# On Relation Betweeen Expected Regret and Conditional Value-at-Risk

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**Abstract**

The paper compares portfolio optimization approaches with expected regret and Conditional Value-at-Risk (CVaR) performance functions. The expected regret is defined as an average portfolio underperformance comparing to a fixed target or some benchmark portfolio. For continuous distributions, CVaR is defined as the expected loss exceeding $\alpha$-Value-at Risk (VaR), i.e., the mean of the worst (1-$\alpha$)100% losses in a specified time period. However, generally, CVaR is the weighted average of VaR and losses exceeding VaR. Optimization of CVaR can be performed using linear programming. We formally prove that a portfolio with a continuous loss distribution, which minimizes CVaR, can be obtained by doing a line search with respect to the threshold in the expected regret. An optimal portfolio in CVaR sense is also optimal in the expected regret sense for some threshold in the regret function. The inverse statement is also valid, i.e., if a portfolio minimizes the expected regret, this portfolio can be found by doing a line search with respect to the CVaR confidence level. A portfolio, optimal in expected regret sense, is also optimal in CVaR sense for some confidence level. The relation of the expected regret and CVaR minimization approaches is explained with a numerical example.

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# 1 Introduction

Modern portfolio optimization theory was originated by Markowitz (1952), who demonstrated that quadratic programming can be used for constructing efficient portfolios. Relatively recently, linear programming techniques, which have superior performance compared to quadratic programming, became popular in finance applications: the mean absolute deviation approach, Konno and Yamazaki (1991), the regret optimization approach, Dembo and King (1992), Dembo and Rosen (1999), and the minimax approach, Young (1998). A reader interested in applications of optimization techniques in finance can find many relevant papers in Ziemba and Mulvey (1998) and in Zenios (1996).

The *expected regret* (see, Dembo and King (1992), Dembo and Rosen (1999)) which is also called the *low partial moment* (see, Harlow (1991)) is defined as the average portfolio underperformance compared to a fixed target or some benchmark portfolio. Although both terms are quite popular, in this paper we use the term "expected regret". A similar concept to the expected regret was utilized by Cariño and Ziemba (1998) in the Russell-Yasuda Kasai financial planning model. In this application, several target thresholds were used and portfolio underperformance was penalized with different coefficients for various thresholds. Probabilistic performance measures similar to the expected regret, such as the conditional expectation constraints and integrated chance constraints described in Prekopa (1995) have been successfully used in various engineering applications outside of financial context. High numerical efficiency of the expected regret approach in portfolio optimization is related to using state-of-the-art linear programming techniques.

This paper establishes relation of the expected regret with *Conditional Value-at-Risk* (CVaR) risk measure. The CVaR risk measure is closely related to *Value-at-Risk* (VaR) performance measure, which is the percentile of the loss distribution. A description of various methodologies for the modeling of VaR can be seen, along with related resources, at URL http://www.gloriamundi.org/. The term Conditional Value-at-Risk was introduced by Rockafellar and Uryasev (2000). For continuous distributions, CVaR is defined as the conditional expected loss under the condition that it exceeds VaR, see Rockafellar and Uryasev (2000). For continuous distributions, this risk measure also is known as Mean Excess Loss, Mean Shortfall, or Tail Value-at-Risk. For continuous distributions, Hurliman (2001) presents ten equivalent definitions of CVaR which were used in different forms in reliability, actuarial science, finance and economics. However, for general distributions, including discrete distributions, CVaR has been defined only recently by Rockafellar and Uryasev (2002) as a weighted average of VaR and losses strictly exceeding VaR. Also, Acerbi et al. (2001), Acerbi and Tasche (2001) redefined expected shortfall similar to CVaR. For general distributions, CVaR, which is a quite similar to the VaR measure of risk has more attractive properties than VaR. CVaR is sub-additive and convex for general distributions, see Rockafellar and Uryasev (2002). Moreover, CVaR is a *coherent* measure of risk in the sense of Artzner et al. (1997). Coherency of CVaR for general distributions was first proved by Pflug (2000); see also Rockafellar and Uryasev (2002), Acerbi et al. (2001), Acerbi and Tasche (2001).

Rockafellar and Uryasev (2000,2002) demonstrated that optimization of CVaR can be performed using linear programming. Several case studies showed that risk optimization with the CVaR performance function and constraints can be done with relatively small computational resources, see Rockafellar and Uryasev (2000,2002), Krokhmal, Palmquist, and Uryasev (1999), Andersson et al. (2001), Bogentoft, Romeijn and Uryasev (2001), Jobst and Zenios (2001).

This paper compares portfolio optimization approaches with expected regret and CVaR utility functions. In order to allow the comparison, regret is specified with $\ell_1$-norm and it is assumed that expectations are calculated using a density function. We show for continuous distributions that an optimal portfolio in the CVaR sense is also optimal in the expected regret sense for some threshold in the regret function. The portfolio which minimizes CVaR can be obtained by adjusting the threshold in the expected regret function and minimizing the expected regret function. An inverse statement is also valid, i.e., a portfolio optimal in the expected regret sense, is also optimal in CVaR sense for some confidence level. If a portfolio minimizes the expected regret, it can be found by solving a one-dimensional minimization problem with respect to CVaR confidence level. We formally prove the statement on the relation of the expected regret and CVaR minimization approaches and explain statements with an example.

# 2 Comparison of expected regret and CVaR

## 2.1 Assumptions and notation

Let $f(\mathbf{x}, \mathbf{y})$ be a loss function, i.e. $-f(\mathbf{x}, \mathbf{y})$ defines the utility of return on investments, associated with a decision vector $\mathbf{x}$ and a random vector $\mathbf{y}$. The decision vector $\mathbf{x}$ can be interpreted in various ways; for instance, it is a portfolio consisting of $n$ instruments with positions belonging to a feasible set $X \subseteq \Re^n$. The random vector $\mathbf{y} \in \Re^m$ accounts for uncertainties in the loss function. To simplify formal analysis, it is supposed that the random vector $\mathbf{y}$ is drawn from a joint density function $p(\mathbf{y})$. For each $\mathbf{x}$, $f(\mathbf{x}, \mathbf{y})$ is a random variable, since it is a function of the random vector $\mathbf{y}$. The distribution function of $f(\mathbf{x},\mathbf{y})$ for a given $\mathbf{x}$,

$$\Psi(\mathbf{x},\zeta) \triangleq \int_{f(\mathbf{x},\mathbf{y})\leq\zeta} p(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

measures the probability of the event that the losses will not exceed a given level of losses, $\zeta$. $\Psi(\mathbf{x},\zeta)$ is a cumulative probability function of $\mathbf{x}$ which is nondecreasing and right-continuous with respect to $\zeta$ in general case. We assume that $\Psi(\mathbf{x},\zeta)$ is continuous with respect to $\zeta$; this is accomplished for a given $\mathbf{x}$ if the probability measure $\int_{f(\mathbf{x},\mathbf{y})=\zeta} p(\mathbf{y})\,\mathrm{d}\mathbf{y}$ is zero for all $\zeta$. This assumption as well as the assumption on the existence of density $p(\mathbf{y})$ is imposed to establish continuity of the function $\Psi(\mathbf{x},\zeta)$ with respect to decision vector $\mathbf{x}$. In some common situations, the required continuity follows from properties of loss $f(\mathbf{x},\mathbf{y})$ and the density $p(\mathbf{y})$; see Uryasev (1995). We denote the $\ell_1$-norm regret function for a portfolio $\mathbf{x}$ as

$$G_\zeta(\mathbf{x}) \triangleq \int_{y\in\Re^m} [f(\mathbf{x},\mathbf{y})-\zeta]^+\,p(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{1}$$

where the integrand may be interpreted as a measure of underperformance of the portfolio with respect to a given benchmark $\zeta$; and the positive part operator, $[\cdot]^+$, is defined as $\max(0,\cdot)$; see Harlow (1991), Dembo and King (1992). VaR for a confidence level $\alpha\in(0,1)$, $\alpha$-VaR, is defined as a minimal value of losses such that the potential losses do not exceed $\zeta$ with probability $\alpha$,

$$\zeta_\alpha(\mathbf{x}) \triangleq \min\{\zeta\in\Re : \Psi(\mathbf{x},\zeta)\geq\alpha\}. \tag{2}$$

Since $\Psi(\mathbf{x},\zeta)$ is continuous by assumption and nondecreasing with respect to $\zeta$, there could exist more than one $\zeta$ such than $\Psi(\mathbf{x},\zeta)=\alpha$; this is the reason for the use of the minimum operator. For continuous distributions considered in this paper, CVaR, with a given confidence level $\alpha$ ($\alpha$-CVaR), is defined as the conditional expectation of losses exceeding VaR

$$\phi_\alpha(\mathbf{x}) \triangleq (1-\alpha)^{-1}\int_{f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y})p(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{3}$$

where the probability that $f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})$ is $1-\alpha$. However, for general distributions, including discrete distributions, CVaR may not be equal to the conditional expectation defined in (3). For general distributions, CVaR is defined as a weighted average of VaR and conditional expectation of losses strictly exceeding VaR.

## 2.2 Formal statements

The minimum expected regret problem on a feasible set $X$ is stated as

$$\min_{\mathbf{x}\in X} \ G_\zeta(\mathbf{x}). \tag{4}$$

Similarly, we formulate the CVaR minimization problem

$$\min_{\mathbf{x}\in X} \ \phi_\alpha(\mathbf{x}), \tag{5}$$

to obtain a decision that minimize risk with probability level $\alpha$.

Rockafellar and Uryasev (2000) showed that $\alpha$-VaR and $\alpha$-CVaR can be characterized in terms of the function

$$F_\alpha(\mathbf{x},\zeta) \ \stackrel{\triangle}{=} \ \zeta + (1-\alpha)^{-1} G_\zeta(\mathbf{x}), \tag{6}$$

which is convex and continously differentiable with respect to $\zeta$ under considered assumptions. They deduced that $\alpha$-VaR is a minimizer of function $F_\alpha(\mathbf{x},\zeta)$ with respect to (w.r.t.) $\zeta$ and $\alpha$-VaR satisfies the equation

$$\zeta_\alpha(\mathbf{x}) \ = \ \inf A_\alpha(\mathbf{x}), \tag{7}$$

where

$$A_\alpha(\mathbf{x}) \ \stackrel{\triangle}{=} \ \operatorname{Arg}\min_{\zeta\in\Re} F_\alpha(\mathbf{x},\zeta) \tag{8}$$

is a nonempty, closed, and bounded interval (since $\Psi(\mathbf{x},\zeta)$ is continuous and nondecreasing with respect to $\zeta$). The fact that $\alpha$-VaR is a minimizer of function $F_\alpha(\mathbf{x},\zeta)$ can be directly proved by differentiating the function $F_\alpha(\mathbf{x},\zeta)$ and equating the derivative to zero. Indeed,

$$\frac{\partial}{\partial\zeta} F_\alpha(\mathbf{x},\zeta) \ = \ 1 + (1-\alpha)^{-1} \frac{\partial}{\partial\zeta} G_\zeta(\mathbf{x}) \ .$$

However, as it was shown by Rockafellar and Uryasev (2000)

$$\frac{\partial}{\partial\zeta} G(\zeta) = \Psi(\mathbf{x},\zeta) - 1 \ .$$

Therefore,

$$\frac{\partial}{\partial\zeta} F_\alpha(\mathbf{x},\zeta) \ = \ 1 + (1-\alpha)^{-1}(\Psi(\mathbf{x},\zeta) - 1) \ = \ (1-\alpha)^{-1}(\Psi(\mathbf{x},\zeta) - \alpha) \ .$$

The derivative $\frac{\partial}{\partial\zeta}F_\alpha(\mathbf{x},\zeta)$ equals zero when $\Psi(\mathbf{x},\zeta)=\alpha$; i.e., $\alpha$-VaR is a minimizer of the function $F_\alpha(\mathbf{x},\zeta)$. This implies that $\alpha$-CVaR of the losses associated with $\mathbf{x}\in X$ can be determined from

$$\phi_\alpha(\mathbf{x}) \;=\; \min_{\zeta\in\Re} F_\alpha(\mathbf{x},\zeta). \tag{9}$$

Indeed,

$$\begin{aligned}\phi_\alpha(\mathbf{x}) \;&=\; (1-\alpha)^{-1}\int_{f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y})p(\mathbf{y})\,\mathrm{d}\mathbf{y} \;=\; \zeta_\alpha(\mathbf{x})+(1-\alpha)^{-1}G_{\zeta_\alpha}(\mathbf{x})\\ &=\; F_\alpha(\mathbf{x},\zeta_\alpha) \;=\; \min_{\zeta\in\Re} F_\alpha(\mathbf{x},\zeta)\;.\end{aligned}$$

This allows to calculate $\alpha$-CVaR without having to calculate $\alpha$-VaR on which its definition depends. Also, Rockafellar and Uryasev (2000) showed that the CVaR minimization problem (5) may be solved by minimizing function $F_\alpha(\mathbf{x},\zeta)$ simultaneously with respect to both arguments,

$$\min_{\mathbf{x}\in X}\phi_\alpha(\mathbf{x}) \;=\; \min_{\mathbf{x}\in X}\min_{\zeta\in\Re}F_\alpha(\mathbf{x},\zeta) \;=\; \min_{(\mathbf{x},\zeta)\in X\times\Re}F_\alpha(\mathbf{x},\zeta). \tag{10}$$

Let us denote by $S_\alpha \triangleq \operatorname{Arg\,min}_{(\mathbf{x},\zeta)\in X\times\Re}F_\alpha(\mathbf{x},\zeta)$ a solution set of the second optimization problem in (10), a solution set of the CVaR optimization problem by $X^C_\alpha \triangleq \operatorname{Arg\,min}_{\mathbf{x}\in X}\phi_\alpha(\mathbf{x})$ , a solution set of the minimum regret problem by $X^R_\zeta \triangleq \operatorname{Arg\,min}_{\mathbf{x}\in X}G_\zeta(\mathbf{x})$ , and a projection of $S_\alpha$ on $\zeta$ line by $A_\alpha \triangleq \{\zeta : \text{there exist } \mathbf{x} \text{ such that } (\mathbf{x},\zeta)\in S_\alpha\}$

In order to explain the algebraic notations, we consider a geometric visualization of the solution sets. For a two dimension example ($n=2$), the relations between the defined sets are illustrated in Figure 1. Let us consider that

$$X \;\triangleq\; \{x_1+x_2=1,\, x_1\geq 0,\, x_2\geq 0\}.$$

Figure 1 displays $X$ and the solution set $S_\alpha$ (the shaded region) belonging to $X\times\Re$. It is shown that $A_\alpha$ is a projection of $S_\alpha$ on $\zeta$ line, and $X^C_\alpha$ is a projection of $S_\alpha$ on $X$. Also, it is shown that the solution set $X^R_{\bar\zeta}$ of the minimum regret problem (4) for some $\bar\zeta\in A_\alpha$ is contained in $X^C_\alpha$, and $\mathbf{x}^*$ and its associated $A_\alpha(\mathbf{x}^*)$ .

Further, we formulate a theorem stating that for each CVaR optimization problem (5), there is a regret optimization problem (4) having the same set of portfolio solutions.

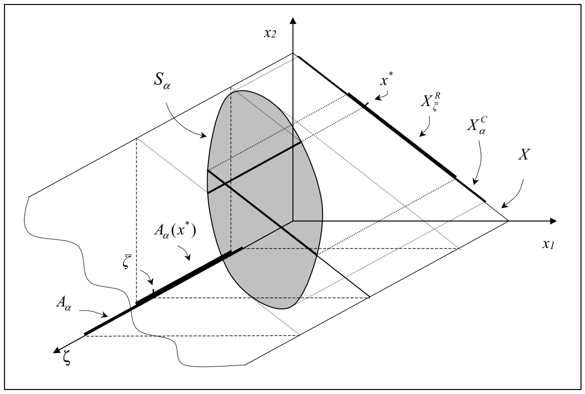


Figure 1: Two dimension example illustrating solution sets relationship.

**Theorem 1 (CVaR ⇒ Regret)** *For any* $\alpha \in (0,1)$ *and* $\mathbf{x}^* \in X^C_\alpha$ *there exists a pair* $(\mathbf{x}^*, \zeta^*) \in S_\alpha$ *such that* $\mathbf{x}^* \in X^R_{\zeta^*}$.

**Proof.** Equality (10) implies (see Theorem 2 in Rockafellar and Uryasev (2000)) that for any $x^* \in X^C_\alpha$ there exists $(\mathbf{x}^*, \zeta^*) \in S_\alpha$; therefore, we need to prove only that $\mathbf{x}^* \in X^R_{\zeta^*}$. Indeed, if

$$(\mathbf{x}^*, \zeta^*) \in \text{Arg} \min_{(\mathbf{x},\zeta) \in X \times \Re} F_\alpha(\mathbf{x}, \zeta),$$

then

$$\mathbf{x}^* \in \text{Arg} \min_{\mathbf{x} \in X} F_\alpha(\mathbf{x}, \zeta^*) = \text{Arg} \min_{\mathbf{x} \in X} \{\zeta^* + (1-\alpha)^{-1} G_{\zeta^*}(\mathbf{x})\} = \text{Arg} \min_{\mathbf{x} \in X} G_{\zeta^*}(\mathbf{x}) = X^R_{\zeta^*}.$$

The theorem is proved. □

The next theorem proves an inverse statement to Theorem 1, i.e., for each regret optimization problem (4) there exists an equivalent CVaR optimization problem (5).

**Theorem 2 (Regret ⇒ CVaR)** *For any* $\zeta \in \Re$ *and* $\mathbf{x}^* \in X^R_\zeta$ *there exists a unique* $\alpha \in (0,1)$ *such that* $\zeta \in A_\alpha(\mathbf{x}^*)$, $(\mathbf{x}^*, \zeta) \in S_\alpha$, *and* $\mathbf{x}^* \in X^C_\alpha$.

**Proof.** Since $\Psi(\mathbf{x}^*, \zeta)$ is continuous with respect to $\zeta$, there exists $\alpha \in (0,1)$ such that $\Psi(\mathbf{x}^*, \zeta) = \alpha$. The derivative of the function $F_\alpha(\mathbf{x}^*, \zeta)$ with respect to $\zeta$ equals (see proof of Theorem 1 in

Rockafellar and Uryasev (2000))

$$\frac{\partial}{\partial \zeta} F_\alpha(\mathbf{x}^*, \zeta) \; = \; (1-\alpha)^{-1}(\Psi(\mathbf{x}^*, \zeta) - \alpha) \; = \; 0$$

The function $F_\alpha(\mathbf{x}^*, \zeta)$ is convex with respect to $\zeta$ (see Theorem 2 in Rockafellar and Uryasev (2000)). Therefore,

$$\zeta \in A_\alpha(\mathbf{x}^*) \; = \; \operatorname{Arg}\min_{\tau \in \Re} F_\alpha(\mathbf{x}^*, \tau), \tag{11}$$

and the first statement of the theorem is proved.

Also, since $\mathbf{x}^* \in X^R_\zeta$, we obtain

$$\mathbf{x}^* \in \operatorname{Arg}\min_{\mathbf{x} \in X} G_\zeta(\mathbf{x}) \; = \; \operatorname{Arg}\min_{\mathbf{x} \in X} \{\zeta + (1-\alpha)^{-1} G_\zeta(\mathbf{x})\} \; = \; \operatorname{Arg}\min_{\mathbf{x} \in X} F_\alpha(\mathbf{x}, \zeta). \tag{12}$$

Inclusions (11) and (12) imply that $(\mathbf{x}^*, \zeta) \in \operatorname{Arg}\min_{(\mathbf{x},\tau) \in X \times \Re} F_\alpha(\mathbf{x}, \tau) = S_\alpha$ , and the second statement of the theorem is proved. Finally, the last statement $\mathbf{x}^* \in X^C_\alpha$ follows from Theorem 2 in Rockafellar and Uryasev (2000). □

The following corollary considers the case when $S_\alpha = X^C_\alpha \times A_\alpha$. For instance, this is valid when the set $A_\alpha$ consists only of one VaR point $\zeta_\alpha(\mathbf{x})$.

**Corollary 1** *If in addition to conditions of Theorem 2, $S_\alpha = X^C_\alpha \times A_\alpha$, then $X^C_\alpha = X^R_\zeta$ for any $\zeta^* \in A_\alpha$*

**Proof.** The statement follows from (12). □

## 3 Numerical example

In this section, we illustrate the formal statements with a numerical example demonstrating the equivalence of the expected regret and CVaR approaches. For numerical calculations, we used the implementation framework described by Rockafellar and Uryasev (2000).

We considered a universe of $n = 1792$ listed stocks for the portfolio optimization problem. The decision vector $\mathbf{x}$ consists of stock positions in the portfolio. Components of the vector $\mathbf{y} \in \Re^m$ are random returns of instruments, hence $m = n$. Distribution of the vector $\mathbf{y} \in \Re^m$ is modeled by $s = 156$ historical weekly scenario returns, $\{\mathbf{y}^1, \ldots, \mathbf{y}^s\}$, with equal discrete probability, $1/s$. The feasible set $X$ is a convex polytope given by the following constraints:

$$
\begin{aligned}
&s^{-1}\sum_{j=1}^{s}\mathbf{x}^T\mathbf{y}^j \geq R && \text{(lower bound on expected return)},\\
&\sum_{i=1}^{n} x_i = 1 && \text{(normalization constraint)},\\
&x_i \geq 0, \quad i = 1, ..., n && \text{(no short positions)}.
\end{aligned}
$$

The loss function for a scenario $j$ is given by $f(\mathbf{x},\mathbf{y}^j) = \mathbf{x}^T(-\mathbf{y}^j)$, which is the negative return on a portfolio $\mathbf{x}$. Generally, the integrals in (1) and (6) cannot be obtained analytically; therefore, numerical procedures are used to estimate them. For this case, we use a statistical approximation method with historical scenarios. The corresponding approximations to $G_\zeta(\mathbf{x})$ and $F_\alpha(\mathbf{x},\zeta)$ are

$$
\widetilde{G}_\zeta(\mathbf{x}) \triangleq \frac{1}{s}\sum_{j=1}^{s}[-\mathbf{x}^T\mathbf{y}^j - \zeta]^+,
$$

and

$$
\widetilde{F}_\alpha(\mathbf{x},\zeta) \triangleq \zeta + (1-\alpha)^{-1}\,\widetilde{G}_\zeta(\mathbf{x}).
$$

Then, the problems of minimizing $\widetilde{G}_\zeta(\mathbf{x})$ on $X$ and minimizing $\widetilde{F}_\alpha(\mathbf{x},\zeta)$ on $X \times \Re$ are convex programming problems, since $f(\mathbf{x},\mathbf{y}^j)$ is convex with respect to $\mathbf{x}$. Both problems can be reduced to linear programming problems using additional variables.

Distribution characteristics of the instrument returns (for the population of instruments) are shown in Table 1.

**Table 1.** Population of instruments: distribution characteristics of average instrument returns in 156 weeks (minimum of mean instrument returns, maximum of mean instrument returns, mean of instrument returns, standard deviation of instrument returns, skewness of average instrument returns, and kurtosis of average instrument returns).

| Minimum | Maximum | Mean | Std. Dev. | Skewness | Kurtosis |
|---|---|---|---|---|---|
| -0.30684 | 0.327555 | 0.003048 | 0.078163 | -0.259866 | 9.568777 |

For the considered portfolio of stocks, we conducted numerical experiments comparing minimum regret and minimum CVaR approaches. We specified a set of minimum regret problems by making a grid in parameter $\zeta$ (fifty values). Further, for value $\bar{\zeta}$ in this grid a sensitivity analysis with respect to $\alpha$ was performed to find a matching CVaR problem with $\min_{\zeta\in\Re}|\zeta-\bar{\zeta}|$. The target weekly return was set to $R = 0.003$. The fifty $\bar{\zeta}$ values produced $\alpha$ ranging approximately between 0.75 and 0.98. The minimum regret and CVaR models were solved using linear programming by previously reformulating the piece-wise linear convex functions $\widetilde{G}_\zeta(\mathbf{x})$ and

$\widetilde{F}_\alpha(\mathbf{x}, \zeta)$ into equivalent linear functions with the additional auxiliary variables. The search procedure ($\zeta^* = \arg\min |\zeta - \bar{\zeta}|$) and the linear programming problem were implemented with GAMS, Brooke et al. (1992), and solved with CPLEX's mathematical programming library, ILOG (1997), in a personal computer with a Pentium-II 300 MHz processor and 128 MB memory. Table 2 depicts a summary of numerical experiments. The numerical results indicated that the expected regret solution can be found quite precisely by solving a minimization problem with respect to $\alpha$ in the CVaR minimization approach. The relative differences of portfolio solution norms between the two considered approaches were less that 1%. Also, there is a close correspondence of $\zeta^*$ and $\bar{\zeta}$ values: the relative difference is less than 5% or the absolute difference is less than 0.0002. Figure 2 depicts a summary of numerical comparisons of minimum regret and CVaR approaches.

## 4 Conclusion

This paper demonstrated that the minimum expected regret and the minimum CVaR approaches are closely related. For the case with $\ell_1$-norm and constant target value, a portfolio optimal in expected regret sense can be obtained by solving a minimization problem with respect to the confidence parameter in the minimum CVaR approach. Also, the inverse statement is valid, i.e., a portfolio optimal in CVaR sense can be obtained by solving a minimization problem with respect to the target value in the minimum expected regret approach. Numerical experiments confirmed the formal mathematical statements. Also, numerical experiments demonstrated that both approaches, minimum regret and minimum CVaR, can be very efficiently implemented using small computational resources, such as a Pentium-II 300 MHz computer.

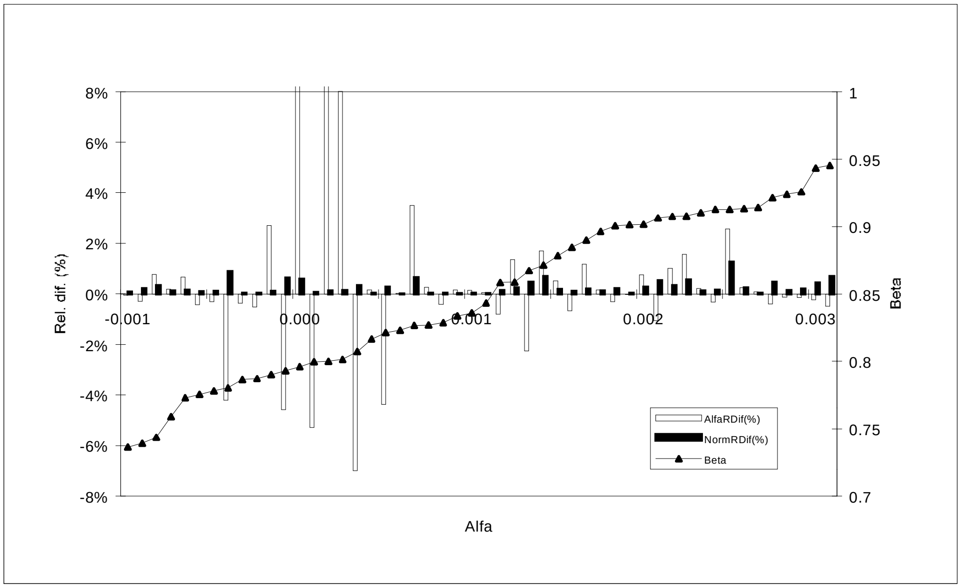


Figure 2: zeta and solution norm relative differences between minimum regret and CVaR for different $\bar{\zeta}$ values and associated $\alpha$.

**Table 2.** Comparison of the minimum expected regret and CVaR solutions. Regret heading: $\bar{\zeta}$ value, solution size (number of non-zero components in the solution), solution norm; CVaR heading: $\alpha$ value, $\zeta$ solution value, solution size, and solution norm; Comparison heading: relative zeta difference and relative solution norm difference.

| Regret | | | CVaR | | | | Comparison | |
|---|---|---|---|---|---|---|---|---|
| $\bar{\zeta}$ | Sol. size | $\|\mathbf{x}^R_{\bar{\zeta}}\|_2$ | $\alpha$ | $\zeta^*$ | Sol. size | $\|\mathbf{x}^C_{\alpha}\|_2$ | $\frac{\zeta^*-\bar{\zeta}}{\lvert\bar{\zeta}\rvert}$ | $\frac{\|\mathbf{x}^C_{\alpha}-\mathbf{x}^R_{\bar{\zeta}}\|_2}{\|\mathbf{x}^R_{\bar{\zeta}}\|_2}$ |
| -0.0011884 | 54 | 0.546686 | 0.7363 | -0.0011891 | 53 | 0.546485 | -0.056% | 0.083% |
| -0.0011033 | 53 | 0.545907 | 0.7393 | -0.0011066 | 52 | 0.546145 | -0.298% | 0.212% |
| -0.0010182 | 51 | 0.541216 | 0.7435 | -0.0010102 | 50 | 0.541307 | 0.780% | 0.327% |
| -0.0009330 | 49 | 0.541990 | 0.7589 | -0.0009313 | 49 | 0.542439 | 0.182% | 0.121% |
| -0.0008479 | 47 | 0.543070 | 0.7728 | -0.0008423 | 46 | 0.542642 | 0.664% | 0.147% |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| -0.0007628 | 48 | 0.542182 | 0.7754 | -0.0007661 | 47 | 0.542092 | -0.433% | 0.107% |
| -0.0006777 | 48 | 0.540022 | 0.7781 | -0.0006798 | 47 | 0.539613 | -0.310% | 0.110% |
| -0.0005925 | 49 | 0.549551 | 0.7803 | -0.0006175 | 48 | 0.548881 | -4.206% | 0.881% |
| -0.0005074 | 47 | 0.551212 | 0.7864 | -0.0005093 | 46 | 0.551060 | -0.365% | 0.047% |
| -0.0004223 | 47 | 0.555201 | 0.7871 | -0.0004244 | 47 | 0.555241 | -0.512% | 0.032% |
| -0.0003372 | 48 | 0.556890 | 0.7901 | -0.0003281 | 48 | 0.557170 | 2.702% | 0.118% |
| -0.0002520 | 48 | 0.560760 | 0.7930 | -0.0002636 | 47 | 0.559461 | -4.579% | 0.630% |
| -0.0001669 | 49 | 0.565039 | 0.7958 | -0.0001496 | 49 | 0.566291 | 10.34% | 0.599% |
| -0.0000818 | 49 | 0.569552 | 0.7996 | -0.0000861 | 49 | 0.569453 | -5.282% | 0.062% |
| 0.0000033 | 49 | 0.570694 | 0.7999 | 0.0000104 | 48 | 0.570732 | 211.0% | 0.130% |
| 0.0000885 | 49 | 0.573080 | 0.8013 | 0.0000956 | 48 | 0.573292 | 8.015% | 0.137% |
| 0.0001736 | 47 | 0.571145 | 0.8073 | 0.0001614 | 46 | 0.571929 | -6.994% | 0.332% |
| 0.0002587 | 47 | 0.570329 | 0.8164 | 0.0002591 | 46 | 0.570296 | 0.162% | 0.037% |
| 0.0003438 | 46 | 0.570130 | 0.8213 | 0.0003288 | 46 | 0.570573 | -4.380% | 0.269% |
| 0.0004290 | 46 | 0.568194 | 0.8230 | 0.0004290 | 46 | 0.568193 | 0.019% | 0.002% |
| 0.0005141 | 45 | 0.572725 | 0.8267 | 0.0005320 | 45 | 0.574391 | 3.492% | 0.646% |
| 0.0005992 | 46 | 0.572726 | 0.8268 | 0.0006008 | 46 | 0.572689 | 0.262% | 0.042% |
| 0.0006843 | 48 | 0.566529 | 0.8286 | 0.0006815 | 48 | 0.566545 | -0.412% | 0.045% |
| 0.0007695 | 46 | 0.566841 | 0.8335 | 0.0007707 | 45 | 0.566861 | 0.168% | 0.023% |
| 0.0008546 | 44 | 0.567263 | 0.8358 | 0.0008558 | 43 | 0.567181 | 0.138% | 0.026% |
| 0.0009397 | 43 | 0.572998 | 0.8430 | 0.0009403 | 43 | 0.573061 | 0.060% | 0.023% |
| 0.0010248 | 43 | 0.569597 | 0.8584 | 0.0010167 | 42 | 0.569538 | -0.796% | 0.139% |
| 0.0011100 | 44 | 0.570064 | 0.8586 | 0.0011250 | 44 | 0.570147 | 1.359% | 0.245% |
| 0.0011951 | 43 | 0.573454 | 0.8672 | 0.0011681 | 43 | 0.572051 | -2.256% | 0.469% |
| 0.0012802 | 43 | 0.570857 | 0.8713 | 0.0013021 | 43 | 0.568413 | 1.708% | 0.695% |
| 0.0013653 | 44 | 0.566464 | 0.8784 | 0.0013723 | 44 | 0.566491 | 0.513% | 0.181% |
| 0.0014504 | 42 | 0.566328 | 0.8846 | 0.0014407 | 42 | 0.566368 | -0.670% | 0.109% |
| 0.0015356 | 45 | 0.566256 | 0.8898 | 0.0015535 | 45 | 0.566520 | 1.170% | 0.194% |
| 0.0016207 | 43 | 0.563752 | 0.8965 | 0.0016232 | 42 | 0.563500 | 0.155% | 0.124% |
| 0.0017058 | 42 | 0.555593 | 0.9004 | 0.0017006 | 41 | 0.556432 | -0.307% | 0.212% |
| 0.0017909 | 41 | 0.547912 | 0.9014 | 0.0017907 | 41 | 0.547814 | -0.014% | 0.037% |
| 0.0018761 | 42 | 0.544711 | 0.9017 | 0.0018901 | 41 | 0.544074 | 0.750% | 0.283% |
| 0.0019612 | 42 | 0.536406 | 0.9063 | 0.0019404 | 41 | 0.536946 | -1.060% | 0.532% |
| 0.0020463 | 42 | 0.532997 | 0.9074 | 0.0020672 | 42 | 0.532065 | 1.021% | 0.342% |
| 0.0021314 | 42 | 0.529311 | 0.9076 | 0.0021650 | 41 | 0.527889 | 1.576% | 0.560% |
| 0.0022166 | 41 | 0.525621 | 0.9101 | 0.0022211 | 41 | 0.525570 | 0.205% | 0.131% |
| 0.0023017 | 40 | 0.520152 | 0.9127 | 0.0022944 | 39 | 0.520477 | -0.318% | 0.149% |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.0023868 | 40 | 0.516449 | 0.9127 | 0.0024482 | 40 | 0.513859 | 2.571% | 1.255% |
| 0.0024719 | 41 | 0.515979 | 0.9134 | 0.0024780 | 41 | 0.516526 | 0.245% | 0.249% |
| 0.0025571 | 39 | 0.509568 | 0.9141 | 0.0025592 | 38 | 0.509469 | 0.083% | 0.042% |
| 0.0026422 | 38 | 0.511696 | 0.9214 | 0.0026316 | 37 | 0.511280 | -0.402% | 0.479% |
| 0.0027273 | 37 | 0.514012 | 0.9238 | 0.0027237 | 37 | 0.514044 | -0.132% | 0.135% |
| 0.0028124 | 37 | 0.519918 | 0.9258 | 0.0028086 | 36 | 0.519317 | -0.135% | 0.198% |
| 0.0028976 | 38 | 0.514950 | 0.9434 | 0.0028908 | 37 | 0.514028 | -0.232% | 0.436% |
| 0.0029827 | 38 | 0.515775 | 0.9453 | 0.0029683 | 37 | 0.517958 | -0.483% | 0.691% |
| 0.0030678 | 38 | 0.503796 | 0.9688 | 0.0030676 | 37 | 0.503834 | -0.005% | 0.023% |

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