Chapter 1

ROBUST DECISION MAKING: ADDRESSING UNCERTAINTIES IN DISTRIBUTIONS*

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Abstract This paper develops a general approach to risk management in military applications involving uncertainties in information and distributions. The risk of loss, damage, or failure is measured by the Conditional Value-at-Risk (CVaR) measure. Loosely speaking, CVaR with the confidence level α estimates the risk of loss by averaging the possible losses over the $(1-\alpha)\cdot 100\%$ worst cases (e.g., 10%). As a function of decision variables, CVaR is convex and therefore can be efficiently controlled/optimized using convex or (under quite general assumptions) linear programming. The general methodology was tested on two Weapon-Target Assignment (WTA) problems. It is assumed that the distributions of random variables in the WTA formulations are not known with certainty. The total cost of a mission (including weapon attrition) was minimized, while satisfying operational constraints and ensuring destruction of all targets with high probabilities. The risk of failure of the mission (e.g., targets are not destroyed) is controlled by CVaR constraints. The case studies conducted show that there are significant qualitative and quantitative differences in solutions of deterministic WTA and stochastic WTA problems.

Keywords: Risk management, Conditional Value-at-Risk, uncertainty, military applications, stochastic programming.

1. Introduction

This paper develops a general approach to managing risk in military applications involving stochasticity and uncertainties in distributions. Various military applications such as intelligence, surveillance, planning, scheduling etc., involve decision making in dynamic, distributed, and uncertain environments. In a large system, multiple sensors may provide incomplete, conflicting, or overlapping data. Moreover, some components or sensors may degrade or become completely unavailable (failures, weather conditions, battle damage). Uncertainties in combat environment induce different kinds of risks that components, sensors or armed units are exposed to, such as the risk to be damaged or destroyed, risk of mission incompleteness (e.g., missing a target) or failure, risk of false target attack etc. Therefore, planning and operating in stochastic and uncertain conditions of a modern combat require robust decision-making procedures. Such procedures must take into account the stochastic nature of risk-inducing factors, and generate decisions that are not only effective on average (in other words, have good "expected" performance), but also safe enough under a wide range of possible scenarios. In this regard, risk management in military applications is similar to practices in other fields such as finance, nuclear safety, etc., where decisions targeted only at achieving the maximal expected performance may lead to an excessive risk exposure. However, in contrast to other applications, distributions of the stochastic risk-inducing factors are often unknown or uncertain in military problems. Uncertainty in distributions of risk parameters may be caused by a lack of data, unreliability of data, or the specific nature of a risk factor (e.g., in different circumstances a risk factor may exhibit different stochastic behavior). Therefore, decision making in military applications must account for uncertainties in distributions of stochastic parameters and be robust with respect to these uncertainties.

In this paper, we propose a general methodology for managing risk in military applications involving various risk factors as well as uncertainties in distributions. The approach is tested with several stochastic versions of the Weapon-Target Assignment problem.

The paper is organized as follows. Section 2 presents key theoretical results on risk management using Conditional Value-at-Risk (CVaR) risk measure, and describes the general approach to controlling risk when distributions of risk factors are uncertain. Section 3 develops various formulations of the stochastic Weapon-Target Assignment (WTA) problem with CVaR constraints. Results of numerical experiments for one-stage and two-stage stochastic WTA problems are presented in Section 4. The Conclusions section summarizes the obtained results and outlines the directions of future research. Finally, the Appendix presents formal definitions and results concerning the risk management using the CVaR risk measure.

2. The General Approach

Presence of uncertainty in a decision-making model leads to the problem of estimation and managing/controlling of risk associated with the stochastic parameters in the model. Over the recent years, risk management has evolved into a sophisticated discipline combining both rigorous and elegant theoretical results and practical effectiveness (this especially applies to the risk management in finance industry). Generally speaking, risk management is a set of activities aimed at reducing or preventing *high losses* incurred from an incorrect decision. The losses (e.g., damages, failures) in a system are quantified by a *loss function* $L(x, \xi)$ that depends upon decision vector x and a stochastic vector ξ standing for uncertainties in the model. Assuming for now that a distribution of the parameter ξ is known, it is possible to determine the distribution of the loss function $L(x, \xi)$ (see Fig. 1.1). Then, the problem of preventing high losses is a problem of controlling and shaping the loss distribution and, more specifically, its right tail, where the high losses reside. To estimate and quantify the losses in the tail of the loss distribution, a *risk* *measure* has to be specified. In particular, a risk measure introduces the ordering relationships for risks, so that one is able to discriminate "less risky" decisions from the "more risky" ones[1]. The appropriate choice of a risk measure is, in most cases, dictated by the nature of uncertainties and risks in the problem at hand. In military applications, for example, one usually deals with the probabilities of events, such as the probability to hit a target, the probability to detect the enemy's aircraft, and so on. Therefore *percentile* risk measures that represent the risk in terms of percentiles of the loss distribution are particularly suitable for the risk management in military applications. Popular percentile risk measures include Value-at-Risk (VaR), Conditional Value-at-Risk, Maximum Loss, and Expected Shortfall. Figure 1.1 displays some of these measures; Value-at-Risk with confidence level α (α-VaR), which is the α-percentile of loss distribution, Maximum Loss ("1.0-percentile" of loss distribution), and α-CVaR, which approximately equals to the expectation of losses exceeding α-VaR.

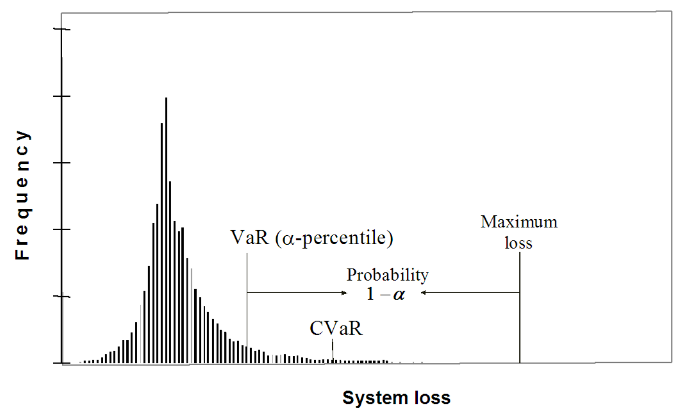

Figure 1.1. Loss function distribution and different risk measures.

We build our approach for risk management in military applications on the CVaR methodology, which is a relatively new development (Rockafellar and Uryasev, 2000, Rockafellar and Uryasev, 2001). This section presents the general framework of risk management using Condi-

[1]Artzner et al., 1999, have introduced a concept of "ideal", or *coherent*, risk measure. A *coherent* risk measure, which satisfies to a set of axioms developed in this paper, is expected to produce "proper" and "consistent" estimates of risk.

tional Value-at-Risk, and extends it to the case when the distributions of stochastic parameters are not certain.

2.1. Risk Management Using Conditional Value-at-Risk

Suppose that the uncertain future is represented by a finite number of future outcomes (scenarios). Then, approximately, Value-at-Risk with confidence level α (α-VaR) is defined as the loss that can be exceeded only in $(1-\alpha)\cdot 100\%$ of worst scenarios. Similarly, one may think of α-CVaR (i.e., Conditional Value-at-Risk with confidence level α) as of the average loss over $(1-\alpha)\cdot 100\%$ of worst cases (see Fig. 1.2). We say "approximately" because $(1-\alpha)\cdot 100\%$ may not be an integer number. Exact definition of α-VaR and α-CVaR is given in the Appendix.

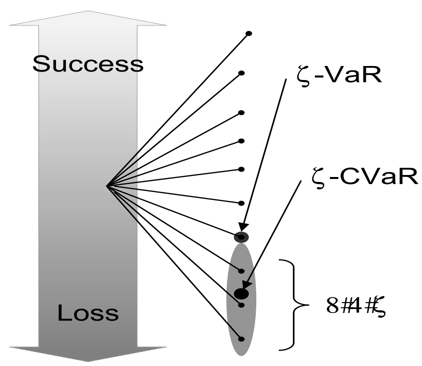

Figure 1.2. A visualization of VaR and CVaR concepts.

These intuitive definitions are correct if, for example, all scenarios are equally probable, and $(1-\alpha)\cdot 100$ is an integer number. The formal definitions of α-VaR and α-CVaR that apply to any loss distribution and value of confidence level α are more complex, and the reader can find them in the Appendix. The Appendix also contains the key formal results concerning calculation and controlling of the Conditional Value-at-Risk for general loss distributions. Here we mention only the most important properties of CVaR and their practical implications.

The Conditional Value-at-Risk function $\mathsf{CVaR}_\alpha[L(x,\xi)]$ has the following properties (Rockafellar and Uryasev, 2000, Rockafellar and Uryasev, 2001, Acerbi and Tasche, 2001):

- CVaR is continuous with respect to confidence level α (other percentile risk measures like VaR, Expected Shortfall, etc., may be discontinuous in α);

- CVaR is convex in α and x, provided that the loss function $L(x,\xi)$ is convex in x (VaR, Expected Shortfall are generally non-convex in x);
- CVaR is *coherent* in the sense of Artzner et al., 1999;
- in case of a continuous loss distribution CVaR equals the conditional expectation of losses exceeding the VaR level.

From the viewpoint of managing and controlling of risk, the most important property of CVaR, which distinguishes it from all other percentile risk measures, is the convexity with respect to decision variables, which permits the use of convex programming for minimizing CVaR. If the loss function $L(x,\xi)$ can be approximated by a piecewise linear function, the procedure of controlling or optimization of CVaR is reduced to solving a Linear Programming (LP) problem.

Assume that there are S possible realizations (scenarios) $\xi_1,...,\xi_S$ of vector ξ with probabilities π_s ($\sum_{s=1}^{S}\pi_s = 1$), then in the optimization problem with multiple CVaR constraints

$$\begin{aligned}
&\max_{x\in X}\ g(x)\\
&\text{subject to}\\
&\mathsf{CVaR}_{\alpha_n}[L(x,\xi)] \le C_n, \quad n = 1,...,N,
\end{aligned}$$

where $g(x)$ is some performance function and X is a convex set, each CVaR constraint may be replaced by a set of inequalities (Rockafellar and Uryasev, 2000, 2001)

$$\begin{aligned}
&L(x,\xi_s) - \zeta_n \le w_{ns}, \quad s = 1,...,S,\\
&\zeta_n + (1-\alpha_n)^{-1}\sum_{s=1}^{S}\pi_s w_{ns} \le C_n,\\
&\zeta_n \in \mathcal{R}, \quad w_{ns} \in \mathcal{R}^+, \quad s = 1,...,S,
\end{aligned} \tag{1}$$

where $\mathcal{R}$ and $\mathcal{R}^+$ are the sets of real and non-negative real numbers correspondingly, and w_{ns} are auxiliary variables. If in the optimal solution the n-th CVaR constraint is active, then the corresponding variable ζ_n is equal to α_n-VaR (i.e., α_n-th percentile of the loss distribution).

In the risk management methodology discussed above the distribution of stochastic parameter ξ is considered to be known. The next subsection extends the presented approach to the case, when the distribution of stochastic parameters in the model is not certain.

2.2. Risk Management Using CVaR in the Presence of Uncertainties in Distributions

The general approach to managing risks in an uncertain environment, where the distributions of stochastic parameters are not known for sure, can be described as follows. Suppose that we have some performance function $F(x, \xi)$, dependent on the decision vector $x \in X$ and some random vector $\xi \in \Xi$, whose distribution is not known for certain. We assume that the actual realization of vector ξ may come from different distributions $\Theta_1, ..., \Theta_N$. The vector ξ stands for the uncertainties in data that make it impossible to evaluate the efficiency $F(x, \xi)$ of the decision for sure. Thus, there always exists a possibility of making an incorrect decision, and, consequently, suffering loss, damage, or failing the mission. If the loss in the system is evaluated by function $L(x, \xi)$, then risk of high losses can be controlled using CVaR constrains. Let formulate the problem of maximizing the expected performance function $F(x, \xi)$ subject to some operational constraints $Ax \leq b$ and CVaR risk constraints. Due to the unknown distribution of vector ξ, we are unable to find the expectation $\mathsf{E}_\Theta[F(x, \xi)]$. Therefore, being on the conservative side, we want the decision x to be optimal with respect to each measure Θ_n, and this leads to the following *max-min* problem:

$$\begin{aligned} &\max_{x \in X} \min_{\Theta_n,\, n=1,...,N} \mathsf{E}_{\Theta_n}[F(x, \xi)] \qquad (2)\\ &\text{subject to}\\ &Ax \leq b,\\ &\mathsf{CVaR}_\alpha[L(x, \xi) \,|\, \Theta_n] \leq C, \quad n = 1, ..., N, \end{aligned}$$

where multiple CVaR constraints with respect to different measures Θ_n control the risk for high losses $L(x, \xi)$ to exceed some threshold C. In formulation (2) we assume that the performance function F is concave in x, and the loss function L is convex in x. These assumptions are not restrictive; on the contrary, they indicate that given more than one decision with equal performance one favors safer decisions over the riskier ones.

Model (2) explains how to handle the risk of generating an incorrect decision in an uncertain environment. In military applications, different types of risks and losses may be explicitly involved, for example, along with loss function $L(x, \xi)$ one may consider a loss function $R(x, \xi)$ for the risk of false target attack. Control for this type of risk can also be included in the model by a similar set of CVaR constraints:

$$\max_{x \in X} \min_{\Theta_n,\, n=1,...,N} \mathsf{E}_{\Theta_N}[F(x, \xi)]$$

$$
\begin{aligned}
&\text{subject to}\\
&Ax \le b,\\
&\mathsf{CVaR}_{\alpha_1}[L(x,\xi)\,|\,\Theta_n] \le C_1, \quad n = 1, ..., N,\\
&\mathsf{CVaR}_{\alpha_2}[R(x,\xi)\,|\,\Theta_n] \le C_2, \quad n = 1, ..., N.
\end{aligned}
$$

In the next sections we test the presented approach to risk management in military applications with the Weapon-Target Assignment problem.

3. Example: Stochastic Weapon-Target Assignment Problem

The Weapon-Target Assignment (WTA) problem considers the optimal assignment of weapons to targets so as to minimize the surviving value of targets. The WTA problem is used in planning environment that features a whole spectrum of uncertainties, such as the number and types of targets in the battle space, their positions, and the probability of a weapon to destroy a target (e.g., probability of kill). To generate robust decisions, one must account for these uncertainties and the corresponding risks. In this section we present two formulations of the stochastic Weapon-Target Assignment problem that address the uncertainties in a weapon's probability of kill and in the number of targets.

3.1. Deterministic WTA Problem

The generic formulation of the Weapon Target Assignment problem is as follows. Given the set of targets and set of available weapons, one must find the optimal assignment of weapons to targets, such that, for example, the damage to the targets is maximized, or the cost of the operation is minimized. The WTA formulation that maximizes the damage to the targets (see, for example, Manne, 1958, denBroeger et al., 1959, Murphey, 1999) leads to a non-linear programming problem (NLP), with linear constraints and is the subject of a future paper. In this paper we adopt another setup, where the total cost of the mission (including battle damage or loss) is minimized, while satisfying constraints on mission accomplishment (i.e., destruction of all targets with some prescribed probabilities). We assume that different weapons have different costs and efficiencies, and, in general, each may have a "multishot" capacity so that it may attack more than one target. In the deterministic setup of the problem we include also the constraint that prescribes how many targets a single weapon can attack.

The deterministic WTA problem is

$$\min_{x} \quad \sum_{k=1}^{K} \sum_{i=1}^{I} c_{ik}\, x_{ik} \tag{3a}$$

subject to

$$\sum_{k=1}^{K} x_{ik} \le m_i, \quad i = 1, ..., I, \tag{3b}$$

$$x_{ik} \le m_i\, v_{ik}, \quad i = 1, ..., I, \quad k = 1, ..., K, \tag{3c}$$

$$\sum_{k=1}^{K} v_{ik} \le t_i, \quad i = 1, ..., I, \tag{3d}$$

$$1 - \prod_{i=1}^{I} (1 - p_{ik})^{x_{ik}} \ge d_k, \quad k = 1, ..., K, \tag{3e}$$

$$x_{ik} \in \mathcal{Z}^{+}, \quad v_{ik} \in \{0, 1\},$$

where

- x_{ik} is the number of shots to be fired by weapon i at target k;
- v_{ik} $= 1$, if weapon i fires at target k, and $v_{ik} = 0$ otherwise;
- c_{ik} is the cost (including the battle loss or damage) of firing one shot from weapon i at target k; c_k includes the relative value of target k with respect to all other targets;
- m_i is the shots capacity for weapon i;
- t_i is the maximal number of targets which can be attacked by weapon i;
- p_{ik} is the probability of destroying target k by firing one shot from weapon i;
- d_k is the minimal required probability for destroying target k;
- $\mathcal{Z}$ is the set of integer numbers, and $\mathcal{Z}^{+}$ is the set of non-negative integers.

The objective function in this problem equals to the total cost of the mission. The first constraint, (3b), states that the munitions capacity of weapon i cannot be exceeded. The second and the third constraints (3c) and (3d) are responsible for not allowing weapon i to attack more than t_i targets, where $t_i \le K$. The last constraint (3e) ensures that after all weapons are assigned, target k is destroyed with probability not less than d_k. Note that this non-linear constraint can be linearized:

$$\sum_{i=1}^{I} \ln(1-p_{ik})\, x_{ik} - \ln(1-d_k) \leq 0. \tag{4}$$

In this way the deterministic WTA problem (3a) can be formulated as a linear integer programming (IP) problem.

3.2. One-Stage Stochastic WTA Problem with CVaR Constraints

In real-life situations many of the parameters in model (3a)–(3e) are not deterministic, but stochastic values. For example, the probabilities p_{ik} of destroying target k may depend upon battle situation, weather conditions, and so on, and consequently, may be treated as being uncertain. Similarly, the cost of firing c_{ik}, which includes battle loss/damage, may also be a stochastic parameter. The number of targets K may be uncertain as well.

First, we consider a one-stage Stochastic Weapon-Target Assignment (SWTA) problem, where the uncertainty is introduced into the model by assuming that probabilities p_{ik} are stochastic and dependent on some random parameter ξ:

$$p_{ik} = p_{ik}(\xi).$$

In accordance to the described methodology of managing uncertainties and risks, we model the stochastic behavior of probabilities p_{ik} using scenarios. Namely, probabilities $p_{ik}(\xi)$ take different values $p_{ik}(\xi_s) = p_{iks}$, $s = 1, ..., S$ under S different scenarios. Such a scenario set may be constructed, for example, by utilizing the historical observations of weapons' efficiency in different environments, or by using simulated data, experts' opinions etc.

We now replace the last constraint in (3a) by a CVaR constraint, where the loss function takes a positive value if the probability of destroying target k is less than d_k:

$$L_k(x,\xi) = \sum_{i=1}^{I} \ln\Big(1 - p_{ik}(\xi)\Big)\, x_{ik} - \ln(1-d_k), \tag{5}$$

and takes a negative value otherwise. Recall that a CVaR constraint with confidence level α bounds the (weighted) average of $(1-\alpha)\cdot 100\%$ highest losses. In our case, allowing small positive values of loss function (5) for some scenarios implies that for these scenarios target k is destroyed with probability slightly less than d_k, which may still be acceptable from a practical point of view.

Except for the constraint on the target destruction probability, the one-stage Stochastic WTA problem is identical to its deterministic predecessor:

$$\min_x \sum_{k=1}^{K}\sum_{i=1}^{I} c_{ik}x_{ik} \tag{6}$$

subject to

$$\sum_{k=1}^{K} x_{ik} \le m_i, \quad i = 1, ..., I,$$

$$x_{ik} \le m_i v_{ik}, \quad i = 1, ..., I, \quad k = 1, ..., K,$$

$$\sum_{k=1}^{K} v_{ik} \le t_i, \quad i = 1, ..., I,$$

$$\mathsf{CVaR}_\alpha \left[L_k(x, \xi)\right] \le C_k, \quad k = 1, ..., K.$$

Here α is the confidence level, C_k are some (small) constants, and all other variables and parameters are defined as before. As demonstrated in (0), for the adopted scenario model with probabilities p_{ik}, the CVaR constraint for the k-th target

$$\mathsf{CVaR}_\alpha \left[L_k(x, \xi)\right] \le C_k$$

is represented by a set of linear inequalities:

$$\sum_{i=1}^{I} \ln(1 - p_{iks})\, x_{ik} - \ln(1 - d_k) - \zeta_k \le w_{sk}, \quad s = 1, ..., S,$$

$$\zeta_k + (1 - \alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_{sk} \le C_k, \tag{7}$$

$$\zeta_k \in \mathcal{R}, \quad w_{sk} \ge 0, \quad s = 1, ..., S, \quad k = 1, ..., K.$$

Thus, the one-stage Stochastic WTA problem can be formulated as a mixed-integer programming (MIP) problem:

$$\min_x \sum_{k=1}^{K}\sum_{i=1}^{I} c_{ik}\, x_{ik} \tag{8}$$

subject to

$$\sum_{k=1}^{K} x_{ik} \le m_i, \quad i = 1, ..., I,$$

$$
\begin{aligned}
& x_{ik} \le m_i\, v_{ik}, \quad i = 1, ..., I, \quad k = 1, ..., K, \\
& \sum_{k=1}^{K} v_{ik} \le t_i, \quad i = 1, ..., I, \\
& \sum_{i=1}^{I} \ln(1 - p_{iks})\, x_{ik} - \ln(1 - d_k) - \zeta_k \le w_{sk}, \\
& \qquad\qquad\qquad s = 1, ..., S, \quad k = 1, ..., K, \\
& \zeta_k + (1 - \alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_{sk} \le C_k, \quad k = 1, ..., K, \\
& x_{ik} \in \mathcal{Z}^+, \quad v_{ik} \in \{0, 1\}, \quad \zeta_k \in \mathcal{R}, \quad w_{sk} \ge 0, \\
& \qquad\qquad\qquad s = 1, ..., S, \quad i = 1, ..., I, \quad k = 1, ..., K.
\end{aligned}
$$

Note that different values of probability p_{ik} represent the uncertainty in the distributions of stochastic parameters discussed in the previous section. Indeed, different values of probability p_{ik} imply different probability measures for the random variable associated with the event of destroying target k by firing one unit of munitions by weapon i. In effect, CVaR constraint (6) is a risk constraint that incorporates multiple probability measures.

3.3. Two-Stage Stochastic WTA Problem with CVaR constraints

In this section we consider a more complex, but also more realistic two-stage Stochastic WTA problem, where the uncertain parameter is the number of targets to be destroyed.

This problem is more realistic since it models the effect of target discovery as being dynamic; that is, not all targets are known at any single instance of time. To address this type of uncertainty, we need to modify notations.

Consider I weapons are deployed in some bounded region of interest in interval of time T with the goal of finding targets and then, once found, attacking those targets. If we delay all assignments of weapon shots to targets until the final time T, then we have a deterministic, “static” WTA problem as in (3a)–(3e). If, on the other hand, we assume that weapons have at least 2 opportunities to shoot during the interval T, then the WTA problem is dynamic. In the later case we have the opportunity to avoid expending all our shots at targets discovered early in T by explicitly modeling the number of undiscovered targets in the objective function.

Assume that K now represents the number of *categories* of targets (the targets may be categorized, for example, by their importance, vulnerability, etc).

We will assume the problem has 2 stages. That is, at any given time, we may always partition all targets into those thus far determined and those that we conjecture to exist but have not yet found. Our conjecture may be based on evidence obtained by prior reconnaissance of the region of interest. At some arbitrary time $0 < \tau < T$ assume that there are n_k detected targets and η_k undetected targets in each category $k = 1, ..., K$. Thus we have two clearly identified stages in our problem: in the first stage one has to destroy the targets known at time τ, in the second stage one must destroy the targets that we conjecture will be found by time T. In other words, one needs to make an assignment of weapons that will allow for the destruction of the targets known at time τ while reserving enough munition capacity for destroying the targets we expect to find in $\tau < t < T$.

Setup of the two-stage stochastic WTA problem can be considered as a part of a moving horizon or quasi-multistage stochastic WTA algorithm, where the WTA problem with many time periods is solved by recursive application of a two-stage algorithm (Murphey, 1999).

To simplify the problem setup, we remove the constraint on the number of targets a single weapon can attack (the second and third constraints in problems (3a)), since this constraint makes the problem combinatorial. Also, we assume that the probabilities p_{ik} are known (*not* random), so that the only stochastic parameters in the two-stage SWTA problem are the numbers of undetected (second-stage) targets η_k, $k = 1, ..., K$.

We model the uncertainty in the number of targets at the second stage, by we introducing a scenario model, where under scenario $s \in \{1, ..., S\}$ there are $\eta_k(s) = \eta_{ks}$ undetected targets in category k.

The first- and second-stage decision variables are defined as follows:

x_{ik} is the number of munitions to be fired by weapon i at a single target in category k during the first stage;

$y_{ik}(s)$ is the number of munitions to be fired by weapon i at a single target in category k during the second stage scenario s.

Note that the same decision is made for all targets within a category, i.e., once weapon i fires, say, 2 missiles at a specific target in category k, it must fire 2 missiles at every other target in this category.

The recursive formulation of the two-stage stochastic WTA problem is

$$\min \left\{ \sum_{k=1}^{K} \sum_{i=1}^{I} n_k c_{ik} x_{ik} + E_\eta[Q(x,\eta)] \right\} \tag{9a}$$

subject to

$$\sum_{k=1}^{K} n_k \, x_{ik} \leq m_i, \quad i = 1, ..., I, \tag{9b}$$

$$\sum_{i=1}^{I} \ln(1-p_{ik}) \, x_{ik} - \ln(1-d_k) \leq \varepsilon_{1k}, \quad k = 1, ..., K, \tag{9c}$$

$$\sum_{k=1}^{K} \varepsilon_{1k} \leq C, \tag{9d}$$

$$x_{ik}, \; \varepsilon_{1k} \geq 0, \quad i = 1, ..., I, \quad k = 1, ..., K,$$

where the recourse function $Q(x, \eta)$ is the solution of the problem

$$Q(x,\eta) = \min_y \left\{ \sum_{k=1}^{K} \sum_{i=1}^{I} \eta_k(s) \, c_{ik} \, y_{ik}(s) + M \sum_{i=1}^{I} \delta_i \right\} \tag{10a}$$

subject to

$$\sum_{k=1}^{K} (n_k x_{ik} + \eta_k(s) \, y_{ik}(s)) \leq m_i + \delta_i, \quad \forall \, i, \tag{10b}$$

$$\sum_{i=1}^{I} \ln(1-p_{ik}) \, y_{ik}(s) - \ln(1-d_k) - \zeta_k \leq w_k(s), \; \forall \, k, \, s, \tag{10c}$$

$$\zeta_k + (1-\alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_k(s) \leq \varepsilon_{2k}, \quad \forall \, k, \tag{10d}$$

$$\sum_{k=1}^{K} (\varepsilon_{1k} + \varepsilon_{2k}) \leq C, \tag{10e}$$

$$y_{ik}(s), \, \delta_i \in \mathcal{Z}^+, \quad w_k(s), \, \varepsilon_{2k} \geq 0, \quad \zeta_k \in \mathcal{R}, \quad M \gg 1.$$

Let us discuss the recourse problem (9a)–(10e). As before, we minimize the total cost of the mission. The first constraint (9b) is the munitions capacity constraint. The second constraint, (9c), allows a first-stage target in category k to survive with (small) error ε_{1k}, and the third constraint (9d) bounds the sum of errors ε_{1k} by some (small) constant C.

In the recourse function (10a) the first constraint (10b) requires the weapon i to not exceed its munitions capacity while destroying the first- and second-stage targets. The possible infeasibility of the munitions capacity constraint can be relaxed using auxiliary variables δ_i that enter the objective function with cost coefficient $M \gg 1$. The second and third constraints (10c)-(10d) form a CVaR constraint that controls the failure of destroying second-stage targets with the prescribed probabilities d_k. Similarly to the deterministic constraint in (9a), CVaR of failure to destroy a second-stage target in category k is bounded by (small) error variable ε_{2k}. The total sum of errors ε_{1k} and ε_{2k} at both stages is bounded by small constant C, which makes possible a tradeoff between the degree of mission accomplishment at the first and second stages.

The extensive form of the two-stage SWTA problem (9a)–(10a) is

$$\min \left\{ \sum_{k=1}^{K} \sum_{i=1}^{I} n_k \, c_{ik} \, x_{ik} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{i=1}^{I} \eta_{ks} \, c_{ik} \, y_{ik}(s) + M \sum_{i=1}^{I} \delta_i \right\} \quad (11)$$

subject to

$$\sum_{k=1}^{K} (n_k \, x_{ik} + \eta_{ks} \, y_{ik}(s)) \leq m_i + \delta_i, \quad \forall\, i,\, s,$$

$$\sum_{i=1}^{I} \ln(1 - p_{ik}) \, x_{ik} - \ln(1 - d_k) \leq \varepsilon_{1k}, \quad \forall\, k,$$

$$\sum_{i=1}^{I} \ln(1 - p_{ik}) \, y_{ik}(s) - \ln(1 - d_k) - \zeta_k \leq w_{ks}, \quad \forall\, k,\, s,$$

$$\zeta_k + (1 - \alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_{ks} \leq \varepsilon_{2k}, \quad \forall\, k,$$

$$\sum_{k=1}^{K} (\varepsilon_{1k} + \varepsilon_{2k}) \leq C,$$

$$x_{ik},\, y_{ik}(s),\, \delta_i \in \mathcal{Z}^+, \quad w_{ks},\, \varepsilon_{1k},\, \varepsilon_{2k} \in \mathcal{R}^+ \quad \zeta_k \in \mathcal{R}, \quad M \gg 1.$$

The two-stage stochastic WTA problem is also a MIP problem.

4. Numerical results

In this section we present and discuss numerical results obtained for both one-stage and two-stage stochastic WTA problems. The algorithms for solving deterministic, one- and two-stage stochastic WTA problems were implemented in C++, and we used CPLEX 7.0 Callable Library to solve the corresponding IP and MIP problems. We used simulated data (sets of weapons and targets, the corresponding costs and probabilities etc.) for testing the implemented algorithms.

4.1. Single-stage deterministic and stochastic WTA problems

For the deterministic and one-stage stochastic WTA problems we used the following data:

- 5 targets ($K = 5$)
- 5 weapons, each with 4 shots ($I = 5,\ \ m_i = 4$)
- any weapon can attack any target ($t_i = 5$),
- probabilities p_{ik} and costs c_{ik} depend only on the weapon index i: $p_{ik} = p_i$, $c_{ik} = c_i$
- all targets have to be destroyed with at least probability 95% ($d_k = 0.95$)
- the confidence levels α_k in CVaR constraint are 0.90
- there are 20 scenarios ($S = 20$) for probabilities $p_{ik}(s)$ in the one-stage SWTA problem; all scenarios are equally probable.

According to the aforementioned, we used simulated data for probabilities p_{iks} and costs c_{ik}. It was assumed that probabilities $p_{iks} = p_{is}$ are uniformly distributed random variables, and the Fig. 4.1 displays the relation between the cost of missile of weapon i and its efficiency (i.e., probability to destroy a target):

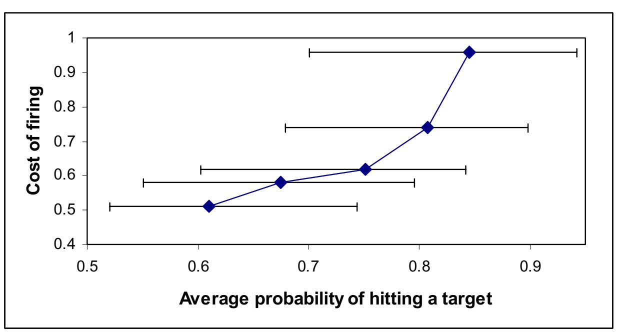

Figure 1.3. Dependence between the cost and efficiency for different types of weapons in one-stage SWTA problem (8) deterministic WTA problem (3a).

On this graph, diamonds represent the average probability of destroying a target by firing one shot from weapon i, and the horizontal segments represent the support for random variable $p_{ik}(\xi) = p_i(\xi)$. The average probabilities

$$\overline{p}_{ik} = \frac{1}{S}\sum_{s=1}^{S} p_{iks}$$

were used for p_{ik} in the deterministic problem (3a).

The efficiency and cost of weapons 1 to 5 increase with the index of weapon, i.e., Weapon 1 is the least efficient and cheapest, whereas Weapon 5 is the most precise, but also most expensive one.

Tables 1.1 and 1.2 represent the optimal solutions (variables x_{ik}) of the deterministic and one-stage stochastic WTA problems.

Table 1.1. Optimal solution of the deterministic WTA problem (3a)

Target	T1	T2	T3	T4	T5	Total shots
Weapon 1	0	2	1	0	1	4
Weapon 2	0	1	2	0	0	3
Weapon 3	1	0	0	1	1	3
Weapon 4	1	0	0	1	1	3
Weapon 5	0	0	0	0	0	0

Table 1.2. Optimal solution of the one-stage stochastic WTA problem (6), (8)

Target	T1	T2	T3	T4	T5	Total shots
Weapon 1	0	1	1	0	1	3
Weapon 2	0	0	1	1	1	3
Weapon 3	2	0	0	1	0	3
Weapon 4	0	1	1	0	1	3
Weapon 5	1	1	0	1	0	3

One can observe the difference in the solutions produced by deterministic and stochastic WTA problems: the deterministic solution does *not* use the most expensive and most precise Weapon 5, whereas the stochastic solution of problem (8) with CVaR constraint uses this weapon. It means that the CVaR-constrained solution of problem (8) represents a more expensive but safer decision.

On a different dataset, we obtained a similar result: the optimal solution of the stochastic problem with CVaR constraints did not use the cheapest and the most unreliable weapon, whereas the deterministic solution used it.

We have also performed testing of the deterministic solution under different scenarios. The deterministic solution failed to destroy more than one target under 13 of 20 scenarios.

This example highlights the importance of using risk management procedures in military decision-making applications involving uncertainties.

4.2. Two-Stage Stochastic WTA Problem

For the two-stage stochastic WTA problems we used the following data:

- 3 categories of targets ($K = 3$)
- 4 weapons, each with 15 shots ($I = 4, \; m_i = 15$)
- probabilities p_{ik} and costs c_{ik} depend only on the weapon index i: $p_{ik} = p_i, \; c_{ik} = c_i$
- all targets have to be destroyed with probability 95% ($d_k = 0.95$)
- the confidence levels α_k in CVaR constraint are equal 0.90
- there are 15 scenarios ($S = 15$) for the number of undetected targets η_{ks} (for each k, the number of undetected targets η_{ks} is a random integer between 0 and 5); all scenarios are equally probable.

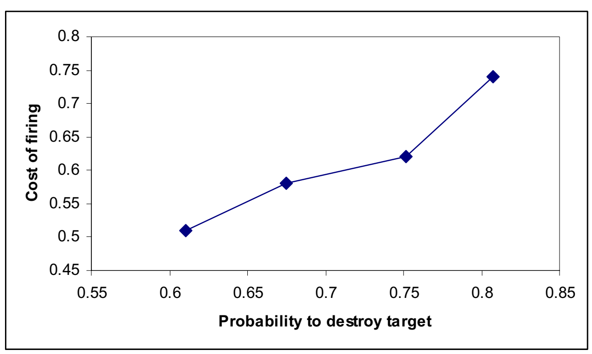

Figure 1.4. Dependence between the cost and efficiency for different types of weapons in two-stage SWTA problem (11).

For the probabilities p_{ik} in the two-stage problem, we used the first four average probabilities from the deterministic WTA problem, and the efficiency-cost dependence is shown in Fig. 4.2.

Tables 1.3 to 1.5 illustrate the optimal solution of the problem (11). Table 1.3 contains the first-stage decision variables x_{ik}, and Tables 1.4 and 1.5 display the second-stage variables $y_{ik}(s)$ for scenarios $s = 1$ and $s = 2$, just for illustrative purposes.

Similarly to the analysis of the one-stage stochastic WTA problem, we compared the scenario-based solution of problem (11) with the solution of the "deterministic two-stage" problem, where the number of second-stage targets in each category is taken as the average over 15 scenarios. The comparison shows that the solution based on the expected information leads to significant munitions shortages in 5 of 15 (i.e., 33%) scenarios, and consequently to failing the mission at the second stage. Recall from the analysis of the one-stage SWTA problem that the solution based on the expected information also exhibited poor robustness with respect to different scenarios. Indeed, solutions that use only the *expected* information, are supposed to perform well *on average*, or *in the long run*. However, in military applications there is *no long run*, and therefore such solutions may not be robust with respect to *many possible scenarios*.

Table 1.3. First-stage optimal solution of the two-stage stochastic WTA problem

Category	K1	K2	K3
# of detected targets	3	5	2
Weapon 1	0	0	0
Weapon 2	0	0	0
Weapon 3	1	1	1
Weapon 4	1	1	1

Thus, solutions of both one-stage and two-stage SWTA problems confirm the general conjecture on the potential importance of exploiting stochastic models and risk management in military applications.

5. Conclusions

We have presented an approach to managing risk in stochastic environments, where distributions of stochastic parameters are uncertain. This approach is based on the methodology of risk management with Conditional Value-at-Risk risk measure developed by Rockafellar and Uryasev, 2000, 2001. Although the presented approach has been used

Table 1.4. First-stage optimal solution of the two-stage stochastic WTA problem (11) for the first scenario

Category	K1	K2	K3
# of undetected targets	1	4	2
Weapon 1	0	0	2
Weapon 2	0	0	1
Weapon 3	1	1	0
Weapon 4	1	1	0

Table 1.5. Second-stage optimal solution of the two-stage stochastic WTA problem (11) for the second scenario

Category	K1	K2	K3
# undetected of targets	3	5	3
Weapon 1	2	0	2
Weapon 2	1	0	1
Weapon 3	0	1	0
Weapon 4	0	1	0

to solve one-stage and two-stage stochastic Weapon-Target Assignment problems, it is quite general and can be applied to wide class of problems with risks and uncertainties in distributions. Among the directions of future research we emphasize consideration of a stochastic WTA problem in NLP formulation, where the damage to the targets is maximized while constraining the risk of false target attack.

6. Appendix. Formal definition of CVaR

Consider a loss function $L(x, \xi)$ depending on a decision vector x and a stochastic vector ξ, and its cumulative distribution function (c.d.f.) $\Psi(x, \zeta)$:

$$\Psi(x, \zeta) = \mathsf{P}[L(x, \xi) \leq \zeta].$$

Then the α-VaR (Value-at-Risk at confidence level α) function $\zeta_\alpha(x)$ corresponding to loss $L(x, \xi)$ is

$$\zeta_\alpha(x) = \min_{\zeta \in \mathcal{R}} \{\Psi(x, \zeta) \geq \alpha\}.$$

Approximately, Conditional Value-at-Risk with confidence level α (α-CVaR) is defined as conditional expectation of losses exceeding the α-

VaR level. This definition is correct for continuously distributed loss functions. However, for loss functions with general non-continuous distributions (including discrete distributions) the α-CVaR function $\phi_\alpha(x)$ is defined as the expected value of random variable z_α (Rockafellar and Uryasev, 2001):

$$\phi_\alpha(x) = \mathsf{CVaR}_\alpha[L(x,\xi)] = \mathsf{E}[z_\alpha],$$

where c.d.f. $\Psi_{z_\alpha}(x,\zeta)$ of z_α has the form

$$\Psi_{z_\alpha}(x,\zeta) = \begin{cases} 0, & \zeta < \zeta_\alpha(x), \\ [\Psi(x,\zeta) - \alpha]/[1-\alpha], & \zeta \geq \zeta_\alpha(x). \end{cases}$$

In (Rockafellar and Uryasev, 2001), it was shown that α-CVaR can be expressed as a convex combination of α-VaR and conditional expectation of losses strictly exceeding α-VaR:

$$\phi_\alpha(x) = \lambda_\alpha(x)\,\zeta_\alpha(x) + [1 - \lambda_\alpha(x)]\,\phi_\alpha^+(x), \tag{12}$$

where

$$\phi_\alpha^+(x) = \mathsf{E}[L(x,\xi) \mid L(x,\xi) > \zeta_\alpha(x)], \tag{13}$$

which is also known as "upper CVaR" or Expected Shortfall, and

$$\lambda_\alpha(x) = [\Psi(x,\zeta_\alpha(x)) - \alpha]/[1-\alpha], \quad 0 \leq \lambda_\alpha(x) \leq 1.$$

Similar to (13), another percentile risk measure, called "lower CVaR", or CVaR$^-$, can be defined:

$$\phi_\alpha^-(x) = \mathsf{E}[L(x,\xi) \mid L(x,\xi) \geq \zeta_\alpha(x)]. \tag{14}$$

Then, as it was shown in (Rockafellar and Uryasev, 2001), the introduced risk functions satisfy the following inequality:

$$\zeta_\alpha(x) \leq \phi_\alpha^-(x) \leq \phi_\alpha(x) \leq \phi_\alpha^+(x).$$

We also note that in the case when behavior of the stochastic parameter ξ can be represented by a scenario model $\{\xi_s,\ s = 1,...,S\}$ with equally probable scenarios ($\pi_s = 1/S$), the concept of CVaR has especially simple and transparent interpretation. Namely, if (for a fixed x) the scenarios $\xi_1,...,\xi_S$ are indexed such that $L(x,\xi_1) \leq \ldots \leq L(x,\xi_S)$, then α-CVaR equals the weighted average of losses for the $\lceil(1-\alpha)S\rceil$ worst scenarios:

$$\phi_\alpha(x) = \frac{1}{(1-\alpha)\,S}\left[(s_\alpha - \alpha S)\,L(x,\xi_{s_\alpha}) + \sum_{s=s_\alpha+1}^{S} L(x,\xi_s)\right],$$

where number s_α is such that

$$S - s_\alpha \leq (1-\alpha)S < S - s_\alpha + 1.$$

References

Acerbi, C., and D. Tasche (2001) "On the Coherence of Expected Shortfall", *Working paper, download from www.gloriamundi.org.*

Artzner, P., Delbaen, F., Eber, J.-M., and D. Heath (1999) "Coherent Measures of Risk", *Mathematical Finance*, 9, 203–228.

denBroeger, G. G., Ellison R. E., and L. Emerling (1959) " On optimum target assignments", *Operations Research*, 7, 322–326.

Golodnikov, A., Knopov, P., Pardalos, P., and S. Uryasev (2000) "Optimization in the Space of Distribution Functions and Applications in the Bayesian Analysis", *Research Report 2000-6, ISE Dept., Univ. of Florida.*

Manne, A. S. (1958) "A Target Assignment Problem", *Operations Research*, 6, 346–351.

Murphey, R. (1999) "An Approximate Algorithm For A Stochastic Weapon Target Assignment Problem", In: *Approximation and Complexity in Numerical Optimization: Continuous and Discrete Problems* (Pardalos, Ed.)

Rockafellar, R. T., and S. Uryasev (2000) "Optimization of Conditional Value-at-Risk", *Journal of Risk*, **2**, 21–41.

Rockafellar, R. T., and S. Uryasev (2001) "Conditional Value-at-Risk for General Loss Distributions", *Research Report 2001-5. ISE Dept., Univ. of Florida.*