to \$40,000,000 (in absolute terms), or 30.77%. Mathematically, the drawdown function for a portfolio is

$$\tilde{f}(\mathbf{x}, t) = \max_{0 \le t \le t} \{v_t(\mathbf{x})\} - v_t(\mathbf{x}), \qquad (1)$$

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[6] Drawdowns are calculated with uncompounded portfolio returns. This is related to the fact that risk measures based on drawdowns of uncompounded portfolios have nice mathematical properties. In particular, these measures are convex in portfolio positions. Suppose that at the initial moment $t = 0$ the portfolio value equals v and portfolio returns in the moments $t = 1,\ldots,T$ equal $r_1,\ldots, r_T$. By definition, the uncompounded portfolio value v_t at time moment t equals $v_t = v\sum_{t=1}^{t} r_t$. We suppose in this case study that the initial portfolio value $v = 1$.

[7] Usually, portfolio value is observed much more frequently. However, for the hedge funds considered in this paper, data are available on monthly basis.

where $\mathbf{x}$ is the vector of portfolio positions, and $v_t(\mathbf{x})$ is the *uncompounded* portfolio value at time t. We assume that the initial portfolio value is equal to 1; therefore, the drawdown is the uncompounded portfolio return starting from the previous maximum point. Figure 2 illustrates the relation between the portfolio value and the drawdown.

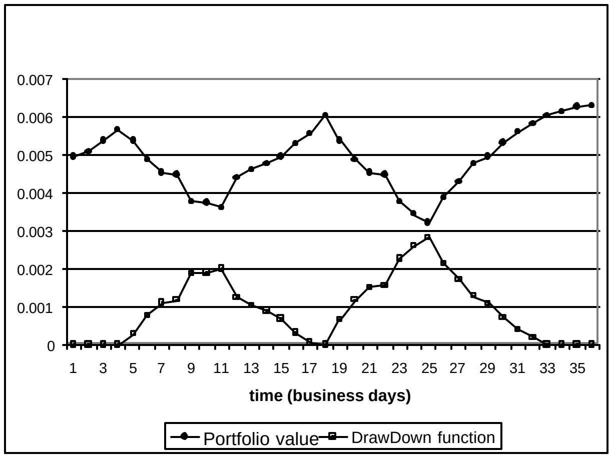


Figure 2. Portfolio value and drawdown.

From these preliminary remarks, it follows that the drawdown quantifies the financial losses in a conservative way: it calculates losses for the most "unfavorable" investment moment in the past as compared to the current moment. This approach reflects quite well the preferences of investors who define their allowed losses in percentages of their initial investments (e.g., an investor may consider it unacceptable to lose more than 10% of his investment). While an investor may excuse short-term drawdowns in his account, he would definitely start worrying about his capital in the case of a long-lasting drawdown. Such drawdown may indicate that something is wrong with that fund, and maybe it is time to move the money to a more successful investment pool. The mutual and hedge fund concerns are focused on keeping existing accounts and attracting new ones; therefore, they should ensure that clients' accounts do not fall into long-term drawdowns.

Thus, one can conclude that *drawdown* accounts not only for the amount of losses, but also for the *duration* of these losses. This highlights the unique feature of the drawdown concept: it is a loss measure "with memory" taking into account the time sequence of losses.

For a specified sample-path, the drawdown function is defined for each time moment. However, in order to evaluate performance of a portfolio on the whole sample-path, we would like to have a function, which aggregates all drawdown information over a given time period into one number. As this function one can pick, for example, the Maximum Drawdown,

$$MaxDD = \max_{0 \le t \le T} \left\{ \tilde{f}(\mathbf{x}, t) \right\},$$

or the Average Drawdown,

$$AverDD = \frac{1}{T}\int_0^T \tilde{f}(\mathbf{x}, t)\, dt \,.$$

However, both these functions may inadequately measure losses. The Maximum Drawdown is based on one "worst case" event in the sample-path. This event may represent some very specific circumstances, which may not appear in the future. The risk management decisions based only on this event may be too conservative.

On the other hand, the Average Drawdown takes into account all drawdowns in the sample-path. However, small drawdowns are acceptable (e.g., 1-2% drawdowns) and averaging may mask large drawdowns.

Chekhlov et al. (2000) suggested a new drawdown measure that combines both the drawdown concept of measuring risks and the CVaR approach in estimating downside losses: Conditional Drawdown-at-Risk. For instance, 0.95-CDaR can be thought of as an average of 5% of the highest drawdowns. Formally, α -CDaR is defined as α -CVaR with drawdown loss function $\tilde{f}(\mathbf{x}, t)$ given by (1) (see mathematical details in Appendix A). The CDaR risk measure holds nice properties of CVaR such as convexity with respect to portfolio positions. Also CDaR can be efficiently treated with linear optimization algorithms (Chekhlov et al., 2000).

2.3. Mean-Absolute Deviation

The Mean-Absolute Deviation (MAD) risk measure was introduced by Konno and Yamazaki (1991) as an alternative to the classical Mean-Variance measure of a portfolio's volatility,

$$MAD = E[\,|\, r_p(\mathbf{x}) - E[r_p(\mathbf{x})]\,|\,] \,,$$

where $r_p(\mathbf{x}) = r_1 x_1 + r_2 x_2 + ... + r_n x_n$ is the portfolio's rate of return, with $r_1, ..., r_n$ being the random rates of return for instruments in the portfolio. Since MAD is a piecewise linear convex function of portfolio positions, it allows for fast efficient portfolio optimization procedures by means of linear programming, in contrast to Mean-Variance approach, which leads to quadratic optimization problems. Konno and Shirakawa (1994) showed that MAD-optimal portfolios exhibit properties, similar to those of Markowitz MV-optimal portfolios, and that one can use MAD as a risk measure in deriving CAPM-type relationships. Later, it was also proved (Ogryczak and Ruszczynski, 1999) that portfolios on the MAD efficient frontier correspond to efficient portfolios in terms of the second-order stochastic dominance.

2.4. Maximum Loss

The Maximum Loss (MaxLoss) of a portfolio in a specified time period is defined as the maximal value over all random loss outcomes (Fig. 1), see for instance, Young (1998). When the distribution of losses is continuous, this risk measure may be unbounded, unless the distribution is "truncated". For example, for normal distribution, the maximum loss is infinitely large. However, for discrete loss distributions, especially for those based on small historical datasets, the MaxLoss is a reasonable measure of risk. We also would like to point out that the Maximum Loss admits an alternative definition as a special case of a -CVaR with a close to 1.

2.5. Market-neutrality

It is generally acknowledged that the market itself constitutes a risk factor. If the instruments in the portfolio are positively correlated with the market, then the portfolio would follow not only market growth, but also market drops. Naturally, portfolio managers are willing to avoid situations of the second type, by constructing portfolios, which are uncorrelated with market, or *market-neutral*. To be market-uncorrelated, the portfolio must have zero *beta*,

$$b_p = \sum_{i=1}^{n} b_i x_i = 0 ,$$

where $x_1,...,x_n$ denote the proportions in which the total portfolio capital is distributed among n assets, and b_i are betas of individual assets,

$$b_i = \frac{\mathrm{Cov}(r_i, r_M)}{\mathrm{Var}(r_M)},$$

here r_M stands for market rate of return. Instruments' betas, b_i, can be estimated, for example, using historical data:

$$b_i = \left(\sum_{j=1}^{J} (r_{M,j} - \overline{r}_M)^2 \right)^{-1} \sum_{j=1}^{J} (r_{i,j} - \overline{r}_i)(r_{M,j} - \overline{r}_M) ,$$

where J is the number of historical observations, and $\overline{r}$ denotes the sample average, $\overline{r} = J^{-1} \sum r_j$. As a proxy for market returns r_M, historical returns of S&P500 index can be used.

In this case study, we investigate the effect of constructing a market-neutral (zero-beta) portfolio, by including a market-neutrality constraint in the portfolio optimization problem. We compare the performance of the optimal portfolios obtained with and without market-neutrality constraint.

2.6. Problem setup

This section presents the "generic" problem formulation, which was used to construct an optimal portfolio. We suppose that some historical sample-path of returns of n instruments is available. Based on this sample-path, we calculate the expected return of the portfolio and the various risk measures for that portfolio. We maximize the expected return of the portfolio subject to different operating, trading, and risk constraints,

$$\max_{x} \; E\left[\sum_{i=1}^{n} r_i x_i\right] \tag{2}$$

subject to

$$0 \le x_i \le 1, \quad i = 1,\dots,n, \tag{3}$$

$$\sum_{i=1}^{n} x_i \le 1, \tag{4}$$

$$\Phi_{\text{Risk}}(x_1,\dots,x_n) \le w, \tag{5}$$

$$-k \le \sum_{i=1}^{n} b_i x_i \le k, \tag{6}$$

where

- x_i is the portfolio position (weight) of asset i ,
- r_i is the (random) rate of return of asset i,
- b_i is market beta of instrument i.

The objective function (2) represents the expected return of the portfolio. The first constraint (3) of the optimization problem imposes limitations on the amount of funds invested in a single instrument (we do not allow short positions). The second constraint (4) is the budget constraint. Constraints (5) and (6) control risks of financial losses. The key constraint in the presented approach is the risk constraint (5). Function $\Phi_{\text{Risk}}(x_1,\dots,x_n)$ represents either a -CVaR, a -CDaR, MAD, or MaxLoss risk measure, and risk tolerance level w is the fraction of the portfolio value that is allowed for risk exposure.

Constraint (6), with b_i representing market's *beta* for instrument i, forces the portfolio to be market-neutral in the "zero-beta" sense, i.e., the portfolio correlation with the market is bounded, and, presumably, such a portfolio would not follow significant market drops. The coefficient k in (6) is a small number that sets the portfolio's beta close to zero. To investigate the effects of imposing a "zero-beta" requirement on the portfolio-rebalancing algorithm, we solved the optimization problem with and without this constraint. Forestalling the events, we can say that constraint (6) significantly improves the out-of-sample performance of the algorithm.

As we pointed out earlier, the four risk measures considered in this paper allow for formulating the risk constraint (5) in terms of linear inequalities, which makes the optimization problem (2)–(6) linear, given the linearity of objective function and other constraints. Exact formulations of the risk constraint (5) for different risk functions can be found in Appendix A. Three of four risk measures considered in the paper are variations of the CVaR concept (CVaR itself, CDaR and MaxLoss). It is interesting to compare the performance of these three risk measures in a realistic setting, as well as to compare them with the more standard MAD measure (recall that MAD methodology stands relatively close to the classical Mean-Variance approach).

3. Case Study: Portfolio of Hedge Funds

The case study investigates investment opportunities and tests portfolio management strategies for a portfolio of hedge funds. As it has been already mentioned, hedge funds are subject to far less regulations as compared with mutual or pension funds. As a result, very little information on hedge funds' activities is publicly available (for example, many funds report their share prices only monthly). On the other hand, fewer regulations and weaker government control provide more room for aggressive, risk-seeking trading and investment strategies. As a consequence, the revenues in this industry are on average much higher than elsewhere, but the risk exposure is also higher (for example, the typical "life" of a hedge fund is about five years, and very few of them perform well in long run). Data availability and sizes of datasets impose challenging requirements on portfolio rebalancing algorithms. Also, the specific nature of hedge fund securities imposes some limitations on using them in trading or rebalancing algorithms. For example, hedge funds are far from being perfectly liquid: hedge funds may *not* be publicly traded or may be closed to new investors. From this point of view, our results contain a rather schematic representation of investment opportunities existing in the hedge fund market and do not give direct recommendations on investing in that market. The goal of this study is to compare the recently developed risk management approaches and to demonstrate their high numerical efficiency in a realistic setting.

The dataset for conducting the numerical experiments was provided to the authors by the Foundation for Managed Derivatives Research. It contained monthly data for more than 500 hedge funds, from which we selected those with significantly long history and some minimum level of capitalization. To pass the selection, a hedge fund should have 66 months of historical data from December 1995 to May 2001, and its capitalization should be at least 5 million U.S. dollars at the beginning of this period. The total number of funds, which satisfied these criteria and accordingly constituted the investment pool for our algorithm, was 301. In this dataset, the field with the titles of hedge funds was unavailable; therefore, we identified the hedge funds with numbers, i.e., HF 1, HF 2, and so on. The historical returns from the dataset were used to generate scenarios for algorithm (2)–(6). Each scenario is a vector of monthly returns for all securities involved in the optimization, and all scenarios are assigned equal probabilities.

We performed separate runs of the optimization problem (2)–(5), with and without constraint (6) for each risk measure (CVaR, CDaR, MAD, MaxLoss) appearing in constraint (5), varying such parameters as confidence levels, risk tolerance levels etc.

The case study consisted from two sets of numerical experiments. The first set, so-called *in-sample* experiments, included the calculation of efficient frontiers and the analysis of the optimal portfolio structure for each of the risk measures. The second set of experiments, *out-of-sample* testing, was designed to demonstrate the performance of our approach in a simulated historical environment.

3.1. In-sample results

Efficient frontier. For constructing the efficient frontier for the optimal portfolio with different risk constraints, we solved the optimization problem (2)–(5) with different risk tolerance levels w in constraint (5), varied from $w = 0.005$ to $w = 0.25$. The parameter a in CVaR and CDaR risk constraints was set to $a = 0.90$. The efficient frontier is presented in Figure 3a, where the portfolio rate of return means expected yearly rate of return. In these runs, the market-neutrality constraint (6) is inactive.

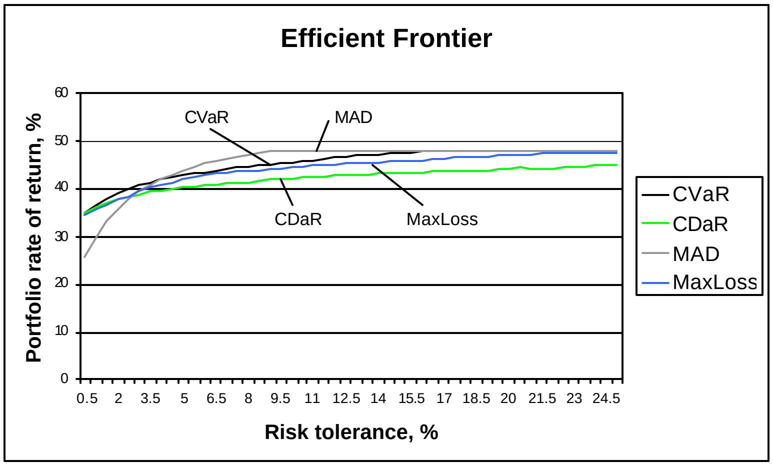


Figure 3a. Efficient frontiers for portfolios with various risk constraints. The market-neutrality constraint is inactive.

Figure 3a shows that three CVaR-related risk measures (CVaR, CDaR and MaxLoss) produce relatively similar efficient frontiers. However, the MAD risk measure produces a distinctively different efficient frontier.

For optimal portfolios, in the sense of problem (2)–(5), there exists an upper bound (equal to 48.13%) for the portfolio's rate of return. Optimal portfolios with CVaR, MAD and MaxLoss constraints reach this bound at different risk tolerance levels, but the CDaR-constrained portfolio does not achieve the maximal expected return within the given range of w values. CDaR is a relatively conservative constraint imposing requirements not only on the magnitude of loses, but also on the time sequence of losses (small consecutive losses may lead to large drawdown, without significant increasing of CVaR, MaxLoss, and MAD).

Figure 3b presents efficient frontiers of optimal portfolio (2)–(5) with the active market-neutrality constraint (6), where coefficient k is equal to 0.01. As one should expect, imposing the extra constraint (6) causes a decrease in in-sample optimal expected return. For example, the "saturation" level of the portfolio's expected return is now 41.94%, and all portfolios reach that level at much lower values of risk tolerance w. However, the market-neutrality constraint almost does not affect the curves of efficient portfolios in the leftmost points of efficient frontiers, which correspond to the lowest values of risk tolerance w. This means that by tightening the risk constraint (5) one can obtain a nearly market-neutral portfolio without imposing the market-neutrality constraint (6).

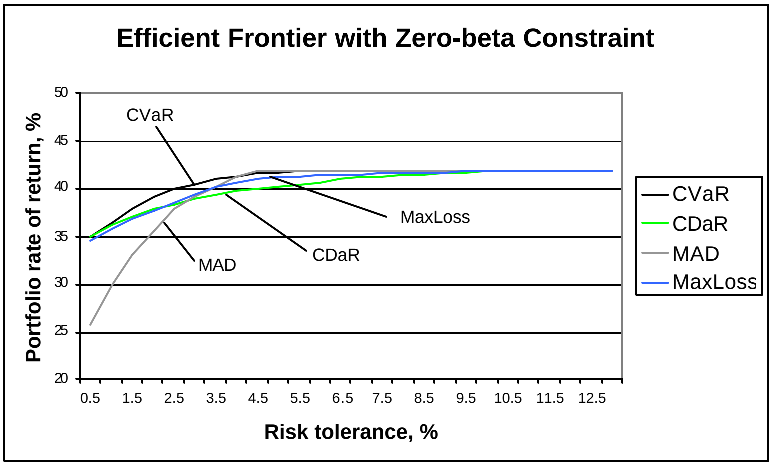


Figure 3b. Efficient frontier for market-neutral portfolio with various risk constraints (k = 0.01).

Optimal portfolio configuration. Let us discuss now the structure of the optimal portfolio with various risk constraints. For this purpose, we selected the corresponding efficient frontiers for four optimal portfolios with expected return of 35% and constraints on CVaR, CDaR, MaxLoss, and MAD (market-neutrality is inactive). For three risk measures (CVaR, CDaR and MaxLoss), the optimal portfolios with expected return of 35% are located in the vicinity of the leftmost points on corresponding efficient frontiers (see Fig. 3). Table 1 presents portfolio weights for the four optimal portfolios. It shows how a particular risk measure selects instruments given the specified expected return. The left column of Table 1 contains the set of the assets, which are chosen by the algorithm (2)–(5) under different risk constraints. Note that among the 301 available instruments, only a few of them are used in constructing the optimal portfolio. Moreover, a closer look at Table 1 shows that nearly two thirds of the portfolio value for all risk measures is formed by three hedge funds HF 209, HF 219 and HF 231 (the corresponding cells are filled with the darker shade of gray). These three hedge funds have stable performance, and each risk measure includes them in the optimal portfolio. Similarly, cells of a lighter shade of gray indicate instruments that are included in the portfolio with smaller weights, but still are approximately evenly distributed among the portfolios. Thus, the instruments HF 93, 100, 209, 219, 231, 258, and 259 constitute the "core" of the optimal portfolio under all risk constraints. The last row in Table 1 lists the total weight of these instruments in the corresponding optimal portfolio. The remaining assets (without highlighting in the table) are the "residual" instruments, which are specific to each risk measure. They may help us to spot differences in instrument selection of each risk constraint. Table 2 displays the residual weights of HF 49, 84, 106, 124, 126, 169, 196, and 298. The "residual" weights are calculated as the instrument's weight with respect to the residual part of the portfolio. For example, in the optimal portfolio with the CDaR constraint, the hedge fund HF 49 represents 11.02% of the total portfolio value, and at the same time it represents 49.06% of the residual $(1.00 - 0.775) \cdot 100\%$ portfolio value. In other words, it occupies almost half of the portfolio assets, not captured by hedge funds in the grayed cells. Also, note that neither of the residual instruments is simultaneously present in all portfolios.

Table 1. Instrument weights in the optimal portfolio with different risk constraints

	CDaR	CVaR	MAD	MaxLoss
HF 49	0.110216	0.043866	0	0.191952
HF 84	0.041898	0	0	0.078352
HF 93	0.081394	0.08754	0.045281	0.062609
HF 100	0.06629	0.073659	0.023899	0.068993
HF 106	0	0	0	0.00191
HF 124	0	0	0.020289	0
HF 126	0	0.008673	0.027908	0
HF 169	0.054298	0.010144	0	0
HF 196	0	0.015627	0.084791	0
HF 209	0.214683	0.224669	0.260824	0.226262
HF 219	0.137165	0.259254	0.169239	0.111746
HF 231	0.183033	0.169207	0.137068	0.177008
HF 258	0.034083	0.014156	0.097597	0.012606
HF 259	0.058684	0.089403	0.133104	0.068562
HF 298	0.018257	0.0038	0	0
	0.775331	0.917889	0.867012	0.727785

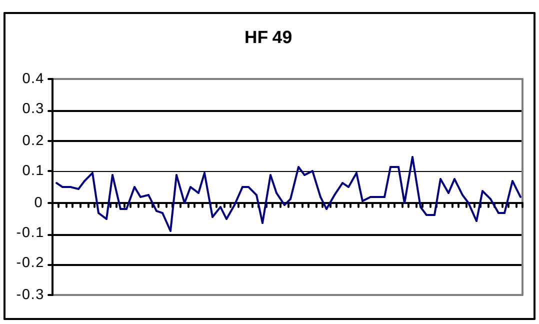
Table 2. Weights of residual instruments in the optimal portfolio with different risk constraints

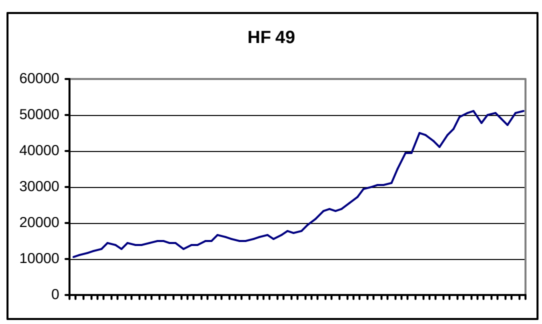
	CDaR	CVaR	MAD	MaxLoss
HF 49	0.490571	0.534229	0	0.705151
HF 84	0.186487	0	0	0.287833
HF 93	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 100	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 106	0	0	0	0.007016
HF 124	0	0	0.152561	0
HF 126	0	0.105624	0.209855	0
HF 169	0.24168	0.123544	0	0
HF 196	0	0.19031	0.637582	0
HF 209	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 219	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 231	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 258	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 259	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 298	0.081261	0.046285	0	0

Figure 4 contains graphs of the historical return and price dynamics for the residual hedge funds. We included these graphs to illustrate the differences in risk constraints and to make some speculations on this subject.

For example, instrument HF 84 is selected by CDaR and MaxLoss risk measures, but is excluded by CVaR and MAD. Note that graph of returns for the instrument HF 84 shows no negative monthly returns exceeding 10%, which is probably acceptable for the MaxLoss risk measure. Also, the price graph for the instrument HF 84 shows that it exhibits few drawdowns; therefore, CDaR picked this instrument. However, MAD probably excluded instrument HF8 because it had a high monthly return of 30% (recall that MAD does not discriminate between high positive returns and high negative returns). It is not clear from the graphs why CVaR rejected the HF 84 instruments. Probably, other instruments had better CVaR-return characteristics from the viewpoint of the overall portfolio performance.

The hedge fund HF 124 has not been chosen by any risk measures, with the exception of MAD. Besides rather average performance, it suffers long-lasting drawdowns (CDaR does not like this), has multiple negative return peaks of -10% magnitude (CVaR does not favor that), and its worst negative return is almost -20% (MaxLoss must protect from such performance drops). The question why this instrument was not rejected by MAD cannot be clearly answered in this case. Do not forget that such a decision is a solution of an optimization problem, and different instruments with properly adjusted weights may compensate each other's shortcomings. This may also be an excuse for MAD not picking the HF 49 fund, whose merits are confirmed by high residual weights of this fund in CDaR, CVaR, and MaxLoss portfolios.





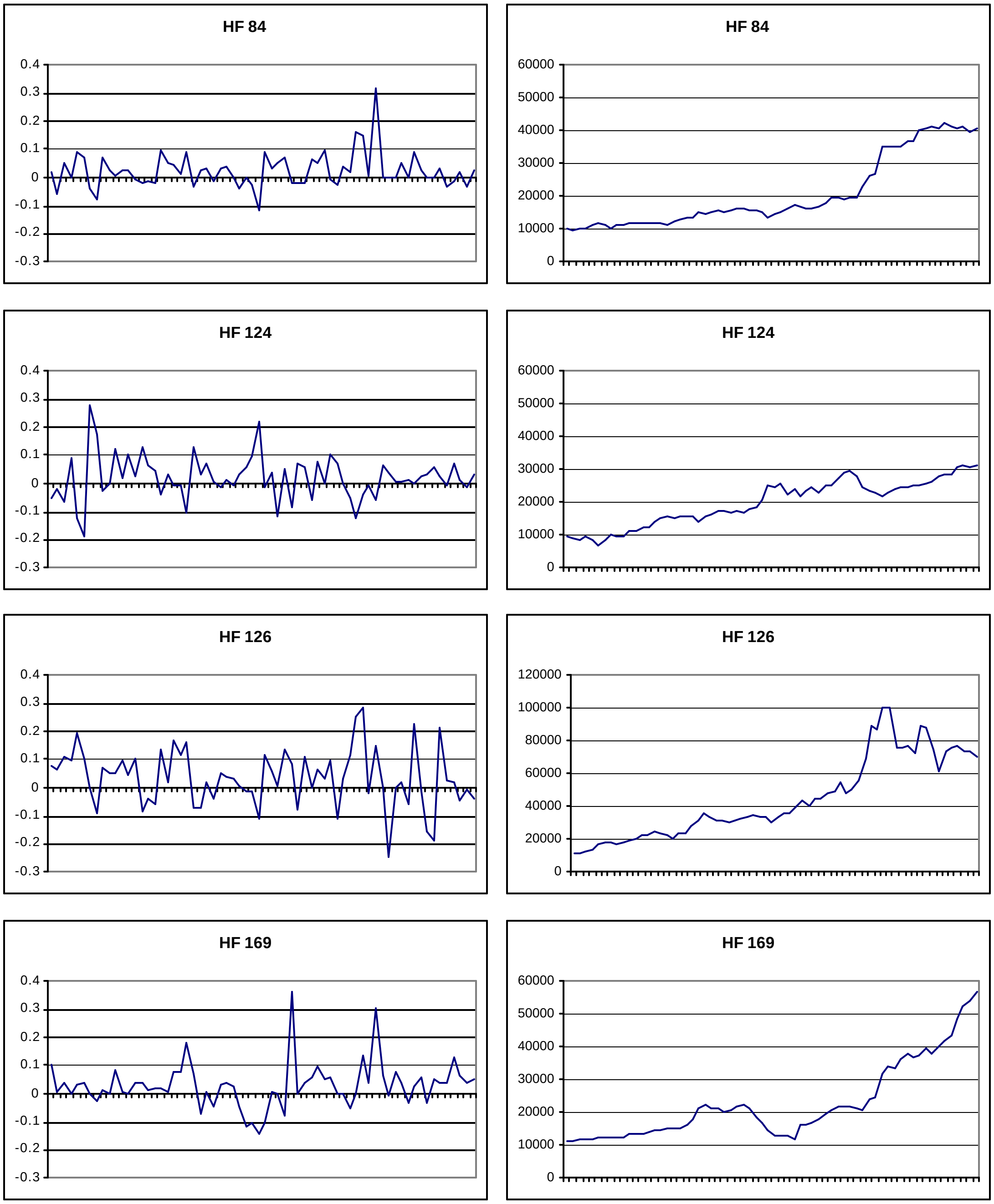


Figure 4. Historical performance in percents of initial value (on the right), and the rate of return dynamics in percentage terms (on the left) for some of the “residual” assets in optimal portfolios.

Fund HF 126 has the highest expected return among “residual” instruments, but it also suffers the most severe drawdowns and has the highest negative return (exceeding -20%) – that’s, probably, why algorithms with CDaR and MaxLoss measures rejected this instrument.

3.2. Out-of-sample calculations

The out-of-sample testing of the portfolio optimization algorithm (2)–(6) sheds light on the “*actual*” performance of the developed approaches. In other words, the question is how well do the algorithms with different risk measures utilize the scenario information based on past history in producing a successful portfolio management strategy? An answer can be obtained, for instance, by interpreting the results of the preceding section as follows: suppose we were back in May 2001, and we would like to invest a certain amount of money in a portfolio of hedge funds to deliver the highest reward under a specified risk level. Then, according to in-sample results, the best portfolio would be the one on the efficient frontier of a particular rebalancing strategy. In fact, such a portfolio offers the best return-to-risk ratio *provided that the historical distribution of returns will repeat in the future*.

To get an idea about the "actual" performance of the optimization approach, we used some part of the data for scenario generation, and the rest for evaluating the performance of the strategy. This technique is referred to as *out-of-sample* testing. In our case study, we perform the out-of-sample testing in two setups: 1) "Real" out-of-sample testing, and 2) "Mixed" out-of-sample testing. Each one is designed to reveal specific properties of risk constraints pertaining to the performance of the portfolio-rebalancing algorithm in out-of-sample runs.

"Real" out-of-sample testing. First, we present the results of a "plain" out-of-sample test, where the older data is considered as the 'in-sample' data for the algorithm, and the newer data are treated as 'to-be-realized' future. First, we took the 12 monthly returns within the time period from December 1995 to November 1996 as the initial historical data for constructing the first portfolio to invest in, and observed the portfolio's "realized" value by observing the historical prices for December 1996. Then, we added one more month, December 1996, to the data which were used for scenario generation (12 months of historical data in total) to generate an optimal portfolio and to allocate to investments in January, 1997, and so on. Note that we did not implement the "moving window" method for out-of-sample testing, where the same number of scenarios (i.e., the most recent historical points) is used for solving the portfolio-rebalancing problem. Instead, we accumulated the historical data for portfolio optimization.

First, we performed out-of-sample runs for each risk measure in constraint (5) for different values of risk tolerance level w (market-neutrality constraint, (6), is inactive). Figures 5a to 5d illustrate historical trajectories of the optimal portfolio under different risk constraints (the portfolio values are given in % relatively to the initial portfolio value). Risk tolerance level w was set to 0.005, 0.01, 0.03, 0.05, 0.10, 0.12, 0.15, 0.17 and 0.20, but for better reading of figures, we report only results with $w = 0.005$, 0.01, 0.05, 0.10, and 0.15. The parameter a in CDaR and CVaR constraints (in the last case a stands for risk confidence level) was set to $a = 0.90$.

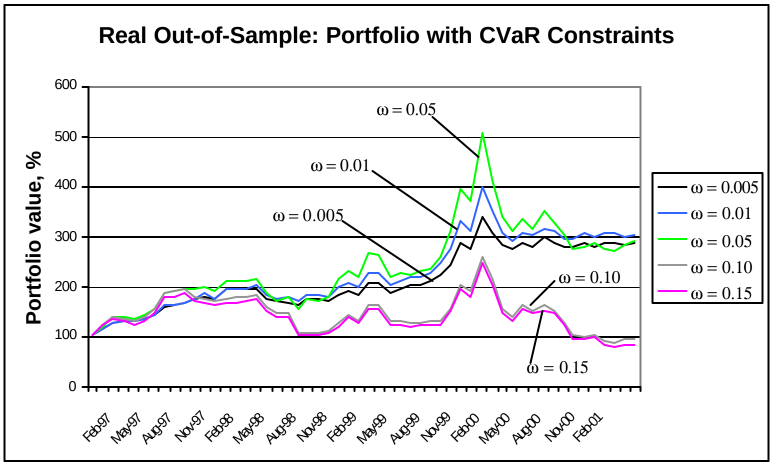


Figure 5a. Historical trajectories of optimal portfolio with CVaR constraints.

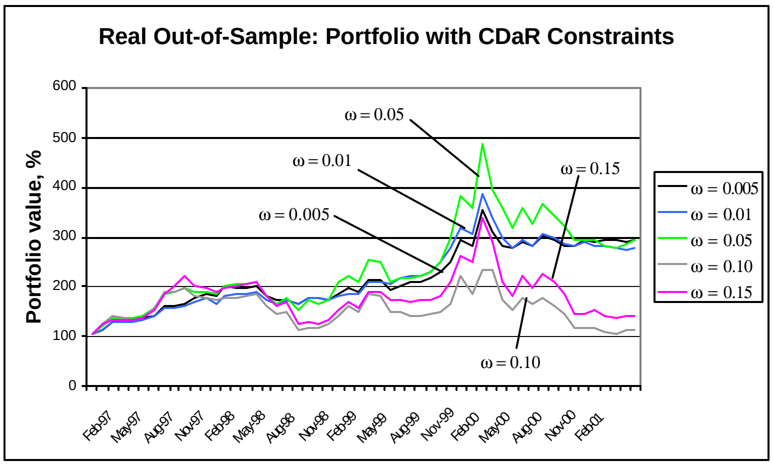


Figure 5b. Historical trajectories of optimal portfolio with CDaR constraints.

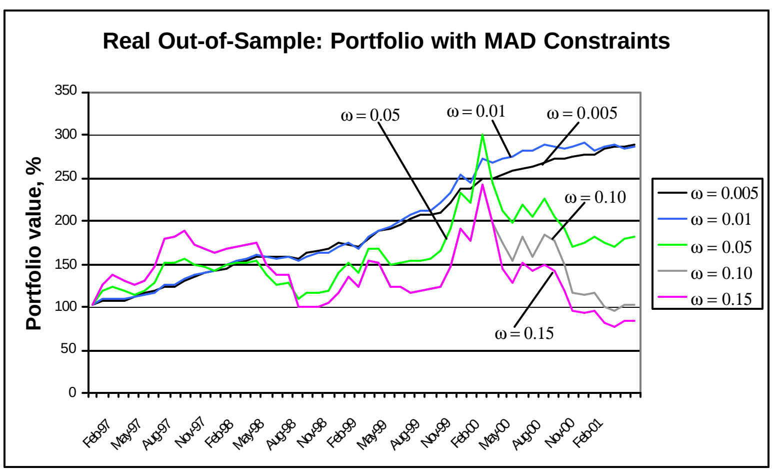


Figure 5c. Historical trajectories of optimal portfolio with MAD constraints.

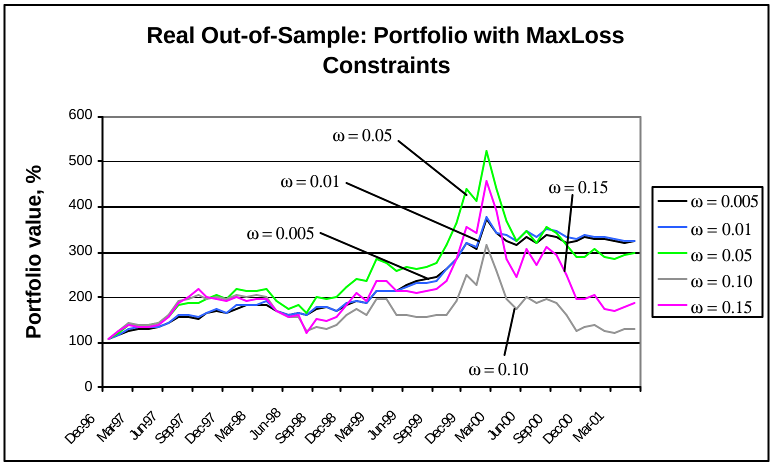


Figure 5d. Historical trajectories of optimal portfolio with MaxLoss constraints.

Figures 5a–5d show that risk constraint (5) has significant impact on the algorithm's out-of-sample performance. Earlier, we had also observed that this constraint has significant impact on the in-sample performance. It is well known that constraining risk in the in-sample optimization decreases the optimal value of the objective function, and the results reported in the preceding subsection comply with this fact. The risk constraints force the algorithm to favor less profitable but safer decisions over more profitable but "dangerous" ones. From a mathematical viewpoint, imposing extra constraints always reduces the feasibility set, and consequently leads to lower optimal objective values. However, the situation changes dramatically for an out-of-sample application of the optimization algorithm. The numerical experiments show that constraining risks *improves* the overall performance of the portfolio rebalancing strategy in out-of-sample runs; *tighter* in-sample risk constraint may lead to *both lower risks and higher out-of-sample returns*. For all considered risk measures, loosening the risk tolerance (i.e., increasing w values) results in increased volatility of out-of-sample portfolio returns and, after exceeding some threshold value, in degradation of the algorithm's performance, especially during the last 13 months (March 2000 – May 2001). For all risk functions in the constraint (5), the most attractive portfolio trajectories are obtained for risk tolerance level $w = 0.005$, which means that these portfolios have high returns (high final portfolio value), low volatility, and low drawdowns. Increasing w to 0.01 leads to a slight increase of the final portfolio value, but it also increases portfolio volatility and drawdowns, especially for the second quarter of 2001. For larger values of w the portfolio returns deteriorate, and for all risk measures portfolio curves with $w = 0.10$ show quite poor performance. Further increasing the risk tolerance to $w = 0.15$ in some cases allows for achieving higher returns at the end of 2000, but after this high peak the portfolio suffers severe drawdowns (see figures for CDaR, MAD, and MaxLoss risk measures).

The next series of Figures 6a–6d illustrates effects of imposing market-neutrality constraint (6) in addition to risk constraint (5). Recall that primary purpose of constraint (6) is making the portfolio uncorrelated with market. The main idea of composing a market-neutral portfolio is protecting it from market drawdowns. Figures 6a to 6d compare the trajectories of market-neutral and without risk-neutrality optimal portfolios. Additional constraining resulted in most cases in further improvement of the portfolio's out-of-sample performance, especially for CVaR and CDaR-constrained portfolios. To clarify how the risk-neutrality condition (6) influences the portfolio's performance, we displayed only figures for lowest and highest values of the risk tolerance parameter, namely for $w = 0.005$ and $w = 0.20$. Coefficient k in (6) was set to $k = 0.01$, and instruments' betas b_i were calculated by correlating with S&P 500 index, which is traditionally considered as a market benchmark. For portfolios with tight risk constraints ($w = 0.005$) imposing market-neutrality constraint (6) straightened their trajectories (reduced volatility and drawdowns), which made the historic curves almost monotone curves with a positive slope. On top of that, CVaR and CDaR portfolios with market-neutrality constraint had a higher final portfolio value, compared to those without market-neutrality. Also, for portfolios with loose risk constraints ($w = 0.20$) imposing market-neutrality constraint had a positive effect on the form of their trajectories, dramatically reducing volatility and drawdowns.

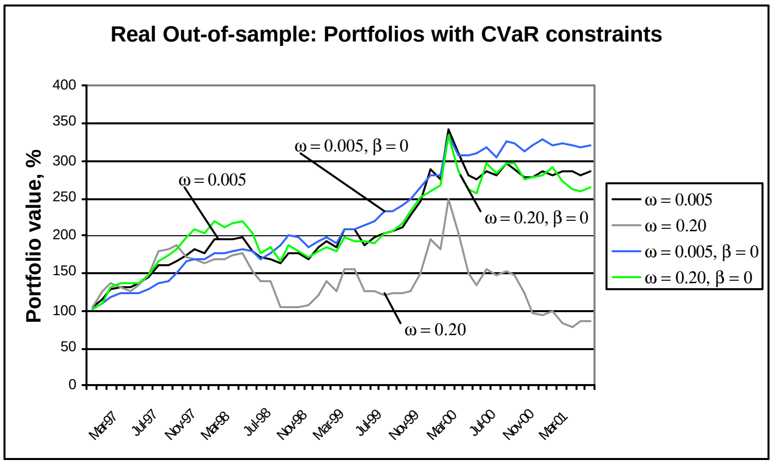


Figure 6a. Historical trajectories of optimal portfolio with CVaR constraints. Lines with $b = 0$ correspond to portfolios with market-neutral constraint.

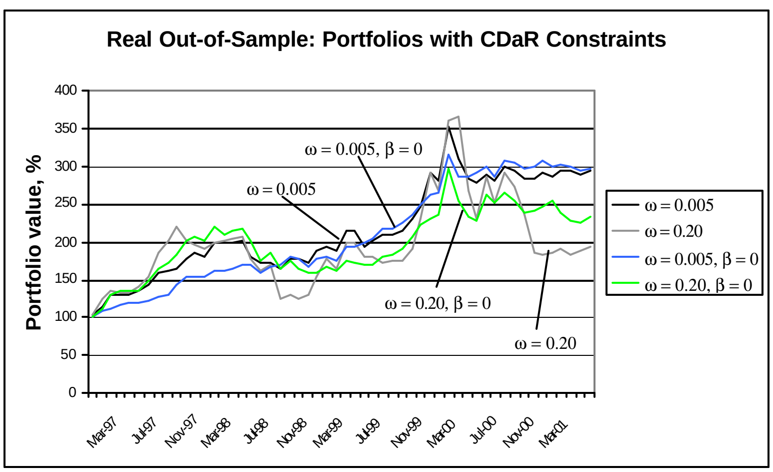


Figure 6b. Historical trajectories of optimal portfolio with CDaR constraints. Lines with $b = 0$ correspond to portfolios with market-neutral constraint.

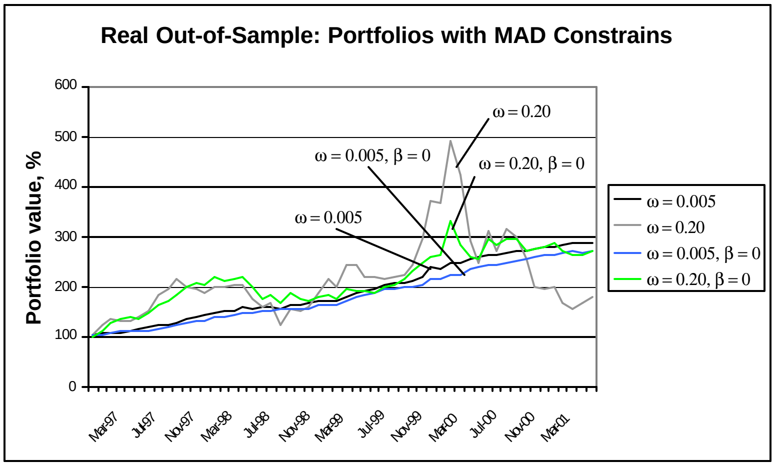


Figure 6c. Historical trajectories of optimal portfolio with MAD constraints.
Lines with $b = 0$ correspond to portfolios with market-neutral constraint.

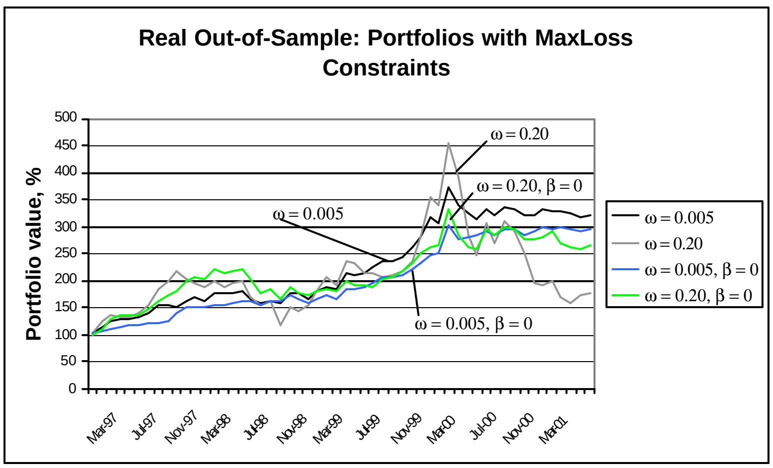


Figure 6d. Historical trajectories of optimal portfolio with MaxLoss constraints.
Lines with $b = 0$ correspond to portfolios with market-neutral constraint.

Finally, Figures 7a and 7b demonstrate the performance of the optimal portfolios versus two benchmarks: 1) S&P500 index; 2) "Best20", representing the portfolio distributed equally among the "best" 20 hedge funds. These 20 hedge funds include funds with the highest expected monthly returns calculated with past historical information. Similarly to the optimal portfolios (2)–(6), the "Best20" portfolio was monthly rebalanced (without risk constraints).

According to Figs. 7a–7b, the market-neutral optimal portfolios as well as portfolios without the market-neutrality constraint outperform the index under risk constraints of all types, which provides an evidence of high efficiency of the risk-constrained portfolio management algorithm (2)–(6). Also, we would like to emphasize the behavior of market-neutral portfolios in "down" market conditions. Two marks on Fig. 7b indicate the points when all four portfolios gained positive returns, while the market was falling down. Also, all risk-neutral portfolios seem to withstand the down market in 2000, when the market experienced significant drawdown. This demonstrates the efficiency and appropriateness of the application of market-neutrality constraint (6) to portfolio optimization with risk constraints (2)–(5).

The "Best20" benchmark evidently lacks the solid performance of its competitors. It not only significantly underperforms all the portfolios constructed with algorithm (2)–(6), but also underperforms the market half of the time. Unlike portfolios (2)–(6), the "Best20" portfolio pronouncedly follows the market drop in the second half of 2000, and moreover, it suffers much more severe drawdowns than the market does. This indicates that the risk constraints in the algorithm (2)–(6) play an important role in selecting the funds.

An interesting point to discuss is the behavior of algorithm (2)–(6) under the MAD risk constraint. Figures 5c and 6c show that a tight MAD constraint makes the portfolio curve almost a straight line, and imposing of market-neutrality constraint (6) does not add much to the algorithm's performance, and even slightly lowers the portfolio's return. At the same time, portfolios with CVaR-type risk constraints (CVaR, CDaR and MaxLoss) do not exhibit such remarkably stable performance, and take advantage of constraint (6). Note that CVaR, CDaR and MaxLoss are *downside* risk measures, whereas the MAD constraint suppresses both high losses and high returns. The market-neutrality constraint (6) by itself also puts symmetric restrictions on the portfolio's volatility; that's why it affects MAD and CVaR-related constraint differently. However, here, we should point out that we just "scratched the surface" regarding the combination of various risk constraints. We have imposed CVaR and CDaR constraints only with one confidence level. We can impose combinations of constraints with various confidence levels including constraining percentiles of high returns and as well as percentiles of high losses. These issues are beyond the scope of the paper.

Summarizing, we emphasize the general inference about the role of risk constraints in the out-of-sample and in-sample application of an optimization algorithm, which can be drawn from our experiments: risk constraints decrease the in-sample returns, while out-of-sample performance may be improved by adding risk constraints, and moreover, stronger risk constraints usually ensure better out-of-sample performance.

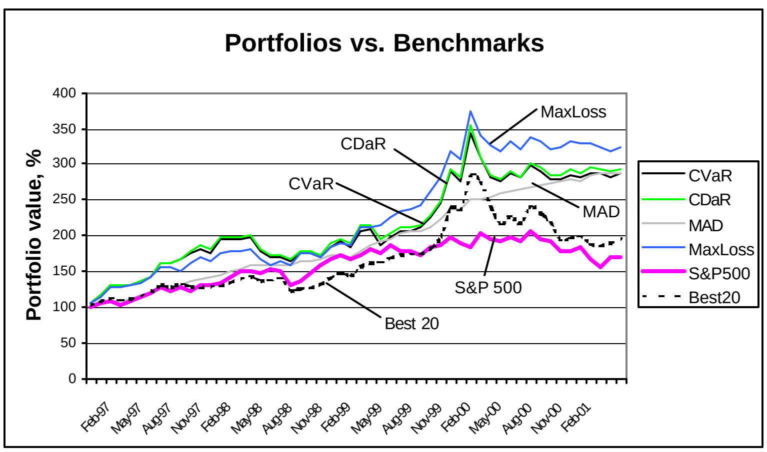


Figure 7a. Performance of the optimal portfolios with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level $w = 0.005$, parameter a = 0.90. Market-neutrality constraint is inactive.

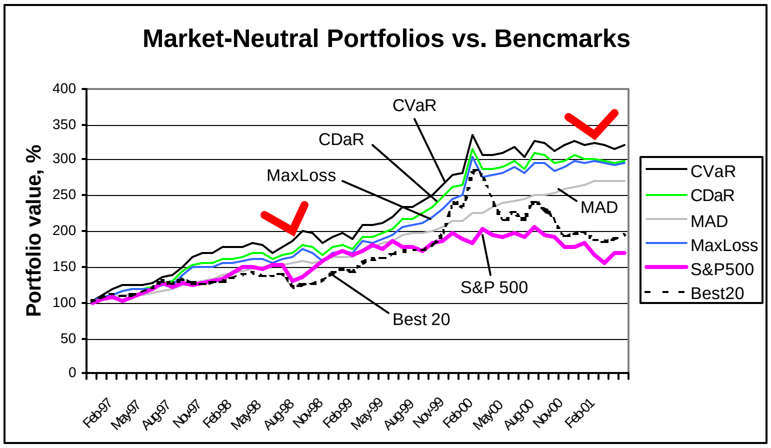


Figure 7b. Performance of market-neutral optimal portfolio with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level is $w = 0.005$, parameter is $a = 0.90$.

<u>"Mixed" out-of-sample test.</u> The second series of out-of-sample tests for the portfolio optimization algorithm (2)–(6) uses an alternative setup for splitting the data in in-sample and out-of-sample portions. Instead of utilizing only *past* information for generating scenarios for portfolio optimization, as it was done before, now we let the algorithm use both *past* and *future* information for constructing scenarios. The design of this experiment is as follows. The portfolio rebalancing procedure was performed every five months, and the scenario model utilized all the historical data except for the 5-month period directly following the rebalancing date. The procedure was started on December 1995. The information for scenario generation was collected from May 1996 to May 2001. The portfolio was optimized using these scenarios and invested on December 1995. After 5 months the money gained by the portfolio was reinvested and, at this time, the scenario model was built on information contained in the entire time interval 12/1995–05/2001 except window 05/1996–09/1996 and so on.

Figures 8a to 8d display dynamics of the optimal portfolio under various risk constraints and with different risk tolerance levels. To avoid overloading the paper with details, we report results only for $w = 0.01$, 0.05 and 0.10. As earlier, we set parameter a in CVaR and CDaR constraints to $a = 0.90$.

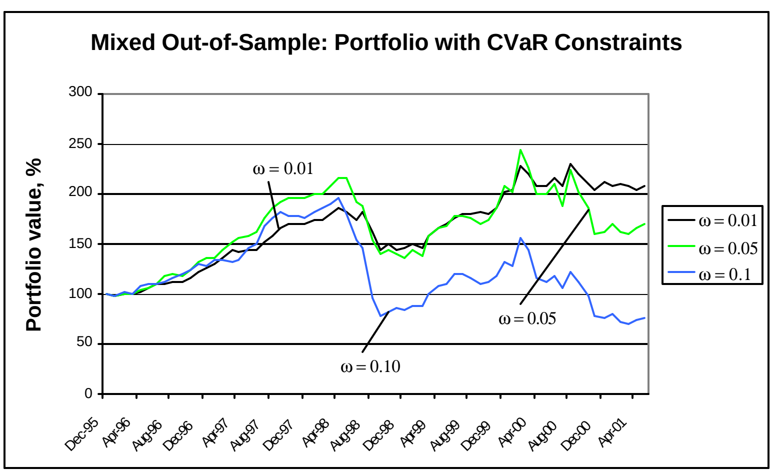


Figure 8a. "Mixed" out-of-sample trajectories of optimal portfolio with CVaR constraints

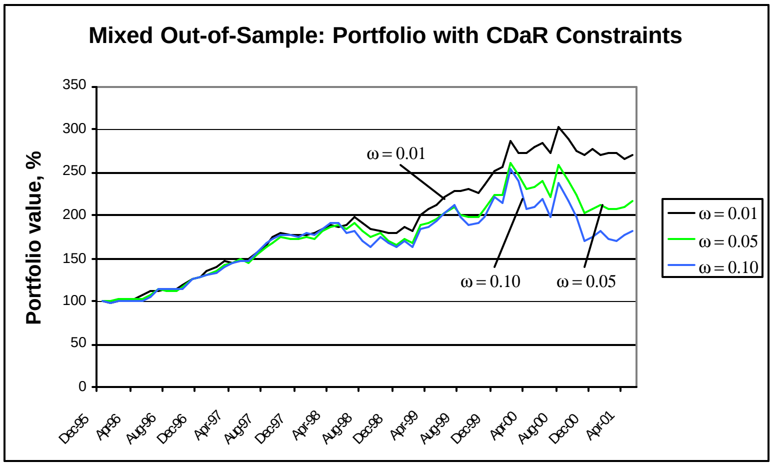


Figure 8b. “Mixed” out-of-sample trajectories of optimal portfolio with CDaR constraints

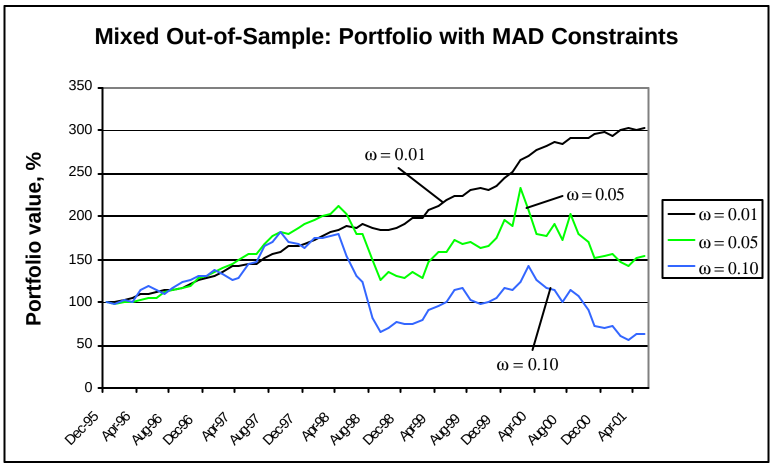


Figure 8c. “Mixed” out-of-sample trajectories of optimal portfolio with MAD constraints

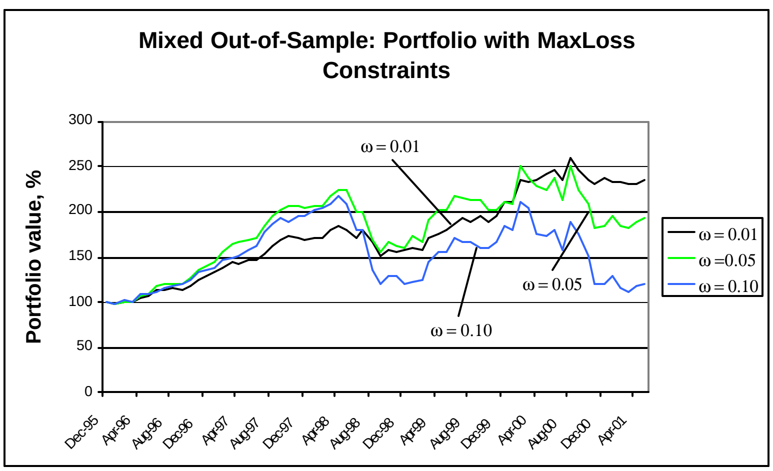


Figure 8d. “Mixed” out-of-sample trajectories of optimal portfolio with MaxLoss constraints

The general picture of these results is consistent with conclusions derived from “real” out-of-sample tests: tightening of risk constrains improve performance of the rebalance algorithms. Lower overall performance of the portfolio optimization strategy under all risk constraints comparing to that in “real” out-of-sample testing is explained by longer rebalancing period. It is well known that more frequent rebalancing may give higher returns (at least, in the absence of transaction costs).

The next four Figures 9a to 9d demonstrate the influence of the market-neutrality constraint on the performance of the portfolio.

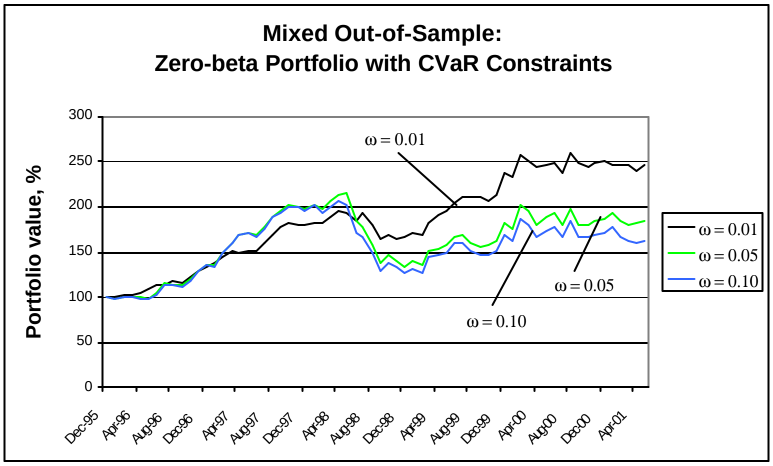


Figure 9a. “Mixed” out-of-sample trajectories of market-neutrality optimal portfolio with CVaR constraints

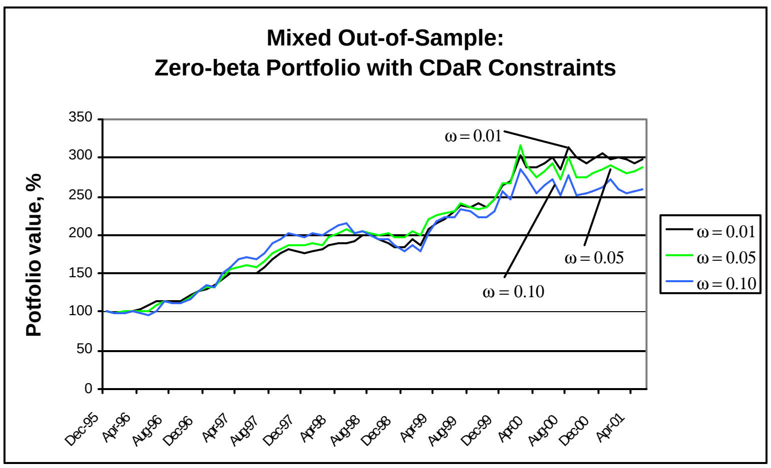


Figure 9b. “Mixed” out-of-sample trajectories of market-neutrality optimal portfolio with CDaR constraints

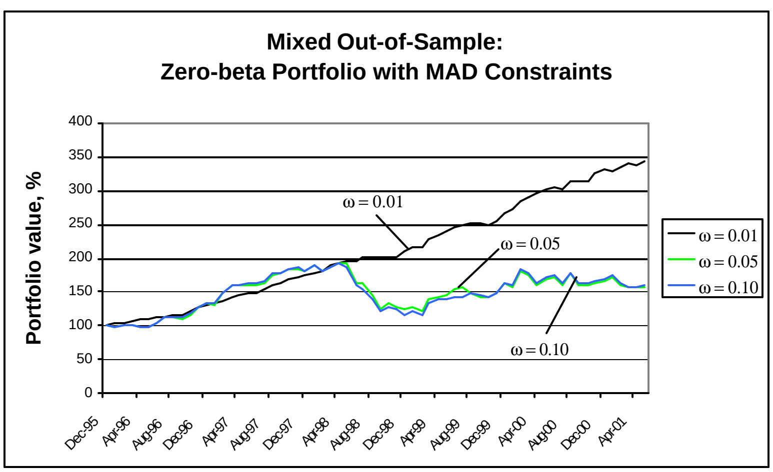


Figure 9c. "Mixed" out-of-sample trajectories of
market-neutrality optimal portfolio with MAD constraints

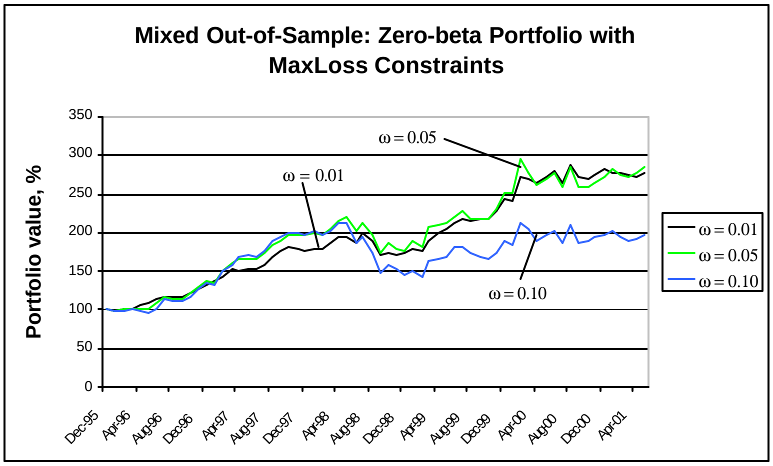


Figure 9d. "Mixed" out-of-sample trajectories of
market-neutrality optimal portfolio with MaxLoss constraints

Imposing of market-neutrality constraint (6) in problem (2)–(5) for the "mixed" out-of-sample testing has a similar impact as in the "real" out-of-sample testing.

4. Conclusions

We have tested the performance of a portfolio allocation algorithm with different types of risk constraints in an application for managing a portfolio of hedge funds. As the risk measure in the portfolio optimization problem, we used Conditional Value-at-Risk, Conditional Drawdown-at-Risk, Mean-Absolute Deviation and Maximum Loss. We also combined the risk constraint based on one of the above measures with the market-neutrality (zero-beta) constraint making the optimal portfolio uncorrelated with the market.

The numerical experiments consist of in-sample and out-of-sample testing. We generated efficient frontiers and compared algorithms with various constraints. The out-of-sample part of experiments was performed in two setups, which differed in constructing the scenario set for the optimization algorithm.

The results obtained are dataset-specific and we cannot make direct recommendations on portfolio allocations based on these results. However, we learned several lessons from this case study. Imposing risk constraints may significantly degrade in-sample expected returns while improving risk characteristics of the portfolio. In-sample experiments showed that for tight risk tolerance levels, all risk constraints produce relatively similar portfolio configurations. Imposing risk constraints may improve the out-of-sample performance of the portfolio-rebalancing algorithms in the sense of risk-return tradeoff. Especially promising results can be obtained by combining several types of risk constraints. In particular, we combined the market-neutrality (zero-beta) constraint and one of the CVaR, CDaR, MAD, and MaxLoss constraints. We found that tightening of risk constraints greatly improves portfolio dynamic performance in out-of-sample tests, increasing the overall portfolio return and decreasing both losses and drawdowns. Also, for portfolios with different risk measures, imposing the market-neutrality constraint adds to the stability of portfolio's return, and reduces portfolio drawdowns. CDaR and CVaR risk measures demonstrated the most solid performance in out-of-sample tests, with CVaR having an advantage in "real" out-of-sample testing, and CDaR performing better in "mixed" out-of-sample setup. However, we found that all risk measures performed quite well in our numerical experiments.

5. Acknowledgements

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Appendix A. Formal definitions of risk measures and risk constraints

A.1. Value-at-risk and Conditional Value-at-Risk

Let $f(\mathbf{x},\mathbf{y})$ be a loss function, where $\mathbf{x}$ is a decision vector (e.g., portfolio positions), and $\mathbf{y}$ is a stochastic vector standing for market uncertainties (in this paper, $\mathbf{y}$ is the vector of returns of instruments in the portfolio). Let $\Psi(\mathbf{x},z)$ be the cumulative distribution function of $f(\mathbf{x},\mathbf{y})$,

$$\Psi(\mathbf{x},z) = \mathsf{P}[\, f(\mathbf{x},\mathbf{y}) \le z\,].$$

Then, the Value-at-Risk (VaR) function $z_a(\mathbf{x})$ with the confidence level a is the a-quantile of $f(\mathbf{x},\mathbf{y})$,

$$z_a(\mathbf{x}) = \min_{z \in \mathbb{R}} \{\, \Psi(\mathbf{x},z) \ge a \,\}.$$

As it was discussed in Section 2.1, CVaR approximately (or exactly in some cases) equals the conditional expectation of values $f(\mathbf{x},\mathbf{y})$ exceeding VaR. For general distributions, Rockafellar and Uryasev (2001) defined a-CVaR function $f_a(\mathbf{x})$ as the a-tail expectation of a random variable z,

$$f_a(\mathbf{x}) = \mathsf{E}_{a-\text{tail}}[z],$$

where the a-tail cumulative distribution functions of z has the form,

$$\Psi_a(\mathbf{x},z) = \mathsf{P}[z \le z] = \begin{cases} 0, & z < z_a(\mathbf{x}), \\ [\Psi(\mathbf{x},z) - a] / [1-a], & z \ge z_a(\mathbf{x}). \end{cases}$$

Along with a-CVaR function $f_a(\mathbf{x})$, we consider also the so called “upper” and “lower” CVaR (a-CVaR$^+$ and a-CVaR$^-$),

$$f_a^+(\mathbf{x}) = \mathsf{E}[\, f(\mathbf{x},\mathbf{y}) \mid f(\mathbf{x},\mathbf{y}) > z_a(\mathbf{x})],$$

$$f_a^-(\mathbf{x}) = \mathsf{E}[\, f(\mathbf{x},\mathbf{y}) \mid f(\mathbf{x},\mathbf{y}) \ge z_a(\mathbf{x})].$$

The introduced CVaR functions satisfy the following inequality:

$$f_a^-(\mathbf{x}) \le f_a(\mathbf{x}) \le f_a^+(\mathbf{x}).$$

Also, a-CVaR can be presented as the convex combination of a-VaR and a-CVaR$^+$,

$$f_a(\mathbf{x}) = l_a(\mathbf{x}) z_a(\mathbf{x}) + [1 - l_a(\mathbf{x})] f_a^+(\mathbf{x}),$$

where

$$l_a(\mathbf{x}) = [\Psi(\mathbf{x}, z_a(\mathbf{x})) - a] / [1-a], \quad 0 \le l_a(\mathbf{x}) \le 1.$$

For a discrete loss distribution, where the stochastic parameter $\mathbf{y}$ can take values $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_J$ with probabilities q_j, $j = 1,...,J$, the a-VaR and a-CVaR functions respectively are

$$z_a(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}_{j_a}),$$

$$f_a(\mathbf{x}) = \frac{1}{1-a}\left[\left(\sum_{j=1}^{j_a} q_j - a\right) f(\mathbf{x}, \mathbf{y}_{j_a}) + \sum_{j=j_a+1}^{J} q_j f(\mathbf{x}, \mathbf{y}_j)\right],$$

where number j_a satisfies

$$\sum_{j=1}^{j_a - 1} q_j < a \le \sum_{j=1}^{j_a} q_j \,.$$

The optimization problem with multiple CVaR constraints

$$\min_{\mathbf{x} \in X} \quad g(\mathbf{x})$$

subject to

$$f_{a_i}(\mathbf{x}) \le w_i, \; i = 1, ..., I,$$

is equivalent to the following problem:

$$\min_{\mathbf{x} \in X,\, z_k \in \mathbb{R} \; \forall k} \quad g(\mathbf{x})$$

$$\text{subject to} \quad z_k + \frac{1}{1-a_k} \sum_{j=1}^{J} q_j \max\{0, \; f(\mathbf{x}, \mathbf{y}_j) - z_k\} \le w_k, \; k = 1, ..., K,$$

provided that the objective function $g(\mathbf{x})$ and the loss function $f(\mathbf{x}, \mathbf{y})$ are convex in $\mathbf{x} \in X$. When the objective and loss functions are linear in $\mathbf{x}$ and constraints $\mathbf{x} \in X$ are given by linear inequalities, the last optimization problem can be reduced to LP, see Rockafellar and Uryasev (2000, 2001).

In the present case study, the loss function is the negative portfolio's return,

$$f(\mathbf{x}, \mathbf{y}) = -\sum_{i=1}^{n} r_i x_i \,, \qquad \text{(A1)}$$

where the vector of instruments' returns $\mathbf{y} = \mathbf{r} = (r_1, ..., r_n)$ is random. Then the risk constraint (5), $f_a(\mathbf{x}) \le w$, where CVaR risk function replaces the function $\Phi_{\text{Risk}}(\mathbf{x})$, reads as

$$z + \frac{1}{(1-a)J} \sum_{j=1}^{J} \max\left\{0, -\sum_{i=1}^{n} r_{ij} x_i - z\right\} \le w \,, \qquad \text{(A2)}$$

where r_{ij} is return of i-th instrument in scenario j, $j = 1, ..., J$. Since the loss function (A1) is linear (and therefore convex), the risk constraint (A2) can be equivalently represented by the following set of linear inequalities,

$$z+\frac{1}{1-a}\frac{1}{J}\sum_{j=1}^{J}w_j \le w,$$
$$-\sum_{i=1}^{n} r_{ij}x_i - z \le w_j,\ \ j=1,\ldots,J, \qquad \text{(A3)}$$
$$z\in\mathbb{R},\quad w_j\ge 0,\quad j=1,\ldots,J.$$

This representation allows for reducing the optimization problem (2)–(6) with the CVaR constraint to a linear programming problem.

A.2. Conditional Drawdown-at-Risk

The Conditional Drawdown-at-Risk is defined in the paper by Chekhlov et al. (2000). We assume that possible realizations of the random vectors describing uncertainties in the loss function is represented by a sample-path (time-dependent scenario), which may be obtained from historical or simulated data. In this paper, it is assumed that we know the sample-path of returns of instruments included in the portfolio. Let r_{ij} be the rate of return of i-th instrument in j-th trading period (that corresponds to j-th month in the case study), $j=1,\ldots,J$. We suppose that the initial portfolio value equals 1. Let x_i, $i=1,\ldots,n$ be weights of instruments in the portfolio. The uncompounded portfolio value at time j equals

$$v_j(\mathbf{x})=\sum_{i=1}^{n}\left(1+\sum_{s=1}^{j}r_{is}\right)x_i\,.$$

The drawdown function $\tilde f(\mathbf{x},\mathbf{r}_j)$ at the time j is defined as the drop in the portfolio value compared to the maximum value achieved before the time moment j,

$$\tilde f(\mathbf{x},\mathbf{r}_j)=\max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k}r_{is}\right)x_i\right\}-\sum_{i=1}^{n}\left(\sum_{s=1}^{j}r_{is}\right)x_i\,.$$

Then, the Conditional Drawdown-at-Risk function $\Delta_a(\mathbf{x})$ is defined as follows. If the parameter a and number of scenarios J are such that their product

$$(1-a)J$$

is an integer number, then $\Delta_a(\mathbf{x})$ is defined as

$$\Delta_a(\mathbf{x})=h_a+\frac{1}{(1-a)J}\sum_{j=1}^{J}\max\left\{0,\ \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k}r_{is}\right)x_i\right\}-\sum_{i=1}^{n}\left(\sum_{s=1}^{j}r_{is}\right)x_i-h_a\right\},$$

where $h_a=h_a(\mathbf{x})$ is the threshold that is exceeded by $(1-a)J$ drawdowns. In this case the drawdown functions $\Delta_a(\mathbf{x})$ is the average of the worst case $(1-a)J$ drawdowns observed in the considered sample-path. If $(1-a)J$ is not integer, then the CDaR function, $\Delta_a(\mathbf{x})$, is the solution of the following optimization problem

$$\Delta_a(\mathbf{x}) = \min_h \left\{ h + \frac{1}{1-a}\frac{1}{J}\sum_{j=1}^{J} \max\left[0, \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - h\right]\right\}.$$

The CDaR risk constraint $\Delta_a(\mathbf{x}) \le w$ has the form

$$h + \frac{1}{1-a}\frac{1}{J}\sum_{j=1}^{J} \max\left[0, \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - h\right] \le w,$$

and it can be reduced to a set of linear constraints similarly to the CVaR constraint.

A.3. Mean-Absolute Deviation

The Mean-Absolute Deviation (MAD) $V(\mathbf{x})$ of portfolio's rate of return equals

$$V(\mathbf{x}) = E\left[\left|\sum_{i=1}^{n} r_i x_i - E\left[\sum_{i=1}^{n} r_i x_i\right]\right|\right],$$

where r_i denotes a random return of i-th asset, see, Konno and Yamazaki (1991). We suppose that $j = 1,\dots,J$ scenarios of returns with probabilities q_j are available. Let us denote by r_{ij} the return of i-th asset in the scenario j. The portfolio's MAD can be written as

$$V(\mathbf{x}) = \sum_{j=1}^{J} q_j \left|\sum_{i=1}^{n} r_{ij} x_i - \sum_{k=1}^{k} q_k \sum_{i=1}^{n} r_{ik} x_i\right|.$$

Given equal scenario probabilities, the MAD constraint $V(\mathbf{x}) \le w$ has the form

$$\frac{1}{J}\sum_{j=1}^{J}\left|\sum_{i=1}^{n} r_{ij} x_i - \frac{1}{J}\sum_{k=1}^{J}\sum_{i=1}^{n} r_{ik} x_i\right| \le w .$$

This constraint admits representation by linear inequalities,

$$\frac{1}{J}\sum_{j=1}^{J}(u_j^+ + u_j^-) \le w,$$

$$\sum_{i=1}^{n} r_{ij} x_i - \frac{1}{J}\sum_{j=1}^{J}\sum_{i=1}^{n} r_{ij} x_i = u_j^+ - u_j^-, \quad j = 1,\dots,J,$$

$$u_j^{\pm} \ge 0, \quad j = 1,\dots,J.$$

A.4. Maximum Loss

Let us suppose that $j = 1,..., J$ scenarios of returns are available (r_{ij} denotes return of i -th asset in the scenario j). The Maximum Loss (MaxLoss) function has the form (see for instance, Young (1998))

$$v\,(\mathbf{x}) = \max_{1 \le j \le J} \left\{ -\sum_{i=1}^{n} r_{ij} x_i \right\}.$$

It is worth noting that a -CVaR function $f_a(\mathbf{x})$ coincides with MaxLoss for values a close to 1. Suppose that scenarios $j = 1,..., J$ have equal probabilities $1/J$. When the confidence level $\tilde{a}$ satisfies $\tilde{a} \ge \frac{J-1}{J}$, the MaxLoss equals to a -CVaR function $v\,(\mathbf{x}) = f_{\tilde{a}}(\mathbf{x})$. The MaxLoss constraint $v\,(\mathbf{x}) \le w$ in (5) can be written as

$$\max_{1 \le j \le J} \left\{ -\sum_{i=1}^{n} r_{ij} x_i \right\} \le w\ .$$

Similar to other considered risk constraints, it can be replaced by a system of linear inequalities

$$w \le w,$$

$$-\sum_{i=1}^{n} r_{ij} x_i \le w, \quad j = 1,..., J.$$