Asia-Pacific Financial Markets (2004) 11: 301–333
DOI: 10.1007/s10690-005-9004-3

Pricing European Options by Numerical Replication: Quadratic Programming with Constraints

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Abstract. The paper considers a regression approach to pricing European options in an incomplete market. The algorithm replicates an option by a portfolio consisting of the underlying security and a risk-free bond. We apply linear regression framework and quadratic programming with linear constraints (input = sample paths of underlying security; output = table of option prices as a function of time and price of the underlying security). We populate the model with historical prices of the underlying security (possibly massaged to the present volatility) or with Monte Carlo simulated prices. Risk neutral processes or probabilities are not needed in this framework.

1. Introduction

Options pricing is a central topic in financial literature. A reader can find an excellent overview of option pricing methods in Broadie and Detemple (2004). The algorithm for pricing European options in discrete time presented in this paper has common features with other existing approaches. We approximate an option value by a portfolio consisting of the underlying stock and a risk-free bond. The stock price is modelled by a set of sample-paths generated by a Monte-Carlo or historical bootstrap simulation. We consider a non-self-financing portfolio dynamics and minimize the sum of squared additions/subtractions of money to/from the hedging portfolio at every re-balancing point, averaged over a set of sample paths. This error minimization problem is reduced to quadratic programming. We also include constraints on the portfolio hedging strategy to the quadratic optimization problem. The constraints dramatically improve numerical efficiency of the algorithm.

Below, we refer to option pricing methods directly related to our algorithm. Although this paper considers European options, some related papers consider American options.

Replication of the option price by a portfolio of simpler assets, usually of the underlying stock and a risk-free bond, can incorporate various market frictions, such as transaction costs and trading restrictions. For incomplete markets, replication-based models are reduced to linear, quadratic, or stochastic programming problems, see, for instance, Bouchaud and Potters (2000), Bertsimas et al. (2001), Dembo and Rosen (1999), Coleman et al. (2004), Naik and Uppal (1994), Dennis (2001), Dempster and Thompson (2001), Edirisinghe et al. (1993), Fedotov and Mikhailov (2001), King (2002), and Wu and Sen (2000).

Analytical approaches to minimization of quadratic risk are used to calculate an option price in an incomplete market, see Duffie and Richardson (1991), Follmer and Schied (2002), Fllmer and Schweizer (1989), Schweizer (1991, 1995, 2001).

Another group of methods, which are based on a significantly different principle, incorporates known properties of the shape of the option price into the statistical analysis of market data. Ait-Sahalia and Duarte (2003) incorporate monotonic and convex properties of European option price with respect to the strike price into a polynomial regression of option prices. In our algorithm we limit the set of feasible hedging strategies, imposing constraints on the hedging portfolio value and the stock position. The properties of the option price and the stock position and bounds on the option price has been studied both theoretically and empirically by Merton (1973), Perrakis and Ryon (1984), Ritchken (1985), Bertsimas and Popescu (1999), Gotoh and Konno (2002), and Levy (1985). In this paper, we model stock and bond positions on a two-dimensional grid and impose constraints on the grid variables. These constraints follow under some general assumptions from non-arbitrage considerations. Some of these constraints are taken from Merton (1973).

Monte-Carlo methods for pricing options are pioneered by Boyle (1977). They are widely used in options pricing: Joy et al. (1996), Broadie and Glasserman (2004), Longstaff and Schwartz (2001), Carriere (1996), Tsitsiklis and Van Roy (2001). For a survey of literature in this area see Boyle et al. (1997) and Glasserman (2004). Regression-based approaches in the framework of Monte-Carlo simulation were considered for pricing American options by Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001, 1999). Broadie and Glasserman (2004) proposed stochastic mesh method which combined modelling on a discrete mesh with Monte-Carlo simulation. Glasserman (2004), showed that regression-based approaches are special cases of the stochastic mesh method.

The pricing algorithm described in this paper combines the features of the above approaches in the following way. We construct a hedging portfolio consisting of the underlying stock and a risk-free bond and use its value as an approximation to the option price. We aimed at making the hedging strategy close to real-life trading. The actual trading occurs at discrete times and is not self-financing at re-balancing points. The shortage of money should be covered at any discrete point. Large shortages are undesirable at any time moment, even if self-financing is present. We consider non-self-financing hedging strategies. External financing of the portfolio or withdrawal is allowed at any re-balancing point. We use a set of sample paths to model the underlying stock behavior. The position in the stock and the amount of money invested in the bond (hedging variables) are modelled on nodes of a discrete grid in time and the stock price. Two matrices defining stock and bond positions on grid nodes completely determine the hedging portfolio on any price path of the underlying stock. Also, they determine amounts of money added to/taken from the portfolio at re-balancing points. The sum of squares of such additions/subtractions of money on a path is referred to as the squared error on a path.

The pricing problem is reduced to quadratic minimization with constraints. The objective is the averaged quadratic error over all sample paths; the free variables are stock and bond positions defined in every node of the grid. The constraints, limiting the feasible set of hedging strategies, restrict the portfolio values estimating the option price and stock positions. We required that the average of total external financing over all paths equals to zero. This makes the strategy "self-financing on average". We incorporated monotonic, convex, and some other properties of option prices following from the definition of an option, a non-arbitrage assumption, and some other general assumptions about the market. We do not make assumptions about the stock process which makes the algorithm distribution-free. Monotonicity and convexity constraints on the stock position are imposed. Such constraints reduce transaction costs, which are not accounted for directly in the model. We aim to prohibit sharp changes in stock and bond positions in response to small changes in the stock price or in time to maturity.

We performed two numerical tests of the algorithm. First, we priced options on the stock following the geometric Brownian motion. Stock price is modelled by Monte-Carlo sample-paths. Calculated option prices are compared with the known prices given by the Black-Scholes formula. Second, we priced options on S&P 500 Index and compared the results with actual market prices. Both numerical tests demonstrated reasonable accuracy of the pricing algorithm with a relatively small number of sample-paths (considered cases include 100 and 20 sample-paths). We calculated option prices both in dollars and in the implied volatility format. The implied volatility matches reasonably well the constant volatility for options in the Black-Scholes setting. The implied volatility for S&P 500 index options (priced with 100 sample-paths) tracks the actual market volatility smile.

The advantage of using the squared error as an objective can be seen from the practical perspective. Although we allow some external financing of the portfolio along the path, the minimization of the squared error ensures that large shortages of money will not occur at any point of time if the obtained hedging strategy is practically implemented.

Another advantage of using the squared error is that the algorithm produces a hedging strategy such that the sum of money added to/taken from the hedging portfolio on any path is close to zero. Also, the obtained hedging strategy requires zero average external financing over all paths. This justifies considering the initial value of the hedging portfolio as a price of an option. We use the notion of "a price of an option in the practical setting" which is the price a trader agrees to buy/sell the option. Therefore, we therefore claim that the initial value of the portfolio can be considered as an estimate of the option price.

We assume an incomplete market in this paper. We use the portfolio of two instruments – the underlying stock and a bond – to approximate the option price and consider many sample paths to model the stock price process. As a consequence, the value of the hedging portfolio may not be equal to the option payoff at expiration on some sample paths. Also, the algorithm is distribution-free, which makes it applicable to a wide range of underlying stock processes. Therefore, the algorithm can be used in the framework of an incomplete market.

Usefulness of our algorithm should be viewed from the perspective of practical options pricing. Commonly used methods of options pricing assume specific type of the underlying stock process. If the process is known, these methods provide accurate pricing. If the stock process cannot be clearly identified, the choice of the stock process and calibration of the process to fit market data may entail significant modelling error. Our algorithm is superior in this case. It is distribution-free and is based on realistic assumptions, such as discrete trading and non-self-financing hedging strategy.

Another advantage of our algorithm is a low back-testing error. Other models do not (directly) account for errors of back-testing on historical paths. Our algorithm can be set up to minimize the back-testing errors on historical paths (which can be considered as the main goal of modelling from practical perspective). Therefore, the algorithm may have a very attractive back-testing performance. This feature is not shared by other models.

The paper is organized as follows. Section 2 introduces the framework and describes the construction of the pricing algorithm. Section 3 present the formulation of the optimization problem and discusses the choices of the objective and the constraints. Section 4 discusses numerical tests of the method. Section 5 concludes the paper. Appendix 1 presents constraints for put options. Appendices 2 and 3 contain proofs of the inequalities considered in the paper.

2. Framework and Notations

2.1. PORTFOLIO DYNAMICS AND SQUARED ERROR

Consider a European option with time to maturity T and strike price X. We suppose that trading occurs at discrete times t_j, $j = 0, 1, \ldots, N$, such that

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_{j+1} - t_j = const, \quad j = 0, 1, \ldots, N-1.$$

We denote the position in the stock at time t_j by u_j, the amount of money invested in the bond by v_j, the risk-free rate by r, and the stock price at time t_j by S_j.

The price of the option at time t_j is approximated by the price c_j of a hedging portfolio consisting of the underlying stock and a risk-free bond. The hedging portfolio is rebalanced at times t_j, $j = 1, \ldots, N-1$. Suppose that at the time t_{j-1} the hedging portfolio consists of u_{j-1} shares of the stock and v_{k-1} dollars invested in the bond.[1] The value of the portfolio right before the time t_j is $u_{j-1}S_j + (1+r)v_{j-1}$. At time t_j the positions in the stock and in the bond are changed to u_j and v_j, respectively, and the portfolio value changes to $u_j S_j + v_j$. We consider a non-self-financing portfolio dynamics by allowing the difference

$$a_j = u_j S_j + v_j - (u_{j-1} S_j + (1+r)v_{j-1}) \tag{1}$$

to be non-zero. The value a_j is the excess/shortfall of the money in the hedging portfolio during the interval $[t_{j-1}, t_j]$. In other words, a_j is the amount of money added to (if $a_j \geq 0$) or subtracted from (if $a_j < 0$) the portfolio during the interval $[t_{j-1}, t_j]$. Thus, the inflow/outflow of money to/from the hedging portfolio is allowed.

We require that the value of the hedging portfolio at expiration be equal to the option payoff $h(S_N)$, $u_N S_N + v_N = h(S_N)$, where

$$h(s) = \begin{cases} \max\{0,\ s - x\} & \text{for call options;} \\ \max\{0,\ x - s\} & \text{for put options.} \end{cases}$$

The non-self-financing portfolio dynamics is given by

$$u_{j+1} S_{j+1} + v_{j+1} = u_j S_{j+1} + (1+r)v_j + a_j, \quad j = 0, \dots, N-1, \tag{2}$$

where the portfolio value at time t_j is $c_j = u_j S_j + v_j,\ j = 0, \dots, N$.

The degree to which a portfolio dynamics differs from a self-financing one is an important characteristic, essential to our approach. In this paper, we define *a squared error* on a path,

$$A = \sum_{j=1}^{N} (a_j e^{-rj})^2, \tag{3}$$

to measure the degree of "non-self-financity". The reasons for choosing this particular measure will be described later on.

2.2. HEDGING STRATEGY

We assume that the composition of the hedging portfolio depends on time and the stock price. We define a hedging strategy as a function determining the composition of a hedging portfolio for any given time and the stock price at that time. If the hedging strategy is defined, the corresponding portfolio management decisions for the stock price path $S_0, S_1, \dots, S_N$ are given by the sequence $(u_0, v_0), (u_1, v_1), \dots, (u_N, v_N)$.

A hedging strategy is modelled on a discrete grid with a set of approximation rules. Consider a grid consisting of nodes $\{(j, k);\ j = 0, \dots, N,\ k = 1, \dots, K\}$ in the time and the stock price. The index j denotes time and corresponds to time t_j; the index k denotes the stock price and corresponds to the stock price $\tilde{S}_k$ (we use the tilde sign for stock values on the grid to distinguish them from stock values on sample-paths). Stock prices $\tilde{S}_k$, $k = 1, \dots, K$ on the grid are equally distanced in the logarithmic scale, i.e.

$$\tilde{S}_1 < \tilde{S}_2 < \cdots < \tilde{S}_K, \ \ln(\tilde{S}_{k+1}) - \ln(\tilde{S}_k) = const.$$

Thus, the node (j, k) of the grid corresponds to time t_j and the stock price $\tilde{S}_k$. To every node (j, k) we assigned two variables U_j^k and V_j^k representing the composition of the hedging portfolio at time t_j with the stock price $\tilde{S}_k$. The pair of matrices

$$\left[U_j^k\right] = \begin{bmatrix} U_0^1 & U_1^1 & \dots & U_N^1 \\ U_0^2 & U_1^2 & \dots & U_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ U_0^K & U_1^K & \dots & U_N^K \end{bmatrix}, \quad \left[V_j^k\right] = \begin{bmatrix} V_0^1 & V_1^1 & \dots & V_N^1 \\ V_0^2 & V_1^2 & \dots & V_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ V_0^K & V_1^K & \dots & V_N^K \end{bmatrix} \tag{4}$$

are referred to as a *hedging strategy*. These matrices define portfolio management decisions on the discrete set of the grid nodes. In order to set those decisions on any path, not necessarily going through grid points, approximation rules are defined.

We model the stock price dynamics by a set of sample paths

$$\left\{\left(S_0, S_1^p, \dots, S_N^p\right) \middle| \ p = 1, \dots, P\right\} \tag{5}$$

where S_0 is the initial price. Let variables u_j^p and v_j^p define the composition of the hedging portfolio on path p at time t_j, where $p = 1, \dots, P$, $j = 0, \dots, N$. These variables are approximated by the grid variables U_j^k and V_j^k as follows. Suppose that $\{S_0, S_1^p, \dots, S_N^p\}$ is a realization of the stock price, where S_j^p denotes the price of the stock at time t_j on path p, $j = 0, \dots, N$, $p = 1, \dots, P$. Let u_j^p and v_j^p denote the amounts of the stock and the bond, respectively, held in the hedging portfolio at time t_j on path p. Variables u_j^p and v_j^p are linearly approximated by the grid variables U_j^k and V_j^k as follows

$$u_j^p = \alpha_j^p U_j^{k(j,p)+1} + \left(1 - \alpha_j^p\right) U_j^{k(j,p)}, \ v_j^p = \alpha_j^p V_j^{k(j,p)+1} + (1 - \alpha_j^p)\, V_j^{k(j,p)}, \tag{6}$$

where $\alpha_j^p = \frac{\ln S_j^p - \ln \tilde{S}_{k(j,p)}}{\ln \tilde{S}_{k(j,p)+1} - \ln \tilde{S}_{k(j,p)}}$, and $k(j, p)$ is such that $\tilde{S}_{k(j,p)} \le S_j^p < \tilde{S}_{k(j,p)+1}$.

According to (1), we define the excess/shortage of money in the hedging portfolio on path p at time t_j by

$$a_j^p = u_{j+1}^p S_{j+1}^p + v_{j+1}^p - \left(u_j^p S_{j+1}^p + (1 + r) v_j^p\right).$$

The squared error $\mathcal{E}_p$ on path p equals

$$\mathcal{E}_p = \sum_{j=1}^{N} \left(a_j^p e^{-rj}\right)^2. \tag{7}$$

We define the *average squared error* $\bar{\mathcal{E}}$ on the set of paths (5) as an average of squared errors $\mathcal{E}_p$ over all sample paths (5)

$$\bar{\mathcal{E}} = \frac{1}{P} \sum_{p=1}^{P} \sum_{j=1}^{N} \left(a_j^p e^{-rj} \right)^2. \tag{8}$$

The matrices $[U_j^k]$ and $[V_j^k]$ and the approximation rule (6) specify the composition of the hedging portfolio as a function of time and the stock price. For any given stock price path one can find the corresponding portfolio management decisions $\{(u_j, v_j) | j = 0, \ldots, N-1\}$, the value of the portfolio $c_j = S_j u_j + v_j$ at any time t_j, $j = 0, \ldots, N$, and the associated squared error.

The value of an option in question is assumed to be equal to the initial value of the hedging portfolio. First columns of matrices $[U_j^k]$ and $[V_j^k]$, namely the variables U_0^k and V_0^k, $k = 1, \ldots, K$, determine the initial value of the portfolio. If one of the initial grid nodes, for example node $(0, \tilde{k})$, corresponds to the stock price S_0, then the price of the option is given by $U_0^{\tilde{k}} S_0 + V_0^{\tilde{k}}$. If the initial point $(t = 0, S = S_0)$ of the stock process falls between the initial grid nodes $(0, k)$, $k = 1, \ldots, K$, then approximation formula (6) with $j = 0$ and $S_0^p = S_0$ is used to find the initial composition (u_0, v_0) of the portfolio. Then, the price of the option is found as $u_0 S_0 + v_0$.

3. Algorithm for Pricing Options

This section presents an algorithm for pricing European options in incomplete markets. Section 3.1, presents the formulation of the algorithm; Sections 3.2–3.4 discuss the choice of the objective and the constraints of the optimization problem.

3.1. OPTIMIZATION PROBLEM

The price of the option is found by solving the following minimization problem.

$$\min \ \bar{\mathcal{E}} = \frac{1}{P} \sum_{j=1}^{N} \sum_{p=1}^{P} \left(\left\{ u_j^p S_j^p + v_j^p - u_{j-1}^p S_j^p - (1+r) v_{j-1}^p \right\} e^{-rj} \right)^2 \tag{9}$$

subject to

$$\frac{1}{P} \sum_{j=1}^{N} \sum_{p=1}^{P} \left\{ u_j^p S_j^p + v_j^p - u_{j-1}^p S_j^p - (1+r) v_{j-1}^p \right\} e^{-rj} = 0$$

$$U_N^k \tilde{S}_k + V_N^k = h(\tilde{S}_k), \ k = 1, \ldots, K,$$

approximation rules (6); constraints (10)–(18) (defined below) for call options; or constraints (19)-(27) (defined in Appendix 1) for put options; free variables: $U_j^k,\ V_j^k,\ j = 0, \ldots, N,\ k = 1, \ldots, K$.

The objective function in (9) is the average squared error on the set of paths (5). The first constraint requires that the average value of total external financing over all paths equals to zero. The second constraint equates the value of the portfolio and the option payoff at expiration. Free variables in this problem are the grid variables U_j^k and V_j^k; the path variables u_j^p and v_j^p in the objective are expressed in terms of the grid variables using approximation (6). The total number of free variables in the problem is determined by the size of the grid and is independent of the number of sample-paths. After solving the optimization problem, the option value at time t_j for the stock price S_j is defined by $u_j S_j + v_j$, where u_j and v_j are found from matrices $[U_j^k]$ and $[V_j^k]$, respectively, using approximation rules (6). The price of the option is the initial value of the hedging portfolio, calculated as $u_0 S_0 + v_0$.

The following constraints (10)–(18) for call options or (19)–(27) for put options impose restrictions on the shape of the option value function and on the position in the stock. These restrictions reduce the feasible set of hedging strategies. Section 3.3 discusses the benefits of inclusion of these constraints in the optimization problem.

Below, we consider the constraints for European call options. The constraints for put options are included in Appendix 1. Proofs of the constraints are provided in Appendices 2 and 3. Most of the constraints are justified in a quite general setting. We assume non-arbitrage and make 5 additional assumptions provided in Appendix 2. Proofs of two constraints on the stock position (horizontal monotinicity and convexity) in the general setting will be addressed in subsequent papers. In this paper we validate these inequalities in the Black-Scholes case.

The notation C_j^k stands for the option value in the node (j, k) of the grid,

$$C_j^k = U_j^k \tilde{S}_j^k + V_j^k.$$

The strike price of the option is denoted by X, time to expiration by T, one period risk-free rate by r.

Constraints on call option value

- "*Immediate exercise constraints*". The value of an option is no less than the value of its immediate exercise[2] at the discounted strike price,

$$C_j^k \geq \left[\tilde{S}_j^k - X e^{-r(T-t_j)}\right]^+ . \tag{10}$$

- *Option price sensitivity constraints.*

$$C_j^{k+1} \leq \gamma_j^k C_j^k + X\left(\gamma_j^k - 1\right) e^{-r(T-t_j)},\ \gamma_j^k = \tilde{S}_j^{k+1}/\tilde{S}_j^k, \quad j = 0, \ldots, N-1,\ k = 1, \ldots, K-1. \tag{11}$$

This constraints bound sensitivity of an option price to changes in the stock price.

- *Monotonicity constraints.*

– *Vertical monotonicity.* For any fixed time, the price of an option is an increasing function of the stock price.

$$\frac{\tilde{S}_j^k}{\tilde{S}_j^{k+1}} C_j^{k+1} \geq C_j^k, \; j = 0, \ldots, N; \; k = 1, \ldots, K-1. \tag{12}$$

– *Horizontal monotonicity.* The price of an option is a decreasing function of time.

$$C_{j+1}^k \leq C_j^k, \; j = 0, \ldots, N-1; \; k = 1, \ldots, K. \tag{13}$$

- *Convexity constraints.* The option value is a convex function of the stock price.

$$\begin{gathered} C_j^{k+1} \leq \beta_j^{k+1} C_j^k + \left(1 - \beta_j^{k+1}\right) C_j^{k+2}, \\ \text{where } \beta_j^{k+1} \text{ is such that } \tilde{S}_j^{k+1} = \beta_j^{k+1} \tilde{S}_j^k + (1 - \beta_j^{k+1}) \tilde{S}_j^{k+2}, \\ j = 0, \ldots, N; k = 1, \ldots, K-2. \end{gathered} \tag{14}$$

Constraints on stock position for call options

Let us define $\hat{k}$, such that $\tilde{S}^{\hat{k}} \leq X < \tilde{S}^{\hat{k}+1}$.

- *Stock position bounds*. The stock position value lies between 0 and 1

$$0 \leq U_j^k \leq 1, \; j = 0, \ldots, N, \; k = 1, \ldots, K. \tag{15}$$

- *Vertical monotonicity.* The position in the stock is an increasing function of the stock price,

$$U_j^{k+1} \geq U_j^k, \; j = 0, \ldots, N; \; k = 1, \ldots, K-1. \tag{16}$$

- *Horizontal monotonicity.* Above the strike price the position in the stock is an increasing function of time; below the strike price it is a decreasing function of time,

$$U_j^k \leq U_{j+1}^k, \; \text{if } k > \hat{k}; U_j^k \geq U_{j+1}^k, \; \text{if } k \leq \hat{k}. \tag{17}$$

- *Convexity constraints.* The position in the stock is a concave function in the stock price above the strike and a convex function in the stock price below the strike,

$$\left(1-\beta_j^{k+1}\right) U_j^{k+2} + \beta_j^{k+1} U_j^k \le U_j^{k+1}, \text{ if } k > \hat{k},$$
$$\left(1-\beta_j^{k-1}\right) U_j^{k-2} + \beta_j^{k-1} U_j^k \ge U_j^{k-1}, \text{ if } k \le \hat{k}, \tag{18}$$

where β_j^l is such that $\tilde{S}_j^l = \beta_j^l \tilde{S}_j^{l-1} + \left(1-\beta_j^l\right) \tilde{S}_j^{l+1}, \ l = (k+1), (k-1).$

3.2. FINANCIAL INTERPRETATION OF THE OBJECTIVE

There are two reasons for considering the average squared error: financial interpretation and accounting for transaction costs. The financial interpretation is discussed here, while the accounting for transaction costs is considered in Section (3.4).

The expected hedging error is an estimate of "non-self-financity" of the hedging strategy. The pricing algorithm seeks a strategy as close as possible to a self-financing one, satisfying the imposed constraints. On the other hand, from a trader's viewpoint, the shortage of money at any portfolio re-balancing point causes the risk associated with the hedging strategy. The average squared error can be viewed as an estimator of this risk on the set of paths considered in the problem.

There are other ways to measure the risk associated with a hedging strategy. For example, Bertsimas et al. (2001) considers a self-financing dynamics of a hedging portfolio and minimizes the squared replication error at expiration. In the context of our framework, different estimators of risk can be used as objective functions in the optimization problem (9) and, therefore, produce different results. However, considering other objectives is beyond the scope of this paper.

3.3. CONSTRAINTS

We use the value of the hedging portfolio to approximate the value of the option. Therefore, the value of the portfolio is supposed to have the same properties as the value of the option. We incorporated these properties into the model using constraints in the optimization problem. The constraints (10)–(14) for call options and (19)–(23) for put options follow under quite general assumptions (see Appendix 2) from the non-arbitrage considerations. The type of the underlying stock price process plays no role in the approach: the set of sample paths (5) specifies the behavior of the underlying stock. For this reason, the approach is distribution-free and can be applied to pricing any European option independently of the properties of the underlying stock price process. Also, as shown in section 5 presenting numerical results, the inclusion of constraints to problem (9) makes the algorithm quite robust to the size of input data. The grid structure is convenient for imposing the constraints, since they can be stated as linear inequalities on the grid variables U_j^k and V_j^k. An important property of the algorithm is that the number of the variables in problem (9) is determined by the size of the grid and is independent of the number of sample paths.

3.4. TRANSACTION COSTS

The explicit consideration of transaction costs is beyond the scope of this paper. We postpone this issue to following papers. However, we implicitly account for transaction costs by requiring the hedging strategy to be "smooth", i.e., by prohibiting significant rebalancing of the portfolio during short periods of time or in response to small changes in the stock price. For call options, we impose the set of constraints (16)–(18) requiring monotonicity and concavity of the stock position with respect to the stock price and monotonicity of the stock position with respect to time (constraints (25)–(27) for put options are presented in Appendix 1). The goal is to limit the variability of the stock position with respect to time and stock price, which would lead to smaller transaction costs of implementing a hedging strategy. The minimization of the average squared error is another source of improving "smoothness" of a hedging strategy with respect to time. The average squared error penalizes all shortages/excesses a_j^p of money along the paths, which tends to flatten the values a_j^p over time. This also improves the "smoothness" of the stock positions with respect to time.

4. Case Study

This section present the results of two numerical tests of the algorithm. First, we price European options on the stock following the geometric Brownian motion and compare the results with prices obtained with the Black-Scholes formula. Second, we price European options on S&P 500 index (ticker SPX) and compare the results with actual market prices.

Tables I, III, and IV report "relative" values of strikes and option prices, i.e. strikes and prices divided by the initial stock price. Prices of options are also given in the implied volatility format, i.e., for actual and calculated prices we found the volatility implied by the Black-Scholes formula.

4.1. PRICING EUROPEAN OPTIONS ON THE STOCK FOLLOWING THE GEOMETRIC BROWNIAN MOTION

We used a Monte-Carlo simulation to create 200 sample paths of the stock process following the geometric brownian motion with drift 10% and volatility 20%. The initial stock price is set to \$ 62; time to maturity is 69 days. Calculations are made for 10 values of the strike price, varying from \$ 54 to \$ 71. The calculated results and Black-Scholes prices for European call options are presented in Table I.

Table 1 shows quite reasonable performance of the algorithm: the errors in the price (Err(%), Table I) are less than 2% for most of calculated put and call options.

Table I. Prices of options on the stock following the geometric brownian motion: calculated versus black-scholes prices

Strike	Calc.	B-S	Err(%)	Calc.Vol.(%)	B-S.Vol.(%)	Vol.Err(%)
			Call options			
1.145	0.0037	0.0038	−3.78	19.63	20.00	−1.86
1.113	0.0075	0.0074	1.35	19.91	20.00	−0.46
1.081	0.0134	0.0133	0.65	19.87	20.00	−0.65
1.048	0.0226	0.0227	−0.04	19.79	20.00	−1.04
1.016	0.0364	0.0361	0.80	19.94	20.00	−0.28
1.000	0.0446	0.0445	0.19	19.82	20.00	−0.92
0.968	0.0651	0.0648	0.47	19.94	20.00	−0.31
0.935	0.0891	0.0892	−0.08	19.59	20.00	−2.07
0.903	0.1166	0.1168	−0.11	19.29	20.00	−3.56
0.871	0.1464	0.1465	−0.07	18.71	20.00	−6.44
			Put options			
1.145	0.1274	0.1276	−0.16	19.73	20.00	−1.36
1.113	0.0995	0.0994	0.04	20.03	20.00	0.17
1.081	0.0738	0.0738	0.05	20.02	20.00	0.12
1.048	0.0514	0.0514	−0.10	19.97	20.00	−0.16
1.016	0.0334	0.0332	0.71	20.14	20.00	0.68
1.000	0.0258	0.0258	0.15	20.02	20.00	0.11
0.968	0.0147	0.0144	1.82	20.19	20.00	0.93
0.935	0.0070	0.0071	−1.60	19.89	20.00	−0.56
0.903	0.0029	0.0031	−5.77	19.71	20.00	−1.45
0.871	0.0010	0.0011	−12.88	19.52	20.00	−2.41

Initial price=$62, time to expiration=69 days, risk-free rate=10%, volatility=20%, 200 sample paths generated by Monte-Carlo simulation. Strike=option strike price(relative), Calc.=obtained option price (relative), BS=Black-Scholes option price (relative), Err=$(Found - BS)/BS$, Calc.Vol.=obtained option price in volatility form, BS.Vol.(%)=Black-Scholes volatility, Vol.Err(%)=$(Calc.Vol. - BS.Vol.)/BS.Vol.$

Also, it can be seen that the volatility is quite flat for both call and put options. The error of implied volatility does not exceed 2% for most call and put options (Vol.Err(%), Table I). The volatility error slightly increases for out-of-the-money puts and in-the-money calls.

4.2. PRICING EUROPEAN OPTIONS ON S&P 500 INDEX

The set of options used to test the algorithm is given in Table II. The actual market price of an option is assumed to be the average of its bid and ask prices. The price of the S&P 500 index was modelled by historical sample-paths. Non-overlapping

Table II. S&P 500 options data set.

Strike	Bid	Ask	Price	Rel.Pr	Strike	Bid	Ask	Price	Rel.Pr
		Call options					Put options		
1500	N/A	0.5	N/A	N/A	1500	311.3	313.3	312.3	0.2638
1325	0.3	0.5	0.4	0.0003	1300	112.7	114.7	113.7	0.0960
1300	0.45	0.8	0.625	0.0005	1275	88.8	90.8	89.8	0.0759
1275	1.15	1.65	1.4	0.0012	1225	46.9	48.9	47.9	0.0405
1250	3.7	4.2	3.95	0.0033	1210	36.9	38.9	37.9	0.0320
1225	8.6	9.6	9.1	0.0077	1200	31	33	32	0.0270
1210	13.2	14.8	14	0.0118	1190	26.1	28.1	27.1	0.0229
1200	17.5	18.9	18.2	0.0154	1175	19.8	21.4	20.6	0.0174
1190	22.1	24.1	23.1	0.0195	1150	12.5	14	13.25	0.0112
1175	30.8	32.8	31.8	0.0269	1125	8	9	8.5	0.0072
1150	48	50	49	0.0414	1100	5.1	5.9	5.5	0.0046
1125	68.3	69.5	68.9	0.0582	1075	3.3	4.1	3.7	0.0031
1100	90.2	92.2	91.2	0.0770	1050	2.2	3	2.6	0.0022
500	682.1	684.1	683.1	0.5771	1025	1.55	2.05	1.8	0.0015

Strike($)=option strike price, Bid($)=option bid price, Ask($)=option ask price, Price($)=option price (average of bid and ask prices), Rel.Pr=relative option price

paths of the index were taken from the historical data set and normalized such that all paths have the same initial price S_0. Then, the set of paths was "massaged" to change the spread of paths until the option with the closest to at-the-money strike is priced correctly. This set of paths with the adjusted volatility was used to price options with the remaining strikes.

Table III displays the results of pricing using 100 historical sample-paths. The pricing error (see Err(%), Table III is around 1.0% for all call and put options and increases for out-of-the-money options. Errors of implied volatility follow similar patterns: errors are of the order of 1% for all options except for deep out-of-the-money options. For deep in-the-money options the volatility error also slightly increases.

4.3. DISCUSSION OF RESULTS

Calculation results validate the algorithm. A very attractive feature of the algorithm is that it can be successfully applied to pricing options when a small number of sample–paths is available. (Table IV shows that in-the-money S&P 500 index options can be priced quite accurately with 20 sample-paths.) At the same time, the method is flexible enough to take advantage of specific features of historical sample-paths. When applied to S&P 500 index options, the algorithm was able to match the volatility smile reasonably well (Figure 1). At the same time, the implied volatility of options calculated in the Black-Scholes setting is reasonably flat

Table III. Pricing options on s&p 500 index: 100 paths

Strike	Calc.	Actual	Err (%)	Calc.Vol.(%)	Act.Vol.(%)	Vol.Err (%)
			Call options			
1.119	0.0002	0.0003	−40.00	13.17	14.14	−6.82
1.098	0.0005	0.0005	−5.28	12.80	12.92	−0.90
1.077	0.0013	0.0012	11.57	12.70	12.40	2.42
1.056	0.0035	0.0033	5.70	13.03	12.80	1.78
1.035	0.0079	0.0077	3.15	13.38	13.18	1.52
1.022	0.0117	0.0118	−0.75	13.43	13.49	−0.48
1.014	0.0156	0.0154	1.32	13.91	13.77	1.03
1.005	0.0195	0.0195	0.01	14.07	14.06	0.01
0.993	0.0269	0.0269	0.18	14.63	14.60	0.23
0.971	0.0416	0.0414	0.50	15.57	15.40	1.09
0.950	0.0589	0.0582	1.12	16.81	16.13	4.25
0.929	0.0775	0.0770	0.62	18.04	17.35	3.94
0.422	0.5789	0.5771	0.33	69.39	N/A	N/A
			Put options			
1.267	0.2633	0.2638	−0.20	22.50	29.02	−22.44
1.098	0.0956	0.0960	−0.47	13.88	15.14	−8.35
1.077	0.0756	0.0759	−0.36	13.71	14.18	−3.32
1.035	0.0406	0.0405	0.33	14.22	14.11	0.77
1.022	0.0319	0.0320	−0.25	14.29	14.35	−0.40
1.014	0.0274	0.0270	1.26	14.75	14.51	1.62
1.005	0.0229	0.0229	−0.01	14.89	14.90	−0.01
0.993	0.0176	0.0174	1.38	15.47	15.30	1.10
0.971	0.0111	0.0112	−0.52	16.43	16.47	−0.28
0.950	0.0070	0.0072	−1.95	17.58	17.72	−0.79
0.929	0.0045	0.0046	−3.42	18.84	19.05	−1.09
0.908	0.0028	0.0031	−10.00	20.02	20.57	−2.68
0.887	0.0015	0.0022	−32.27	20.46	22.24	−7.99
0.866	0.0011	0.0015	−26.00	22.46	23.78	−5.54

Initial price= 1183.77, time to expiration=49 days, risk-free rate=2.3%. Stock price is modelled with 100 sample paths. Grid dimensions: $P = 15$, $N = 49$. Strike=option strike price (relative), Calc.=calculated option price (relative), Actual=actual option price (relative), Err=(Calc.-Actual)/Actual, Calc.Vol.=calculated option price in volatility form, Act.Vol.(%)=actual option price in volatility terms, Vol.Err(%)=(Calc.Vol.-Act.Vol.)/Act.Vol.

(Figure 2). Therefore, one can conclude that the information causing the volatility smile is contained in the historical sample-paths. This observation is in accordance with the prior known fact that the non-normality of asset price distribution is one of causes of the volatility smile.

Figures 3 and 4 present distributions of total external financing ($\sum_{j=1}^{N} a_j^p e^{-rj}$) on sample paths and distributions of discounted money inflows/outflows ($a_j^p e^{-rj}$)

Table IV. Pricing options on S&P 500 index: 20 paths

Strike	Calc.	Actual	Err(%)	Calc.Vol.(%)	Act.Vol.(%)	Vol.Err(%)
			Call options			
1.119	0.0005	0.0003	45.00	14.95	14.14	5.78
1.098	0.0010	0.0005	88.80	14.48	12.92	12.09
1.077	0.0020	0.0012	66.86	13.95	12.40	12.50
1.056	0.0047	0.0033	41.80	14.39	12.80	12.38
1.035	0.0092	0.0077	19.84	14.43	13.18	9.42
1.022	0.0132	0.0118	11.41	14.47	13.49	7.26
1.014	0.0160	0.0154	4.03	14.20	13.77	3.13
1.005	0.0195	0.0195	0.00	14.06	14.06	0.00
0.993	0.0264	0.0269	−1.66	14.28	14.60	−2.15
0.971	0.0393	0.0414	−5.01	13.67	15.40	−11.23
0.950	0.0548	0.0582	−5.76	12.01	16.13	−25.52
0.929	0.0737	0.0770	−4.35	8.39	17.35	−51.65
0.422	0.5790	0.5771	0.34	N/A	N/A	N/A
			Put options			
1.267	0.2633	0.2638	−0.19	23.45	29.02	−19.16
1.098	0.0959	0.0960	−0.13	14.82	15.14	−2.11
1.077	0.0762	0.0759	0.40	14.67	14.18	3.45
1.035	0.0415	0.0405	2.49	14.92	14.11	5.72
1.022	0.0332	0.0320	3.69	15.20	14.35	5.93
1.014	0.0278	0.0270	2.74	15.03	14.51	3.54
1.005	0.0229	0.0229	0.01	14.90	14.90	0.01
0.993	0.0168	0.0174	−3.31	14.90	15.30	−2.63
0.971	0.0089	0.0112	−20.72	14.58	16.47	−11.48
0.950	0.0030	0.0072	−58.73	12.99	17.72	−26.73
0.929	0.0000	0.0046	−100.00	4.38	19.05	−77.00
0.908	0.0000	0.0031	−100.00	6.07	20.57	−70.50
0.887	0.0000	0.0022	−100.00	7.68	22.24	−65.48
0.866	0.0000	0.0015	−100.00	8.98	23.78	−62.21

Initial price= 1183.77, time to expiration=49 days, risk-free rate=2.3%. Stock price is modelled with 20 sample paths. Grid dimensions: $P = 15$, $N = 49$. Strike=option strike price (relative), Calc.=calculated option price (relative), Actual=actual option price (relative), Err=(Calc.-Actual)/Actual, Calc.Vol.=calculated option price in volatility form, Act.Vol.(%)=actual option price in volatility terms

at re-balancing points for Black-Scholes and SPX call options. We summarize statistical properties of these distributions in TableV.

Figures 3 and 4 also show that the obtained prices satisfy the non-arbitrage condition. With respect to pricing a single option, the non-arbitrage condition is understood in the following sense. If the initial value of the hedging portfolio is

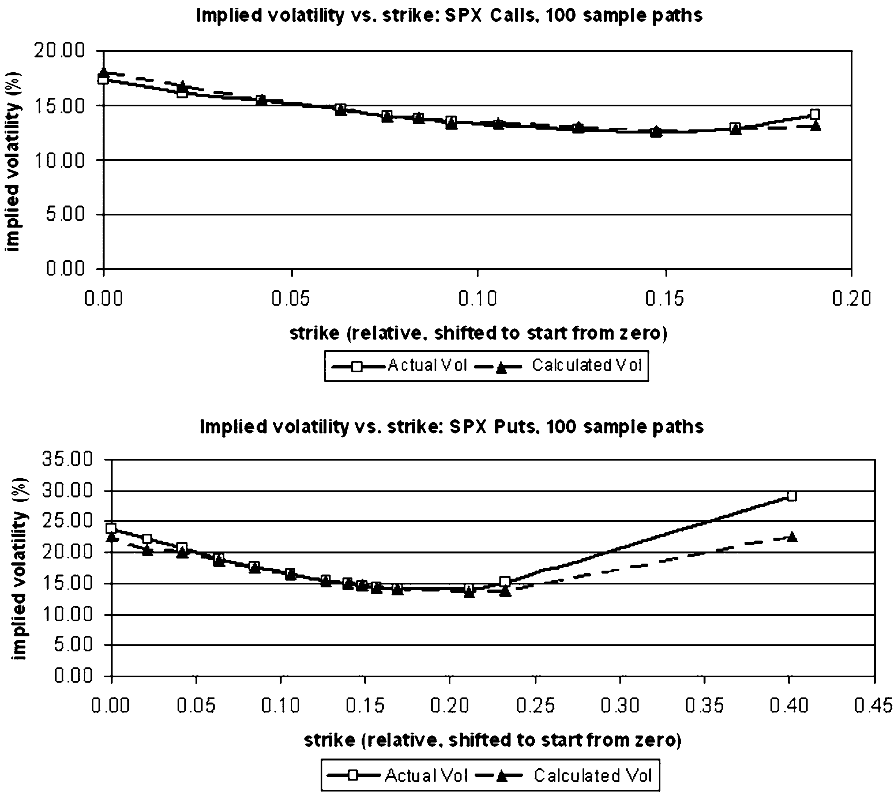


Figure 1. Tracking market volatility smile: Call and Put options on S&P 500 index. Based on prices in columns *Calc.Vol(%)* and *Act.Vol(%)* of Table III. Calculated Vol(%) = implied volatility of calculated options prices (100 sample-paths), Actual Vol(%) = implied volatility of market options prices, strike price is shifted left by the value of the lowest strike.

considered as a price of the option, then at expiration the corresponding hedging strategy should outperform the option payoff on some sample paths, and underperform the option payoff on some other sample paths. Otherwise, the free money can be obtained by shorting the option and buying the hedging portfolio or vise versa. The algorithm produces the price of the option satisfying the non-arbitrage condition in this sense. The value of external financing on average is equal to zero over all paths. The construction of the squared error implies that the hedging strategy delivers less money than the option payoff on some paths and more money that the option payoff on other paths. This ensures that the obtained price satisfies the non-arbitrage condition.

The pricing problem is reduced to quadratic programming, which is quite efficient from the computational standpoint. For the grid consisting of P rows (the stock price axis) and N columns (the time axis), the number of variables in the

Table V. Summary of cashflow distributions for obitained hedging strategies presented on Figures 3 and 4

	Black-Scholes Call		SPX Call	
	Total financing	Re-bal. cashflow	Total financing	Re-bal. cashflow
mean	0	0	0	0
st.dev.	0.6274	0.0449	16.1549	1.2730
median	0.0770	−0.0008	0.2695	−0.0314

Total financing ($) = the sum of discounted inflows/outflows of money on a path; Re-bal. cashflow ($) = discounted inflow/outflow of money on re-balancing points. Black-Scholes Call: Initial price = $62, strike = $62 time to expiration=70, risk-free rate = 10%, volatility = 20%. Stock price is modelled with 200 Monte-Carlo sample paths. SPX Call: Initial price = $1183.77, strike price = $1190 time to expiration = 49 days, risk-free rate = 2.3%. Stock price is modelled with 100 sample paths.

Table VI. Calculation times of the pricing algorithm: CPLEX 9.0 on Pentium 4, 1.7GHz, 1GB RAM

# of paths	P	N	Building time (sec)	CPLEX time (sec)	Total time (sec)
20	20	49	0.8	8.2	9
100	25	49	1.6	12.6	14.2
200	25	70	5.5	31.7	37.2

# of paths = number of sample-paths, P = vertical size of the grid, N = horizontal size of the grid, Building time = time of building the model (preprocessing time), CPLEX time = time of solving optimization problem, Total time = total time of pricing one option.

problem (9) is $2PN$ and the number of constraints is $O(NK)$, regardless of the number of sample paths. Table VI presents calculation times for different sizes of the grid with CPLEX 9.0 quadratic programming solver on Pentium 4, 1.7 GHz, 1GB RAM computer.

In order to compare our algorithm with existing pricing methods, we need to consider options pricing from the practical perspective. Pricing of actually traded options includes three steps.

1. **Choosing stock process and calibration.** The market data is analyzed and an appropriate stock process is selected to fit actually observed historical prices. The stock process is calibrated with currently observed market parameters (such as implied volatility) and historically observed parameters (such as historical volatility).
2. **Options pricing.** The calibrated stock process is used to price options. Analytical methods, lattices, dynamic programming, Monte-Carlo simulation, and other methods are used for pricing.

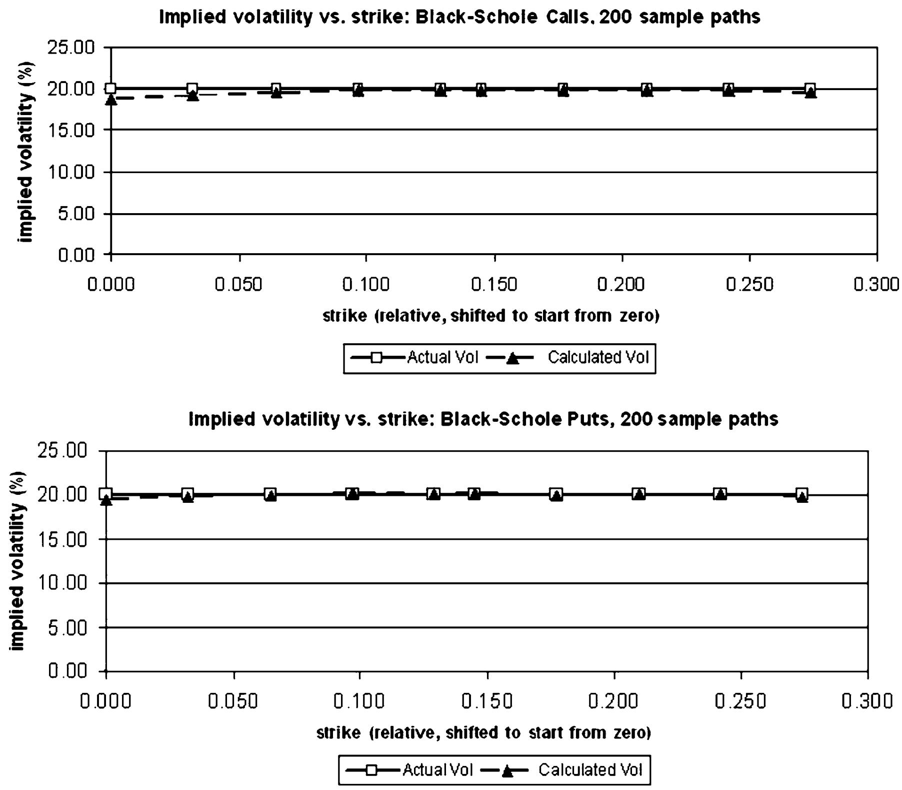


Figure 2. Flat volatility: Call and Put options in Black-Scholes setting. Based on prices in columns *Calc.Vol(%)* and *B-S.Vol(%)* of Table I. Calculated Vol(%) = implied volatility of calculated options prices (200 sample-paths), Actual Vol(%) = flat volatility implied by Black-Scholes formula, strike price is shifted left by the value of the lowest strike.

3. **Back-testing.** The model performance is verified on historical data. The hedging strategy, implied by the model, is implemented on historical paths.

Commonly used approaches for practical pricing of options are based on specific underlying stock processes (Black–Scholes model, stochastic volatility model, jump-diffusion model, etc). We will refer to these methods as process-specific methods. In order to judge the advantages of the proposed algorithm against the process-specific methods, we will compare them step by step.

Comparison at step 1. Choosing the model may entail modelling error. For example, stocks are approximately follow the geometric Brownian motion. However, the Black-Scholes prices of options would fail to reproduce the market volatility smile.

Our algorithm does not rely on some specific model and does not have errors related to the choice of the specific process. Also, we have realistic assumptions, such as discrete trading, non-self-financing hedging strategy, and possibility to introduce transaction costs (this feature is not directly presented in the paper).

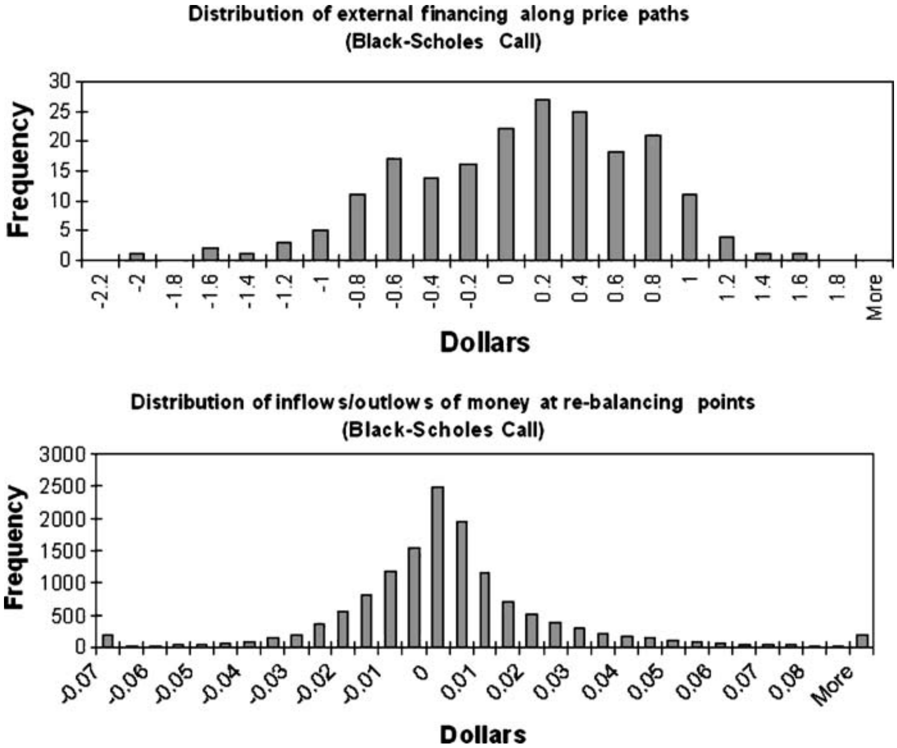
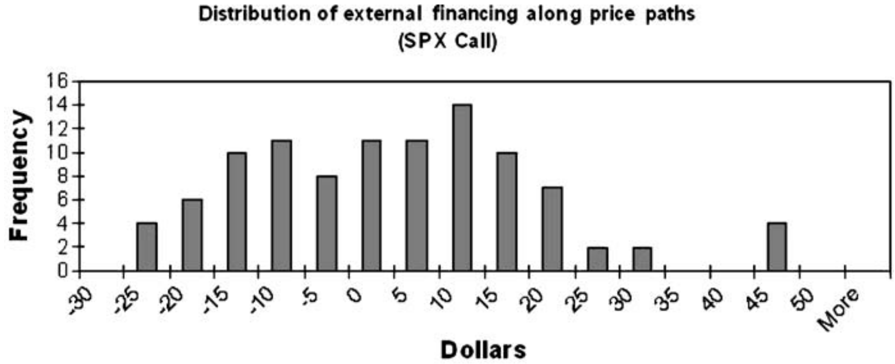


Figure 3. Black-Scholes call option: Distributions of the total external financing ($\sum_{j=1}^{N} a_j^p e^{-rj}$) on sample paths and discounted inflows/outflows ($a_j^p e^{-rj}$) at re-balancing points. Initial price = \$62, strike = \$62 time to expiration = 70, risk-free rate =10%, volatility = 20%. Stock price is modelled with 200 Monte-Carlo sample paths.

Calibration of process-specific methods usually require a small amount of market data. Our algorithm competes well in this respect. We impose constraints reducing feasible set of hedging strategies, which allows pricing with a small number of sample-paths.

Comparison at step 2. If the price process is identified correctly, the process-specific methods may provide an accurate pricing. Our algorithm may not have any advantages in such cases and may provide similar prices (by using sample-paths generated from the same process); see for instance numerical results for Black-Scholes setting in this paper.

Comparison at step 3. To perform back-testing, the hedging strategy, implied by a pricing method, is validated with historical price paths. The back-testing hedging error is a measure of practical usefulness of the algorithm.



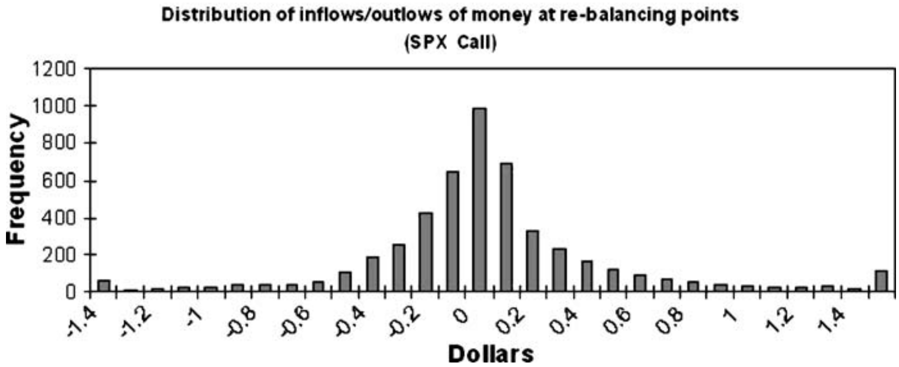


Figure 4. SPX call option: Distributions of the total external financing ($\sum_{j=1}^{N} a_j^p e^{-rj}$) on sample paths and discounted inflows/outflows ($a_j^p e^{-rj}$) at re-balancing points. Initial price = \$1183.77, strike price = \$1190 time to expiration = 49 days, risk-free rate = 2.3%. Stock price is modelled with 100 sample paths.

The major advantage of our algorithm is that the errors of back–testing in our case can be much lower than the errors of process-specific methods. The reason being, our algorithm can be set up to minimize the back–testing errors on historical paths (which can be considered as the main goal of modelling from practical perspective). Minimization of the squared error on historical paths ensures that the need of additional financing to practically hedge the option on historical paths is the lowest possible. None of the process-specific methods possess this property.

5. Conclusion and Future Research

We presented an approach to pricing European options in incomplete markets. The pricing problem is reduced to minimization of the expected quadratic error subject to constraints. To price an option we solve the quadratic programming problem and find a hedging strategy minimizing the risk associated with it. The hedging strategy is modelled by two matrices representing the stock and the bond positions in the portfolio depending upon time and the stock price. The constraints on the option value impose the properties of the option value following from general non–arbitrage considerations. The constraints on the stock position incorporate requirements on "smoothness" of the hedging strategy. We tested the approach with options on the stock following the geometric Brownian motion and with actual market prices for S&P 500 index options.

This paper is the first in the series of papers devoted to implementation of the developed algorithm to various types of options. Our target is pricing American-style and exotic options and treatment actual market conditions such as transaction costs, slippage of hedging positions, hedging options with multiple instruments and other issues. In this paper we established basics of the method; the subsequent papers will concentrate on more complex cases.

Appendix 1: Constraints for Put Options

This appendix presents constraints in optimization problem (9) for pricing European put options.

Constraints on value of Put options

- Immediate exercise" constraints

$$P_j^k \geq \left[Xe^{-r(T-t_j)} - \tilde{S}_j^k\right]^+. \tag{19}$$

- Option price sensitivity constraints

$$P_j^k \leq \gamma_j^k P_j^{k+1} + X\left(1-\gamma_j^k\right)e^{-r(T-t_j)},\ \gamma_j^k = \tilde{S}_j^k/\tilde{S}_j^{k+1},$$
$$j = 0, \ldots, N-1,\ k = 1, \ldots, K-1. \tag{20}$$

- *Monotonicity constraints*

  - *Vertical monotonicity*

$$P_j^k \geq P_j^{k+1},\ j = 0, \ldots, N;\ k = 1, \ldots, K-1. \tag{21}$$

  - *Horizontal monotonicity*

$$P_{j+1}^k \leq P_j^k + X\left(e^{-r(T-t_{j+1})} - e^{-r(T-t_j)}\right),$$
$$j = 0, \ldots, N-1;\ k = 1, \ldots, K. \tag{22}$$

- *Convexity constraints*

$$P_j^{k+1} \leq \beta_j^{k+1} P_j^k + \left(1 - \beta_j^{k+1}\right) P_j^{k+2}$$
$$\text{where } \beta_j^{k+1} \text{ is such that } S_j^{k+1} = \beta_j^{k+1} S_j^k + \left(1 - \beta_j^{k+1}\right) \tilde{S}_j^{k+2} \tag{23}$$
$$j = 0, \ldots, N;\ k = 1, \ldots, K - 2.$$

Constraints on stock position for put options

In the following constraints, $\hat{k}$ is such that $\tilde{S}^{\hat{k}} \leq X < \tilde{S}^{\hat{k}+1}$.

- **Stock Position Bounds**

$$0 \leq U_j^k \leq 1,\ \ j = 0, \ldots, N;\ k = 1, \ldots, K. \tag{24}$$

- **Vertical monotonicity**

$$U_j^{k+1} \geq U_j^k,\ \ j = 0, \ldots, N; k = 1, \ldots, K - 1. \tag{25}$$

- **Horizontal monotonicity**

$$U_j^k \leq U_{j+1}^k,\ \text{if } k > \hat{k}, U_j^k \geq U_{j+1}^k,\ \text{if } k \leq \hat{k} \tag{26}$$

- **Convexity constraints**

$$\left(1 - \beta_j^{k+1}\right) U_j^{k+2} + \beta_j^{k+1} U_j^k \leq U_j^{k+1},\ \text{if } k > \hat{k},$$
$$\left(1 - \beta_j^{k-1}\right) U_j^{k-2} + \beta_j^{k-1} U_j^k \geq U_j^{k-1},\ \text{if } k \leq \hat{k},$$
$$\text{where } \beta_j^l \text{ is such that } \tilde{S}_j^l = \beta_j^l \tilde{S}_j^{l-1} + (1 - \beta_j^l) \tilde{S}_j^{l+1}, \tag{27}$$
$$l = (k+1), (k-1).$$

Appendix 2: Justification of Constraints on Option Values

This appendix proves inequalities on put and call option values under certain assumptions. Properties of option values under various assumptions were thoroughly studied in financial literature. In optimization problem (9) we used the following constraints holding for options in quite a general case. We assume non-arbitrage and make technical assumptions 1–5 (used by Merton in (1973)) for deriving properties of call and put option values. Some of the considered properties of option values are proved by Merton (1973). Other inequalities are proved by the authors.

The rest of the appendix is organized as follows. First, we formulate and prove inequalities (10)–(14) for call options. Some of the considered properties of option values are not included in the constraints of the optimization problem (9), they are used in proofs of some of constraints (10)–(14). In particular, weak and strong scaling properties and two inequalities preceding proofs of option price sensitivity constraints and convexity constraints are not included in the set of constraints.

Second, we consider inequalities (19)–(23) for put options. We provide proofs of vertical and horizontal option price monotonicity; proofs of other inequalities are similar to those for call options.

We use the following notations. $C(S_t, T, X)$ and $P(S_t, T, X)$ denote prices of call and put options, respectively, with strike X, expiration T, when the stock price at time t is S_t. When appropriate, we use shorter notations C_t and P_t to refer to these options.

Similar to Merton (1973), we make the following assumptions to derive inequalities (10)–(14) and (19)–(23).

Assumption 1. Current and future interest rates are positive.
Assumption 2. No dividends are paid to a stock over the life of the option.
Assumption 3. Time homogeneity assumption.
Assumption 4. The distributions of the returns per dollar invested in a stock for any period of time is independent of the level of the stock price.
Assumption 5. If the returns per dollar on stocks i and j are identically distributed, then the following condition hold. If $S_i = S_j$, $T_i = T_j$, $X_i = X_j$; then $Claim_i(S_i, T_i, X_i) = Claim_j(S_j, T_j, X_j)$, where $Claim_i$ and $Claim_j$ are options (either call or put) on stocks i and j respectively.

Below are the proofs of inequalities (10)–(14).

1. *"Immediate exercise" constraints,* (Merton, 1973)

$$C_t \geq \left[S_t - X \cdot e^{-r\cdot(T-t)}\right]^+.$$

□ Put–Call parity, $C_t - P_t + X \cdot e^{-r\cdot(T-t)} = S_t$, and non-negativity of a put option price ($P_t \geq 0$) imply $C_t \geq S_t - Xe^{-r\cdot(T-t)}$. This inequality combined with $C_t \geq 0$ gives $C_t \geq Max(0, S_t - X \cdot e^{-r\cdot(T-t)}) = [S_t - X \cdot e^{-r\cdot(T-t)}]^+$. ■

2. *Scaling property*

(a) *Weak scaling property*, (Merton, 1973)

For any $k > 0$ consider two stock price processes $S(t)$ and $k \cdot S(t)$. For these processes, the following inequality is valid $C(k \cdot S_t, T, k \cdot X) = k \cdot C(S_t, T, X)$, where S_t is the value of the process $S(t)$ at time t.

□ At expiration T, the price of the first stock is S_T, the value of the second stock is $k \cdot S_T$. By definition, the values of call options written on the first stock (with strike X) and on the second stock (with strike $k \cdot X$) are $C(S_t, T, X) = Max[0, S_T - X]$ and $C(k \cdot S_t, T, k \cdot X) = Max[0, k \cdot S_T - k \cdot X]$, respectively.

From $Max[0, k \cdot S_T - k \cdot X] = k \cdot Max[0, S_T - X]$ and non-arbitrage considerations, it follows that $C(k \cdot S_t, T, k \cdot X) = k \cdot C(S_t, T, X)$. ■

(b) *Strong scaling property,* (Merton, 1973)
Under Assumptions 4 and 5, the call option price $C(S, T, X)$ is homogeneous of degree one in the stock price per share and exercise price. In other words, if $C(S, T, X)$ and $C(k \cdot S, T, k \cdot X)$ are option prices on stocks with initial prices S and $k \cdot S$ and strikes X and $k \cdot X$, respectively, then $C(k \cdot S, T, k \cdot X) = k \cdot C(S, T, X)$.

□ Consider two stocks with initial prices S_1 and S_2; define $k = S_2/S_1$. Let $z_i(t)$ be the return per dollar for stock i, $i = 1, 2$. Consider two call options, A and B, on stock 2. Option A is written on $1/k$ shares of stock 2 and has strike price X_1; option B is written on one share of stock 2 and has strike $X_2 = k \cdot X_1$. Prices $C_2(S_1, T, X_1)$ and $C_2(S_2, T, X_2)$ of these options are related as $C_2(S_2, T, X_2) = C_2(k \cdot S_1, T, k \cdot X_1) = k \cdot C_2(S_1, T, X_1)$, according to the weak scaling property.

Now consider an option C with the strike X_1 written on one share of the stock 1. Denote its price by $C_1(S_1, T, X_1)$. Options A and C have equal initial prices $S_1 = \frac{1}{k} S_2$, time to expiration T, and X_1. Moreover, the distribution of returns per dollar $z_i(t)$ for stocks $i = 1, 2$ are the same. Hence, from assumption 5, $C_1(S_1, T, X_1) = C_2(S_1, T, X_1)$, and, therefore, $C_2(S_2, T, X_2) = k \cdot C_1(S_1, T, X_1)$, which concludes the proof. ■

3. *Option price sensitivity constraints*

(a) First, we derive an inequality taken from Merton (1973). In part b) we apply it to obtain the sensitivity constraint on the call option price.

For any X_1, X_2 such that $0 \leq X_1 \leq X_2$, the following inequality holds

$$C(S_t, T, X_1) \leq C(S_t, T, X_2) + (X_2 - X_1) \cdot e^{-r \cdot (T-t)}.$$

□ Consider two portfolios. Portfolio A contains one call option with strike X_2 and $(X_2 - X_1) \cdot e^{-r \cdot (T-t)}$ dollars invested in bonds. Portfolio B consists of one call option with strike X_1. Both options are written on the stock following the process S_t. At expiration, the value of portfolio A is $\max\{0, S_T - X_2\} + X_2 - X_1$, the value of portfolio B is $\max\{0, S_T - X_1\}$. The value of portfolio A is always greater than the value of portfolio B at expiration. This statement with non-arbitrage considerations implies that $C(S_t, T, X_2) + (X_2 - X_1) \cdot e^{r \cdot (T-t)} \geq C(S_t, T, X_1)$. ■

(b) Consider two options with strike X and initial prices S_2 and S_1, $S_2 \geq S_1$. Denote $\gamma = S_2/S_1$. The following inequality takes place,

$$C(S_2, T, X) \leq \gamma C(S_1, T, X) + X(\gamma - 1) e^{-r(T-t)}.$$

□ Let $\alpha = \frac{1}{\gamma} = \frac{S_1}{S_2}$. Using inequality presented in a), we write $C(S_1, T, \alpha X) \leq$

$C(S_1, T, X) + (X - \alpha X)e^{-r(T-t)}$. Applying scaling property to the left-hand side of this inequality yields $C(S_1, T, \alpha X) = C(S_1 \frac{S_2}{S_1} \frac{S_1}{S_2}, T, \alpha X) = C(S_2 \cdot \alpha, T, \alpha X) = \alpha C(S_2, T, X)$. Therefore,

$$\alpha C(S_2, T, X) \leq C(S_1, T, X) + X(1 - \alpha)e^{-r(T-t)}.$$

Dividing by α and substituting $1/\alpha = \gamma$ we get $C(S_2, T, X) \leq \gamma C(S_1, T, X) + X(\gamma - 1)e^{-r(T-t)}$. ■

4. *Vertical option price monotonicity*

For two options with strike X and initial prices S_1 and S_2, $S_2 \geq S_1$, there holds

$$C(S_1, T, X) \leq \frac{S_1}{S_2} \cdot C(S_2, T, X).$$

□ For any strike $X_1 \leq X$, from non-arbitrage assumptions we have $C(S_1, T, X) \leq C(S_1, T, X_1)$. Applying scaling property to the right-hand side gives$C(S_1, T, X) \leq \frac{X_1}{X} C(S_1 \frac{X}{X_1}, T, X)$. By setting $X_1 = \frac{S_1}{S_2} X \leq X$, we get $C(S_1, T, X) \leq \frac{S_1}{S_2} \cdot C(S_2, T, X)$. ■

5. *Horizontal option price monotonicity*

Let $C(t, S, T, X)$ denote the price of a European call option with initial time t, initial price at time t equal to S, time to maturity T, and strike X. Under the assumptions 1, 2 and 3 for any $t, u, t < u$, the following inequality holds,

$$C(t, S, T, X) \geq C(u, S, T, X).$$

□ Similar to $C(t, S, T, X)$, define $A(t, S, T, X)$ to be the value of American call option with parameters t, S, T, and X meaning the same as in $C(t, S, T, X)$. Time homogeneity Assumption 2 implies that two options with different initial times, but equal initial and strike prices and times to maturity should have equal prices: $A(t, S, T, X) = A(u, S, T + u - t, X)$. On the other hand, non-arbitrage considerations imply $A(u, S, T + u - t, X) \geq A(u, S, T, X)$. Combining the two inequalities yields $A(t, S, T, X) \geq A(u, S, T, X)$. Since the value of an American call option is equal to the value of the European call option under Assumption 1, the above inequality also holds for European options: $C(t, S, T, X) \geq C(u, S, T, X)$.■

6. *Convexity* (Merton, 1973)

(a) C is a convex function of its exercise price: for any $X_1 > 0$, $X_2 > 0$ and $\lambda \in [0, 1]$

$$C(S, T, \lambda \cdot X_1 + (1 - \lambda) \cdot X_2) \leq \lambda \cdot C(S, T, X_1) + (1 - \lambda) \cdot C(S, T, X_2).$$

□ Consider two portfolios. Portfolio A consists of λ options with strike X_1 and $(1-\lambda)$ options with strike X_2; portfolio B consists of one option with strike $\lambda \cdot X_1 + (1-\lambda) \cdot X_2$. Convexity of function $\max\{0, x\}$ implies that the value of portfolio A at expiration in no less than the value of portfolio B at expiration. $\lambda \max\{0, S_T - X_1\} + (1-\lambda)\max\{0, S_T - X_2\} \geq \max\{0, S_T - (\lambda \cdot X_1 + (1-\lambda) \cdot X_2)\}$. Hence, from non-arbitrage assumptions, portfolio A costs no less than portfolio B: $\lambda \cdot C(S, T, X_1) + (1-\lambda) \cdot C(S, T, X_2) \geq C(S, T, \lambda \cdot X_1 + (1-\lambda) \cdot X_2)$. ■

(b) Under the Assumption 4, option price $C(S, T, X)$ is a convex function of the stock price: for any $S_1 > 0$, $S_2 > 0$ and $\lambda \in [0, 1]$ there holds,

$$C(\lambda \cdot S_1 + (1-\lambda) \cdot S_2, T, X) \leq \lambda \cdot C(S_1, T, X) + (1-\lambda) \cdot C(S_2, T, X).$$

□ Denote $S_3 = \lambda S_1 + (1-\lambda)S_2$. Choose X_1, X_2 and α such that $X_1 = X/S_1$, $X_2 = X/S_2$, $\alpha = \lambda S_1/S_3 \in [0, 1]$, and denote $X_3 = \alpha X_1 + (1-\alpha)X_2$.

Consider an inequality $C(1, T, X_3) \leq \alpha \cdot C(1, T, X_1) + (1-\alpha) \cdot C(1, T, X_2)$ following from convexity of option price with respect to the strike price (proved in a)). Since

$$\alpha S_3 = \lambda S_1,\ (1-\alpha)S_3 = \left(1 - \frac{\lambda S_1}{S_3}\right) S_3 = S_3 - \lambda S_1 = (1-\lambda)S_2, \tag{28}$$

multiplying both sides of the previous inequality by S_3 gives $S_3 \cdot C(1, T, X_3) \leq \lambda \cdot S_1 \cdot C(1, T, X_1) + (1-\lambda) \cdot S_2 \cdot C(1, T, X_2)$. Further, using the weak scaling property, we get $C(S_3, T, S_3 \cdot X_3) \leq \lambda \cdot C(S_1, T, S_1 \cdot X_1) + (1-\lambda) \cdot C(S_2, T, S_2 \cdot X_2)$. Using definitions of X_1 and X_2 and expanding S_3X_3 as

$$\begin{aligned} S_3(\alpha X_1 + (1-\alpha)X_2) &= S_3 X\left(\frac{\alpha}{S_1} + \frac{1-\alpha}{S_2}\right) \\ &= S_3 X\left(\frac{\lambda S_1}{S_3}\frac{1}{S_1} + \frac{S_3 - \lambda S_1}{S_3}\frac{1}{S_2}\right) = S_3 X\left(\frac{\lambda}{S_3} + \frac{1-\lambda}{S_3}\right) = X, \end{aligned}$$

we arrive at $C(S_3, T, X) \leq \lambda \cdot C(S_1, T, X) + (1-\lambda) \cdot C(S_2, T, X)$, as needed. ■

Constraints on European put option values are presented below. We state them in the same order as the constraints for call options. Proofs are given for vertical option price monotonicity constraints; other inequalities can be proved using put-call parity and considerations similar to those in the proofs of corresponding inequalities for call options.

1. *"Immediate exercise" constraints*

$$P_t \geq \left[X \cdot e^{-r\cdot(T-t)} - S_t\right]^+.$$

2. *Scaling property*

(a) *Weak scaling property*

For any $k>0$, consider two stock price processes $S(t)$ and $k \cdot S(t)$. For these processes the following inequality holds: $P_1(k \cdot S_t, T, k \cdot X) = k \cdot P_2(S_t, T, X)$, where P_1 and P_2 are options on the first and the second stocks respectively.

(b) *Strong scaling property*

Under the Assumptions 4 and 5, put option value $P(S, T, X)$ is homogeneous of degree one in the stock price and the strike price, i.e., for any $k > 0$, $P(k \cdot S, T, k \cdot X) = k \cdot P(S, T, X)$.

3. *Option price sensitivity constraints*

(a) For any $X_1, X_2, 0 \leq X_1 \leq X_2$, the following inequality is valid,

$$P(S_t, T, X_2) \leq P(S_t, T, X_1) + (X_2 - X_1) \cdot e^{r \cdot (T-t)}.$$

(b) For initial stock prices S_1 and S_2, $S_1 \leq S_2$

$$P(S_1, T, X) \leq \gamma P(S_2, T, X) + X(1 - \gamma)e^{-r(T-t)},$$

where $\gamma = S_1/S_2$.

4. *Vertical option price monotonicity*

(a) For any $\alpha \in [0, 1]$ the following inequality is valid:

$$P(S, X \cdot \alpha) \leq \alpha \cdot P(S, X).$$

□ Consider portfolio A consisting of one option with strike $\alpha \cdot X$, and portfolio B consisting of α options with strike X. We need to show that portfolio B always outperforms portfolio A. This follows from non-arbitrage consideration since at expiration the value of portfolio B is greater or equal to the value of portfolio A: $[X \cdot \alpha - S_T]^+ \leq \alpha \cdot [X - S_T]^+,\ 0 < \alpha < 1.$ ■

(b) For any $S_1, S_2, S_1 \leq S_2$, there holds $P(S_2, T, X) \leq P(S_1, T, X)$.

□ Consider an inequality $P(S_1, \alpha X) \leq \alpha P(S_1, X),\ 0 < \alpha < 1$, proved above. Set $\alpha = S_1/S_2 \in [0, 1]$. Applying the weak scaling property, we get

$$P\left(S_1 \frac{1}{\alpha} \alpha, T, \alpha X\right) \leq \alpha P(S_1, T, X),$$
$$P\left(S_1 \frac{1}{\alpha}, T, X\right) \leq P(S_1, T, X),$$
$$P(S_2, T, X) \leq P(S_1, T, X). \blacksquare$$

5. *Horizontal option price monotonicity*

Under Assumptions 1, 2, and 3, for any initial times t and u, $t < u$, the following inequality is valid:

$$P(t, S, T, X) \geq P(u, S, T, X) + X \cdot \left(e^{-r\cdot(T-t)} - e^{-r\cdot(T-u)}\right),$$

where $P(\tau, S, T, X)$ is the price of a European put option with initial price τ, initial price at time τ equal to S, time to maturity T, and strike X.

6. *Convexity*
   (a) $P(S, T, X)$ is a convex function of its exercise price X
   (b) Under Assumption 4, $P(S, T, X)$ is a convex function of the stock price.

Appendix 3: Justification of Constraints on Stock Position

This appendix proves/validates inequalities (15)–(18) and (24)–(27) on the stock position. Stock position bounds and vertical monotonicity are proven in the general case (i.e. under Assumptions 1–5 of Appendix 2 and the non-arbitrage assumption); horizontal monotonicity and convexity are justified under the assumption that the stock process follows the geometric Brownian motion.

The notation $C(S, T, X)$ ($P(S, T, X)$) stands for the price of a call (put) option with the initial price S, time to expiration T, and the strike price X. The corresponding position in the stock (for both call and put options) is denoted by $U(S, T, X)$.

First, we present the proofs of inequalities (15)–(18) for call options.

1. *Vertical monotonicity (Call options)*
   $U(S, t, X)$ is an increasing function of S.

□ This property immediately follows from convexity of the call option price with respect to the stock price, proved in Appendix 2 (property 6(b) for call options). ■

2. *Stock position bounds (Call options)*

$$0 \leq U(S, T, X) \leq 1$$

□ Since the option price $C(S, t, X)$ is an increasing function of the stock price S, it follows that $U(S, t, X) = C'_s(S, t, X) \geq 0$.
Now we need to prove that $U(S, t, X) \leq 1$. We will assume that there exists such S^* that $C'_s(S^*) \geq \alpha$ for some $\alpha > 1$ and will show that this assumption contradicts the ineqiality[3] $C(S, t, X) \leq S$.

Since $U(S, t, X)$ increases with S, for any $S \geq S^*$ we have $U(S, t, X) \geq \alpha$, $\int_{S^*}^{S} U(s, t, X)ds \geq \int_{S^*}^{S} \alpha ds$, $C(S, t, X) - C(S^*, t, X) \geq \alpha S - \alpha S^*$, $C(S, T, x) \geq C(S^*, t, X) - \alpha S^* + \alpha S$, $C(S, t, X) \geq S + (\alpha - 1)S + C(S^*, t, X) - \alpha S^*$.
Let $f(s) = (\alpha - 1)s + C(S^*) - \alpha S^*$. Since $(\alpha - 1) > 0$, there exists such $S_1 > S^*$

that $f(S_1) > 0$. This implies $C(S_1, t, X) > S_1$ which contradicts inequality $C(S, t, X) \leq S$. ■

The previous inequalities were justified in a quite general setting of Assumptions 1–5 (see Appendix 2) and a non-arbitrage assumption. We did not manage to prove the following two groups of inequalities (horizontal monotonicity and convexity) in this general setting. The proofs will be provided in further papers. However, here we present proofs of these inequalities in the Black-Scholes setting.

3. *Horizontal monotonicity (Call options)*
$U(S, t, X)$ is an increasing function of t when $S \geq X$,
$U(S, t, X)$ is a decreasing function of t when $S < X$.
□ We will validate these inequalities by analyzing the Black-Scholes formula and calculating the areas of horizontal monotonicities for the options used in the case study. The Black-Scholes formula for the price of a call option is

$$C(S, T, X) = S\,N(d_1) - Xe^{rT}N(-d_2),$$

where S is the stock price, T is time to maturity, r is a risk-free rate, σ is the volatility,

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{Z^2}{2}}\,dZ, \tag{29}$$

and d_1 and d_2 are given by expressions

$$d_1 = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{Se^{rT}}{X}\right) + \frac{1}{2}\sigma\sqrt{T},$$
$$d_2 = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{Se^{rT}}{X}\right) - \frac{1}{2}\sigma\sqrt{T}.$$

Taking partial derivatives of $C(S, T, X)$ with respect to S and t, we obtain

$$C_s{}'(S, T, X) = U(S, T, X) = (S, T, X) = N(d_1),$$
$$C_{st}{}''(S, T, X) = U_t{}'(S, T, X)$$
$$= \frac{\exp\left\{-\frac{\left(T\left(r+\frac{\sigma^2}{2}\right)+\ln\left(\frac{S}{X}\right)\right)^2}{2T\sigma^2}\right\}\left(-T(2r+\sigma^2)+2\ln\left(\frac{S}{X}\right)\right)}{4\sqrt{2\pi}\,T^{\frac{3}{2}}\sigma}.$$

The sign of $U_t{}'(S, T, X)$ is determined by the sign of the expression $F(S) = -T(2r + \sigma^2) + 2\ln(\frac{S}{X})$. $F(S) \geq 0$ (implying $U_t{}'(S, T, X) \geq 0$) when $S \geq L$ and $F(S) \leq 0$ (implying $U_t{}'(S, T, X) \leq 0$) when $S \leq L$, where $L = X \cdot e^{T(r+\sigma^2/2)}$.

Table VII. Numerical values of inflexion points of the stock position as a function of the stock price for some options

Expir. (days)	Strike ($)	Inflexion ($)	Error (%)
0	62	60.126	3.02
35	62	61.056	1.52
69	62	61.975	0.04
0	54	52.368	3.02
35	54	53.178	1.52
69	54	53.974	0.05
0	71	68.855	3.02
35	71	69.919	1.52
69	71	70.967	0.05

Expir.(days) = time to expiration, Strike($) = strike price of the option, Inflexion($) = inflexion point, Error(%) = (Strike-Inflexion)/Strike.

For the values of $r = 10\%$, $\sigma = 31\%$, $T = 49$ days L differs from X less than 2.5%. For all options considered in the case study the value of implied volatility did not exceed 31% and the corresponding value of L differs from the stike price less than 2.5%. Taking into account resolution of the grid, we consider the approximation of L by X in the horizonal monotonicity constraints to be reasonable. ■

4. *Convexity (Call options)*

$U(S, t, X)$ is a concave function of S when $S \geq X$,
$U(S, t, X)$ is a convex function of S when $S < X$.
□ We used MATHEMATICA to find the second derivative of the Black-Scholes option price with respect to the stock price ($U_{SS}''(S, t, X)$). The expression of the second derivative is quite involved and we do not present it here. It can be seen that $U_{SS}''(S, t, X)$ as a function of S has an inflexion point. Above this point $U(S, t, X)$ is concave with respect to S and below this point $U(S, t, X)$ is convex with respect to S. We calculated inflexion points for some options and presented the results in the Table VII.

The Error(%) column contains errors of approximating inflexion points by strike prices. These errors do not exceed 3% for a broad range of parameters. We conclude that inflexion points can be approximated by strike prices for options considered in the case study. ■

Next, we justify the constraints (24)–(27) for put options.

1. *Vertical monotonicity (Put options)*

$U(S, t, X)$ is an increasing function of S.

□ This property immediately follows from convexity of the put option price with respect to the stock price, proved in Appendix 2 (property 6(b) for put options). ■

2. *Stock position bounds (Put options)*

$$-1 \leq U(S, T, X) \leq 0$$

□ Taking derivative of the put-call parity $C(S, T, X) - P(S, T, X) + X \cdot e^{-rT} = S$ with respect to the stock price S yields $C_s{}'(S, T, X) - P_s{}'(S, T, X) = 1$. This equality together with $0 \leq C_s{}'(S, T, X) \leq 1$ implies $-1 \leq P_s{}'(S, T, X) \leq 0$, which concludes the proof. ■

3. *Horizontal monotonicity (Put options)*
$U(S, t, X)$ is an increasing function of t when $S \geq X$,
$U(S, t, X)$ is a decreasing function of t when $S < X$.
□ Taking the derivatives with respect to S and T of the put-call parity yields $C_{st}{}''(S, T, X) = P_{st}{}''(S, T, X)$. Therefore, the horizontal monotonic properties of $U(S, T, X)$ for put options are the same as the ones for call options. ■

4. *Convexity (Put options)*
$U(S, t, X)$ is a concave function of S when $S \geq X$,
$U(S, t, X)$ is a convex function of S when $S < X$.
□ Put-call parity implies that $C_{SS}{}''(S, T, X) = P_{SS}{}''(S, T, X)$. Therefore, the convexity of put options is the same as the convexity of call options. ■

NOTES

1. Below, the number of shares of the stock and the amount of money invested in the bond are referred to as positions in the stock and in the bond.
2. European options do not have the feature of immediate exercise. However, the right part of constraint (10) coincides with the immediate exercise value of an American option having the current stock price $\tilde{S}_j^k$ and the strike price $Xe^{-r(T-t_j)}$.
3. This inequality can be proven by considering a portfolio consisting of one stock and one shorted call option on this stock. At expiration, the portfolio value is $S_T - \max\{0, S_T - X\} \geq 0$ for any S_T and $X \geq 0$. Non-arbitrage assumption implies that $S \geq C(S, t, X)$.

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