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A sample-path approach to optimal position liquidation

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Abstract We consider the problem of optimal position liquidation where the expected cash flow stream due to transactions is maximized in the presence of temporary or permanent market impact. A stochastic programming approach is used to construct trading strategies that differentiate decisions with respect to the observed market conditions, and can accommodate various types of trading constraints. As a scenario model, we use a collection of sample paths representing possible future realizations of state variable processes (price, trading volume etc.), and employ a heuristical technique of sample-path grouping, which can be viewed as a generalization of the standard nonanticipativity constraints.

Keywords Optimal trading · Market impact · Stochastic programming · Sample paths

Introduction

This paper presents a numerical technique for optimal transaction implementation, e.g., determining the best way to sell (buy) a specified number of shares on the market within a prescribed period of time.

The main challenge of constructing optimal trading strategies is commonly found in preventing or minimizing losses caused by the so-called *market impact*, or *price slippage*, which affect payoffs during market transactions. The market impact phenomenon manifests itself by adverse price movements during trades and is caused by disturbances of market equilibrium. As a rather simplistic explanation, consider an investor executing a market order to sell a block of shares. If at the moment of execution no one is willing to buy this block for

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the current market price, the seller will be forced to offer a lower price per share in order to accomplish the transaction, thereby suffering a loss due to the *market impact*. In reality the function of price adjustments during transactions is performed by market intermediaries.

Traditionally, two types of market impact are considered: *temporary* and *permanent*. Permanent market impact denotes the changes in prices that are caused by the investor's trades and persist throughout the entire period of the investor's trading activity. If price fluctuations caused by the investor's transaction are unobservable by the time of her next trade, the market impact is said to be temporary. Naturally, the magnitude of market impact and, consequently, the loss due to adverse price movements, depend not only on the size of trades, but also on the time windows between the transactions. For a detailed discussion of the issues related to market impact, see, among others, Chan and Lakonishok (1995), Kein and Madhavan (1995), Kraus and Stoll (1972). Some of the latest developments on the optimal transaction implementation and optimal trading policies are reflected in papers by Bertsimas and Lo (1998), Bertsimas, Lo, and Hummel (1999), Rickard and Torre (1999), Almgren and Chriss (2000), and Almgren (2003).

Bertsimas and Lo (1998) employed a dynamic programming approach for devising optimal trading strategies that minimize the expected cost of trading a block of $\bar{S}$ shares within a fixed number of periods T. They derived analytical expressions for best-execution strategies in the standard framework of discrete random walk models, under assumption that market impact is linear in the number of shares traded. To gain an insight on how information component can influence the optimal strategy, authors introduced a serially correlated "information" variable in the price process of the security. Extension of this methodology to the case of multiple assets and optimal execution for portfolios is presented in Bertsimas, Lo, and Hummel (1999).

Almgren and Chriss (2000) constructed risk-averse trading strategies using the classical Markowitz mean-variance methodology. With permanent and temporary market impact functions being also linear in the number of shares traded, the risk of the trading strategy is associated with the volatility of the market value of the position. A continuous-time approximation of this approach, as well as non-linear and stochastic temporary market impact functions, was considered by Almgren (2003). An application of fuzzy set theory to optimal transaction execution, which helps to mimic non-rational "human" behaviour of traders, is discussed in Rickard and Torre (1999).

However, a common drawback of the discussed methodologies can be seen in the inability of the trading strategy to respond dynamically to realized or changed market conditions. Instead of executing a prescribed sequence of trades based on the parameters of the security's price process, we would like to construct a trading strategy that differentiates decisions with respect to both actual realizations of market conditions at the moment of transaction and the past history of decisions. Also, the trading strategy should be able to incorporate different types of constraints such as investor's risk preferences, institutional regulations, etc.

In view of the above, the objective of the present research endeavor is to develop a dynamic optimal transaction implementation strategy that (1) allows for a differentiated response to varying market conditions; (2) is applicable under different forms of market impact; (3) allows for incorporation of different types of constraints; and (4) is numerically tractable and capable of accepting a range of historical or simulated data as an input.

This paper considers the problem of optimal position liquidation where the expected cash flow stream due to selling a block of shares in the market is maximized. The problem is formulated and solved in a stochastic programming framework so as to obtain multi-stage decision-making algorithms with appropriate response to different realizations of uncertainties at each time moment. A key feature of our approach is a sample-path scenario model that represents the uncertain price process of the security by a set of its possible future trajectories (*sample paths*). This scenario model allows for improved computational efficiency of the corresponding stochastic programming problems; however, special care should be taken to avoid anticipativity of the resulting solutions (for a discussion of non-anticipativity in stochastic programs see, e.g., Birge and Louveaux (1997)). To this end, we propose a heuristical technique of sample-path grouping, which can be viewed as a generalization of the standard non-anticipativity constraints (Birge and Louveaux, 1997), and can be flexibly used in practice.

The sample-path scenario model is a relatively new technique in the realm of multi-stage decision making problems, where the dominant scenario models are the classical multinomial trees or lattices. Sample-path based simulation models have been employed for pricing of American-style options (see Titley, 1993; Boyle, Broadie, and Glasserman, 1997; Broadie and Glasserman, 1997; Carriere, 1996; Barraquand and Martineau, 1995; and others), where the optimal decision policy consists of a single binary decision, i.e., exercise-or-not the security contract. In this sense, the problem of optimal transaction execution is more complex since the optimal strategy is a sequence of non-binary decisions.

The rest of the paper is organized as follows. The next section introduces a general formulation of the optimal liquidation problem, definitions of market impact, etc. Section 2 discusses the optimal position liquidation under temporary and permanent market impacts, and illustrates incorporation of risk constraints in the developed trading strategies. Finally, Section 3 presents a case study and numerical results. The Appendix contains proofs of the propositions presented in the text.

1 General definitions and problem statement

In this section we introduce formal mathematical definitions and develop several formulations for the optimal closing problem. The formal setup for the optimal liquidation problem is as follows. Suppose that at time $t = 0$ there is an open position in some financial instrument (stock, bond, option etc.), which has to be liquidated (closed) within the predefined time interval $0 \le t \le T$. Assume also that trades are only allowed at the specified time moments $t = 1, 2, \dots, T$ (integer indexing is used for briefness of notation only; by writing $t = 0, 1, 2, \dots$, we understand $t = t_0, t_1, t_2, \dots$, with time moments $t_0, t_1, t_2, \dots$, not necessarily equally spaced). Information on future market conditions at $t = 1, 2, \dots, T$ is represented by a set of J sample paths (Fig. 1a). The objective is to generate an optimal trading policy that maximizes the expected cash flow stream incurred from liquidating the position. Throughout the paper, it is implicitly assumed that a long position is being liquidated.

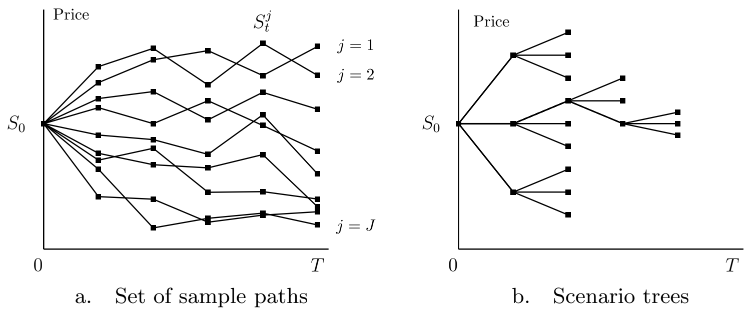


Fig. 1 Sample paths and and scenario trees

1.1 Sample-path scenario model

Traditional models for evolution of uncertainties in multi-stage decision-making problems are represented by scenario trees (Fig. 1b) and lattices. These classical concepts have proved to be effective tools in dealing with multistage problems, especially within analytical framework. However, in many real-life financial applications that require solving large-scale optimization problems, usage of the tree- or lattice-based scenario models may lead to considerable computational difficulties, known as the "curse of dimensions." One of the most popular alternative approaches is to model the uncertain future as a collection of sample paths, each representing a possible future price trajectory of a financial instrument or group of instruments (Fig. 1a). This type of scenario model is supposed to overcome the "curse of dimensions" that often defeats large-scale instances of optimization models based on scenario trees or lattices.

A significant deficiency of the multinomial trees or lattices is a poor balance between randomness and the tree/lattice size. A small number of branches per node in a tree/lattice clearly makes a poor approximation of the uncertain future, whereas increasing of this parameter results in exponential growth of the overall number of scenarios in the tree.

In contrast, a sample-path scenario model represents the uncertain value of stochastic parameter at each time by a (large) number of sample points (nodes) belonging to different sample paths of the scenario set. Increasing the number of nodes at each time step for better accuracy results in a *linear* increase of the number of sample paths in the set. Similarly, increasing the number of time periods in the model leads to a linear increase of the number of nodes, as opposed to *exponential* growth in the number of nodes in scenario trees.

Besides superior scalability, sample-path concept allows for effortless incorporation of historical data into the scenario model, which is a vital attribute from a practical standpoint. There has been an increasing interest to the usage of sample paths for description of uncertain market environment in problems of finance and financial engineering during the recent years. For the most part, this approach has been employed in the area of pricing of derivatives (Titley, 1993, Boyle, Broadie, and Glasserman, 1997; Broadie and Glasserman, 1997; Carriere, 1996; Barraquand and Martineau, 1995; etc.). Recently, the sample-path framework was applied to solving dynamic asset and liability management problems (Bogentoft, Romeijn, and Uryasev 2001; Hibiki, 1999, 2001). In the present paper we consider the concept of sample-path scenario sets and corresponding optimization techniques in application to a problem of optimal transaction execution in presence of market imperfections and friction.

1.2 A generic problem formulation

Let $\mathfrak{S}$ be a collection of sample paths

$$\mathfrak{S} = \left\{ \left(\mathbf{S}_0, \mathbf{S}_1^j, \mathbf{S}_2^j, \dots, \mathbf{S}_T^j \right) \mid j = 1, \dots, J \right\},$$

where each term $\mathbf{S}_t^j$ stands for, generally, a vector of relevant market parameters, such as the mid-price of the security S_t^j, bid-ask spread s_t^j, volume V_t^j, etc. at time t according to a sample path j:

$$\mathbf{S}_t^j = \left(S_t^j, s_t^j, V_t^j, \dots \right).$$

Vector $\mathbf{S}_t^j$ may also include other information observable at time t, e.g., major financial indices, interest rates etc. In the simplest case, $\mathbf{S}_t^j$ may only contain the price S_t^j of the security.

We define a *trading strategy,* corresponding to the sample-path collection $\mathfrak{S}$, as a set

$$\Xi = \left\{ \left(\xi_0, \xi_1^j, \ldots, \xi_T^j\right) \mid 1 = \xi_0 \geq \xi_1^j \geq \cdots \geq \xi_T^j = 0, \ j = 1, \ldots, J \right\}, \tag{1}$$

where ξ_t^j is the normalized value of the position at time t on path j. The instant proceeds incurred from transaction at time t are determined by the *payoff function* $p_t(\cdot)$, whose general form is

$$p_t\left(\xi_\tau^j ; \mathbf{S}_\tau^j \,\middle|\, \tau \leq t\right), \quad j = 1, \ldots, J, \quad t = 1, \ldots, T.$$

Let the payoff function $p_t(\cdot)$ incorporate discounting and transaction costs, market slippage etc. The explicit form of the function $p_t(\cdot)$ and its impact on properties of the optimal liquidation problem is discussed below; now we stress that payoff at time t cannot depend on the values of $\mathbf{S}_\tau^j$ and ξ_τ^j for $\tau > t$.

The objective of our trading strategy is to maximize the expected cash flow stream incurred from selling the asset:

$$\max_{\Xi} \ Z = \mathsf{E}_{\mathfrak{S}} \left[\sum_{t=1}^{T} p_t\left(\xi_\tau^j ; \mathbf{S}_\tau^j \,\middle|\, \tau \leq t\right) \right]. \tag{2}$$

Problem (2) is a stochastic programming formulation of the optimal closing problem based on the sample-path scenario model. Here Ξ denotes the set of all possible trading strategies (1), and $\mathsf{E}_{\mathfrak{S}}$ is the expectation operator defined on $\mathfrak{S}$ (ordinarily, all paths are assumed to be equiprobable, thus $\mathsf{E}_{\mathfrak{S}}$ may be replaced by average over J paths). It will be seen later that the generic formulation (2) is far from being perfect; but before diving into further details, we discuss the form and properties of the payoff function $p_t(\cdot)$ in (2), which has a major impact on the properties of the problem of optimal closing in general.

1.3 Modeling the market impact

Generally, the payoff on path j at time t is a function of the preceding series $\mathbf{S}_\tau^j$ and position values ξ_τ^j:

$$p_t(\cdot) = p_t\left(\xi_\tau^j ; \mathbf{S}_\tau^j \mid \tau \leq t\right) \triangleq p_t\left(\xi_0, \xi_1^j, \ldots, \xi_t^j ; \ \mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_t^j\right).$$

Dependence of the payoff function $p_t(\cdot)$ on the preceding decisions $\xi_0, \xi_1^j, \ldots, \xi_{t-1}^j$ constitutes the generally non-linear effect of *permanent market impact*, when the trading activity of the market player contributes to changes in prices that do not vanish during the trading period. If the price at current moment t is unaffected by the preceding trades, but does depend on the size of current transaction, it is said that only *temporary market impact* is present. First, we consider the temporary market impact as it allows for computationally more tractable formulations of the optimal liquidation problem.

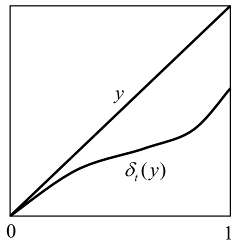


Fig. 2 Impact function $\delta_t(\cdot)$

Temporary market impact Let the payoff $p_t(\cdot)$ at time t depend only on the current transaction size $\Delta\xi_t$

$$p_t(\cdot) = p_t\big(\Delta\xi_t^j;\, \mathbf{S}_\tau^j \,\big|\, \tau \le t\big), \quad \Delta\xi_t^j = \xi_{t-1}^j - \xi_t^j.$$

This form of the payoff function is appropriate when the changes in the price caused by the transaction at time t are negligibly small by the next time moment, $t+1$. Assume further that function $p_t(\cdot)$ admits the representation

$$p_t\big(\Delta\xi_t^j\,;\, \mathbf{S}_\tau^j \,\big|\, \tau \le t\big) = S_t^j\, \delta\big(\Delta\xi_t^j\,;\, \mathbf{S}_\tau^j \,\big|\, \tau \le t\big), \tag{3}$$

where the price S_t^j is always positive: $S_t^j > 0$. The term $\delta(\Delta\xi_t^j\,;\, \mathbf{S}_\tau^j \mid \tau \le t)$ in (3) captures the effects of market friction. It may depend parametrically on the information observable at path j up to time t, e.g., volume V_t^j, prices $S_0, \dots, S_t^j$, etc.[1] To lighten the notation, we suppress the term $\mathbf{S}_\tau^j \mid \tau \le t$ in $\delta(\cdot)$ and introduce subscript t:

$$\delta\big(\Delta\xi_t^j\,;\, \mathbf{S}_\tau^j \,\big|\, \tau \le t\big) \triangleq \delta_t\big(\Delta\xi_t^j\big).$$

As a function of the trade size $\Delta\xi = y$, $\delta_t(y)$ satisfies

(i) $\delta_t\colon [0,1] \mapsto [0,1], \quad \delta_t(0) = 0.$

In other words, function $\delta_t(y)$ determines the fraction of the quoted price S_t received per share when selling portion y of the position. In perfect frictionless market, the price per share received during liquidation equals to the quoted price, therefore in frictionless market $\delta_t(y) = y$, $y \in [0,1]$. We assume that in the presence of temporary market impact, proportional transaction costs[2] etc., $\delta_t(\cdot)$ further satisfies

(ii) $\delta_t(y) \le y$ for all $y \in [0,1]$;
(iii) $\delta_t(y_1) < \delta_t(y_2)$ for all $0 \le y_1 < y_2 \le 1$;
(iv) $\delta_t(y)$ is continuous on $[0,1]$.

Condition (ii) states that payoff in market with friction is always less than that in frictionless market, condition (iii) ensures that larger trades lead to higher revenues (see Fig. 2). Finally, condition (iv) ascertains that infinitesimal changes in the transaction size do not incur finite changes in the payoff. We do not consider fixed transaction costs, since they have diminished

[1] The current volume of trades V_t may be defined as the number of trades between $t-1$ and t.

[2] We neglect fixed transaction costs and fees, which are usually much smaller comparing to the losses caused by market impact.

considerably during the past several years, and usually are much smaller comparing to the losses caused by market impact.

Summarizing the aforesaid, in the case of temporary market impact we write the objective function (2) in the form

$$Z = \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T} S_t^j \delta_t\big(\Delta\xi_t^j\big), \tag{4}$$

where $S_t^j = \text{const} > 0$, and $\delta_t(\cdot)$ satisfies conditions (i)–(iv).

Permanent market impact A more complex model of market response to the trading activity includes permanent market impact as the dependence of the current market prices on the preceding transactions of the market player. To model the permanent market impact, we assume the following price dynamics for S_t^j

$$\check{S}_t^j = S_t^j - S_t^j \gamma_t\big(\Delta\xi_t^j\big) - \sum_{\tau=1}^{t-1} S_\tau^j\, \theta_\tau\big(\Delta\xi_\tau^j\big), \tag{5}$$

where $S_\tau^j, \tau = 1, \ldots, t$, is the "undisturbed" price trajectory that would realize in the absence of the investor's trades, and $\check{S}_t^j$ is the actual price per share received by the investor for liquidating portion $\Delta\xi_t^j$ of the position at moment t on path j. Functions $\gamma_t(\Delta\xi)$ and $\theta_t(\Delta\xi)$ are equal to the percentage drop in market price due to the temporary and permanent market impact, correspondingly. Similarly to the above notation $\delta_t(\cdot)$, functions $\gamma_t(\cdot)$ and $\theta_t(\cdot)$ may contain, in general, values $\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_t^j$ as parameters:

$$\gamma_t\big(\Delta\xi_t^j\big) \triangleq \gamma\big(\Delta\xi_t^j\,;\,\mathbf{S}_\tau^j \,\big|\, \tau \le t\big), \quad \theta_t\big(\Delta\xi_t^j\big) \triangleq \theta\big(\Delta\xi_t^j\,;\,\mathbf{S}_\tau^j \,\big|\, \tau \le t\big).$$

Using (5), the total payoff attained over a path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_T^j)$ is

$$\begin{aligned} Z = \sum_{t=1}^{T} \check{S}_t^j \Delta\xi_t^j &= \sum_{t=1}^{T} \left\{ S_t^j \left(\Delta\xi_t^j - \Delta\xi_t^j \gamma_t\big(\Delta\xi_t^j\big)\right) - \Delta\xi_t^j \sum_{\tau=1}^{t-1} S_\tau^j \theta_\tau\big(\Delta\xi_\tau^j\big) \right\} \\ &= \sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \Delta\xi_t^j \gamma_t\big(\Delta\xi_t^j\big) - \theta_t\big(\Delta\xi_t^j\big) \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\}. \end{aligned} \tag{6}$$

The first term in braces in (6) attributes to the profit of selling the portion $\Delta\xi_t^j$ of the position in perfect frictionless market. The second and third summands in (6) represent the losses due to the effects of temporary and permanent market impacts, correspondingly. For consistency with the above discussion of temporary market impact, we assume that $\gamma_t(\cdot)$ is such that the function

$$y - y\,\gamma_t(y)$$

satisfies conditions (i)–(iv). This allows us to think of $\gamma_t(\cdot)$ as a non-negative non-decreasing function on $[0,\ 1]$. Similarly, the permanent market impact function $\theta_t(\cdot)$ is assumed to be non-negative and non-decreasing. We suppose that function $\theta_t(\cdot)$ satisfies

$$\theta_t(\Delta\xi) = 0, \quad 0 \le \Delta\xi \le \lambda_t \quad \text{for some} \quad 0 < \lambda_t < 1.$$

This condition reflects the presumption that small enough trades $\Delta\xi_t$ should not cause permanent changes in market prices given the time windows $t_k - t_{k-1}$ between transactions. A plausible form of $\theta_t(\cdot)$ is as follows:

$$\theta_t(\Delta\xi) = \max\{0,\ \varkappa\gamma_t(\Delta\xi - \lambda_t)\}, \tag{7}$$

where $\varkappa = \text{const} \in (0,\ 1]$.

1.4 Non-anticipativity and path grouping

Section 1.2 presented a generic formulation (2) of the optimal closing problem. In this formulation, each sample path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_T^j)$, $j = 1, \ldots, J$, is assigned its own set of decision variables $(\xi_0, \xi_1^j, \ldots, \xi_T^j)$. This leads to a fundamental flaw in model (2), known in stochastic programming as *anticipativity*. As an illustration, consider a special case of problem (2) corresponding to optimal liquidation in frictionless market:

$$\max_{\Xi}\ Z = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\ \Delta\xi_t^j. \tag{8}$$

Problem (8) is obtained from (2) by letting $p_t(\xi_\tau^j; \mathbf{S}_\tau^j \mid \tau \le t) = S_t^j\,\Delta\xi_t^j$ and replacing the expectation operator $\mathsf{E}_{\mathfrak{S}}$ with the average over the set of sample paths. It is easy to see that the optimal solution of (8) is given by

$$\xi_t^j = \begin{cases} 1, & t \le t_j - 1, \\ 0, & t \ge t_j, \end{cases} \quad \text{where} \quad t_j = \arg\max_{t=1,\ldots,T}\left\{S_t^j\right\}, \quad j = 1, \ldots, J. \tag{9}$$

The optimal trading strategy (9) is to *wait until the maximum price across a trajectory is achieved, and then liquidate the position.* Clearly, such a strategy is contingent on the perfect knowledge of the future, i.e., it is *anticipative.*

Broadly speaking, the anticipativity in multi-stage decision making models is the ability to take advantage of the information that is unobservable at the moment of generating the decision. In our example, it is the ability of the algorithm to predict the maximum price over a trajectory.

In multi-stage decision-making problems based on scenario tree models, the anticipativity is circumvented by means of the so-called *non-anticipativity constraints* (see, for example, Birge and Louveaux, 1997). These constraints essentially require that *scenarios that share the same history until some moment t must also share the same sequence of decisions until that moment.* Indeed, the scenario tree model (Fig. 1b) can be viewed as a sample-path set of special form, where several (different) paths coincide until the point where they "branch" into the corresponding number of "leaves." Thus, it is natural to require that decisions on such paths must also coincide until the time when paths "split," which in our case would yield

$$\xi_\tau^{j_1} = \xi_\tau^{j_2}, \quad \tau = 1, \ldots, t \quad \text{for } j_1, j_2, t \quad \text{such that } \mathbf{S}_\tau^{j_1} = \mathbf{S}_\tau^{j_2}, \quad \tau = 1, \ldots, t. \tag{10}$$

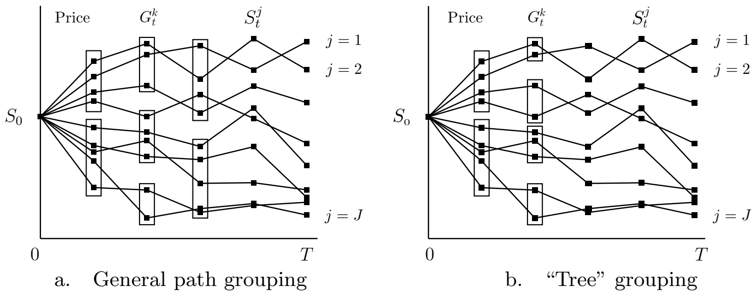


Fig. 3 Path grouping

In a general sample-path scenario model there may be no paths that share the same history, as depicted in Fig. 1b. Consequently, non-anticipativity constraints cannot be stated in form (10). Below we discuss alternative approaches to formulating the non-anticipativity constraints for sample-path scenario models.

Path grouping By analogy with the standard non-anticipativity constraints (10) for scenario trees, one may require identical decisions to be made on scenarios that possess *similar* properties at a given time t. This leads to the idea of *path grouping*, also known as "path bundling" (see similar approaches in Titley, 1993; Hibiki, 1999, 2001; Bogentoft, Romeijn, and Uryasev 2001). At each time t, the J sample paths in the collection $\mathfrak{S}$ are partitioned into K_t groups G_t^k, $k = 1, \ldots, K_t$ (Fig. 3a)

$$\pi(\mathfrak{S}) = \left\{ G_t^k \;\middle|\; \text{for all } \ t = 1, \ldots, T : \ \{1, \ldots, J\} = \bigcup_{k=1}^{K_t} G_t^k, \right. \tag{11}$$
$$\left. G_t^{k_1} \cap G_t^{k_2} = \varnothing, \quad k_1 \neq k_2 \right\},$$

where each group G_t^k contains paths with supposedly similar or approximately equal numerical attributes (price, volume, etc.) at time t. Then, identical decisions are enforced on all paths within a group G_t^k; in our case, for example, the same position level ξ_t^j within a group G_t^k may be required:

$$\xi_t^j = x_t^{k(j,t)} \quad \forall\, j \in G_t^k, \quad t = 1, \ldots, T. \tag{12}$$

Here $x_t^{k(j,t)}$ is a new decision variable equal to the value of position variables ξ_t^j within the group G_t^k, and $k(j,t)$ is the function that returns the index k of set G_t^k containing path j at moment t, according to the partitioning $\pi(\mathfrak{S})$. We denote this as

$$k(j,t) \propto \pi(\mathfrak{S}).$$

Hibiki (1999, 2001) introduced a "tree-like" grouping, where each group G_t^k is sub-partitioned into several groups $G_{t+1}^{k_1}, \dots, G_{t+1}^{k_r}$ at the next time moment, $t+1$ (see Fig. 3b). Formally,

$$\pi_{\text{tree}}(\mathfrak{S}) = \big\{\pi(\mathfrak{S}) \mid \text{for all } \; t = 2, \dots, T, \; k = 1, \dots, K_t \\ \text{there exists } \; k^* \; \text{ such that } \; G_t^k \subseteq G_{t-1}^{k^*}, \; 1 \le k^* \le K_{t-1}, \; K_1 < J\big\}. \tag{13}$$

The last condition in (13), $K_1 < J$, is imposed to exclude the degenerated case of $K_t = J, \; t = 1, \dots, T$, which leads to essentially anticipative solution of type (9). A sample-path collection $\mathfrak{S}$ equipped with tree-like path grouping and constraints (12) can be considered as a generalization of the traditional scenario tree model (Fig. 1a) with non-anticipativity constraints (12). In practice, however, it may not be easy to construct a tree-like partitioning that preserves similarity among paths within a group across all time periods, for an arbitrary collection of sample paths.

We propose a formulation of non-anticipativity constraints for a sample-path scenario model that generalizes the path grouping approach, as compared to Titley (1993) and Hibiki (1999, 2001), in two respects. First, we consider a general path grouping that allows for "path intermixing" (i.e., paths belonging to different groups at time $t-1$ may be assigned to the same group at time t, see Fig. 3a):

$$\exists \; t, k, j_1, j_2 : \quad j_1, j_2 \in G_t^k, \quad G_{t-1}^{k(j_1, t-1)} \neq G_{t-1}^{k(j_2, t-1)}.$$

Second, we allow constraints (12) to have a more general form:

$$\forall j \in G_t^k : \quad \xi_t^j = f\big(x_t^{k(j,t)}, \xi_{t-1}^j\big), \tag{14}$$

where function $f(\cdot)$ is non-constant in $x_t^{k(j,t)}$. As this constraint defines the way the decision variables ξ_t^j are determined, we call (14) the *decision rule.*

The rationale behind the proposed path grouping method and decision rule (14) is as follows. Sample paths assigned to a common group G_t^k by (11) are expected to possess similar, but maybe not identical state parameters (price, volume, etc.) at time t. Hence, it is natural to require the decision variables ξ_t^j corresponding to those paths to take similar, but maybe not exactly the same values. Next, non-anticipativity of the solution under the decision rule (14) is achieved via dependence of the decision variables ξ_t^j within group G_t^k on the variable x_t^k unique to that group. This can be viewed as a generalization of decision rule (12), and, consequently, the standard non-anticipativity constraints (10). For example, in the special case of tree-like path grouping (13) the suggested decision rule (14) is equivalent to (12):

Proposition 1. [3] *If partition* $\pi(\mathfrak{S})$ *satisfies the "tree" grouping condition* (13)*, then the generalized group decision rule* (14) *is equivalent to* (12).

It follows from Proposition 1 that for a sample-path set in the form of a scenario tree (Fig. 1b), the path grouping (11) and decision rule (14) are naturally reducible to the classical non-anticipativity constraints (Birge and Louveaux, 1997).

[3] Proofs of all propositions are furnished in the Appendix.

However, the flexibility provided by the path grouping (11) should be executed cautiously: it is possible to construct a path grouping that does not guarantee non-anticipativity. An example of path grouping that can lead to anticipative solutions of type (9) is served by the extreme case of $K_t = J$ at each time period t (in this case, however, no two variables ξ_t^j can depend on the same x_t^k). On the other extreme, having only one group per period ($K_t = 1$) eliminates anticipativity, but also eliminates the desired dynamic character of the trading strategy. Although it is difficult to derive optimal parameters of path grouping (11) that would preserve both the non-anticipativity and the desired dynamic properties of the solution, it is possible to perform a numerical "calibration" of the path partitioning for a certain class of sample-path sets that will yield non-anticipative strategies of satisfactory flexibility. The particular form of path grouping implemented in this study and its validation are discussed in Section 3.

Likewise, the form of function $f(\cdot)$ in (14) is not arbitrary; along with preserving feasibility etc., it has to satisfy certain conditions implied by the nature of the problem at hand. In our case, such conditions are dictated by the structure of optimal trading strategy in frictionless market, and are discussed in the next section.

2 Optimal position liquidation under different types of market impact

This section presents formulations and properties of optimal position liquidation problem under different forms of market impact. According to the foregoing discussion, the optimal liquidation problem based on a sample-path scenario model with non-anticipativity constraints can be written as

$$\max_{\Xi} \left\{ Z \;\middle|\; \xi_t^j = f(x_t^{k(j,t)}, \xi_{t-1}^j), \quad k(j,t) \propto \pi(\mathfrak{S}) \right\}, \tag{15}$$

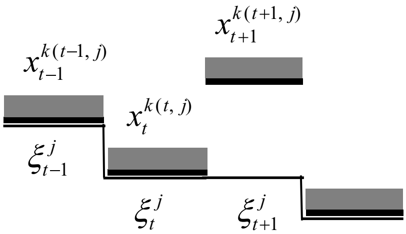
where the objective Z has form (8), (4), or (6) in the case of zero, temporary, or permanent market impacts respectively. The form of the decision rule $\xi_t^j = f(x_t^{k(j,t)}, \xi_{t-1}^j)$ employed in this paper is discussed below.

2.1 Optimal liquidation in frictionless market

In the assumption of absence of fixed transaction costs (recall that we have agreed to consider proportional transaction costs as a part of temporary market impact), the case of zero market impact reduces to idealized setting of frictionless market. This simplest form of market model does, nevertheless, have important implications in the scope of the developed approach, and can also be used to bridge the areas of optimal trading and pricing of derivatives. First, we observe that in frictionless market with finite set of trading moments, an optimal trading strategy that maximizes the expected payoff from the transaction has a binary (0–1) structure. This observation will help us to narrow the selection for function $f(\cdot)$ in (14) and (15).

The binary structure of optimal closing under zero impact is easy to see by drawing an analogy between trading in frictionless market and exercising of options. Indeed, selling a block of shares of a security in frictionless market is essentially identical to executing a call option with zero strike price written on the corresponding number of shares. In both cases, the trader receives profit equal to the current price of the security times the number of shares sold. Noting this, the problem of finding optimal liquidation strategy that maximizes the expected payoff in frictionless market can be restated as the problem of exercising n American (more

Fig. 4 The "lawn-mower" principle



precisely, Bermudan)[4] options with zero strike price and expiration date T, each written on 1 share of the security, where n is the number of shares in the initial position. Then, if at time moment t it is optimal to exercise such an option, it is optimal to exercise all n options, which is equivalent to liquidating the entire position at time t. It has to be emphasized that since no restrictions are imposed on the price process of the security, an early optimal exercise of American (Bermudan) call option is possible.

The binary structure of the optimal trading strategy in frictionless market should not be confused with the anticipativity flaw of the optimal solution of problem (8). Quite the contrary, our argument implies that if, at some time t, it is optimal to sell some portion of the position, then it is also optimal to sell the entire position, since it is impossible to predict another opportunity for liquidation that is at least as good as the current one.

The specific 0–1 structure of optimal trading strategy in frictionless market imposes a restriction on the form of the function $f(\cdot)$ in (14) that can be used in the context of the problem of optimal position liquidation. Below we discuss selection of the decision rule in our problem.

The "lawn-mower" strategy Given that the objective (8) of the problem with zero impact (15) is linear and the set Ξ of trading strategies is a polytop, the requirement of existence of an optimal 0–1 strategy under zero impact indicates favorably toward a piecewise-linear form of the function $f(\cdot)$. Trivial examples include

$$\xi_t^j = x_t^{k(j,t)} \quad \text{and} \quad x_t^{k(j,t)} = \Delta\xi_t^j = \xi_{t-1}^j - \xi_t^j.$$

In what follows, we consider a less trivial case of the group decision rule (14), which we call the "lawn-mower" strategy:

$$\forall j \in G_t^k\colon \quad \xi_t^j = \min\{\xi_{t-1}^j,\ x_t^{k(j,t)}\}, \quad 0 \le x_t^{k(j,t)} \le 1. \tag{16}$$

Definition (1) of trading strategy set Ξ and (16) imply that at the final time T,

$$x_T^{k(j,T)} = 0, \quad j = 1, \dots, J. \tag{17}$$

Observe that the position value ξ_{t-1}^j is allowed to change (decrease) at time t *only* by being "trimmed" to the level $x_t^{k(j,t)}$. If the position level ξ_{t-1}^j is not high enough to be "trimmed" by $x_t^{k(j,t)}$, it remains unchanged at time t. This resembles lawn mowing, with position variables ξ_t^j being the "grass" and decision variables $x_t^{k(j,t)}$ being the "blades" (see Fig. 4).

[4] A Bermudan option is an American-style option with a finite set of exercise opportunities.

By solving problem (15) with the "lawn-mower" decision rule (16) one finds an optimal allocation of thresholds $x_t^{k(j,t)}$ that determines the following optimal trading strategy: *if scenario j belongs to the group G_t^k, then sell the portion of position in excess of x_t^k; if the current position value is less than x_t^k, do nothing.*

In the case of zero market impact, the optimal liquidation problem with the "lawn-mower" decision rule (16) reads as

$$\max_{\Xi} \quad \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \, \Delta\xi_t^j \tag{18}$$

$$\text{s.t.} \quad \xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}, \qquad t = 1, \dots, T, \;\; j = 1, \dots, J,$$

$$0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \quad t = 1, \dots, T, \;\; j = 1, \dots, J.$$

Problem (18) constitutes a special case of the optimal closing problem with temporary market impact. The methods for solving (18) will be discussed in the next subsection. Here we demonstrate that the optimal liquidation problem (18) is feasible and well-posed, i.e., for any allocation of thresholds x_t^k the corresponding trading strategy is unique.

Proposition 2. *Under the "lawn-mower" decision rule* (16) *each variable ξ_t^j is a single-valued function of variables $x_1^{k(j,1)}, \dots, x_t^{k(j,t)}$.*

Proposition 3. *For any path partitioning $\pi(\mathfrak{S})$ problem (18) is feasible.*

Note that the above propositions concern the feasible region of problem (18). Since optimization problems for different forms of market impact only differ by objective functions, statements of Propositions 2 and 3 hold also for problems with temporary and permanent market impacts. Finally, we show that the "lawn-mower" decision rule (16) admits a binary optimal strategy in frictionless market.

Proposition 4. *The optimal liquidation problem under zero market impact* (18) *has an optimal 0–1 solution.*

Remark 1. Proposition 4 implies that if the optimal solution of (18) is unique, then it is 0–1.

The introduced forms of optimal liquidation problem (15) and (18) admit incorporation of different types of constraints into the trading strategy. Additional constraints may reflect the investor's preferences regarding the desired structure of the optimal trading strategy, or make the strategy compliant with institutional regulations etc. Section 2.4, for example, discusses imposing of risk constraints on the feasible set of problem (18).

The next subsection demonstrates how to approximate the solution of (18) using convex or linear programming, which will allow for efficient solving of optimal position liquidation problem with additional constraints. In comparison, the dynamic programming approach to optimal transaction execution (see Bertsimas and Lo (1998)) makes the incorporation of additional constraints in the trading strategy less straightforward.

2.2 Optimal liquidation under temporary market impact

In the presence of temporary market impact, the optimal position liquidation problem based on the "lawn-mower" principle is obtained from (15) by using expression (4) for the objective function and (16) for the decision rule:

$$\max_{\Xi} \quad \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\, \delta_t(\Delta\xi_t^j) \tag{19}$$

$$\text{s.t.} \quad \xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}, \qquad t = 1,\dots,T,\ \ j = 1,\dots,J,$$

$$0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \qquad t = 1,\dots,T,\ \ j = 1,\dots,J.$$

Note that the feasible region of (19) is non-convex due to the "lawn-mower" constraint $\xi_t^j = \min\{\xi_{t-1}^j,\ x_t^{k(j,t)}\}$. The non-convexity of the constraint set is quite undesirable from a computational point of view, as it usually leads to computationally expensive optimization algorithms. However, the separable form of the objective function (19) and properties (i)–(iv) of the function $\delta_t(\cdot)$ allow us to transform (19) into an equivalent problem with convex feasible region, thereby improving its numerical tractability. First, let us extend the domain of function $\delta_t(\cdot)$ to $[-1,\ 1]$:

(v) $\delta_t(y) = y \quad \text{for} \quad y \in [-1, 0)$

Then, consider the following problem with a non-linear objective and linear constraints

$$\max_{\Xi} \quad \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0, \delta_t(u_t^j)\} \tag{20}$$

$$\text{s.t.} \quad \xi_t^j \le x_t^{k(j,t)}, \qquad t = 1,\dots,T,\ \ j = 1,\dots,J,$$

$$u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, \qquad t = 1,\dots,T,\ \ j = 1,\dots,J,$$

$$0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \qquad t = 1,\dots,T,\ \ j = 1,\dots,J.$$

Note that in accordance with definition (1) of trading strategy set Ξ and condition (17), one has to put

$$u_1^j = 1 - x_t^{k(j,t)}, \quad u_T^j = \xi_{T-1}^j.$$

Proposition 5. *Optimization problems* (19) *and* (20) *are equivalent in the sense that optimal values of their objectives are equal, and the sets of their optimal solutions in variables* ξ_t^j *and* $x_t^{k(j,t)}$ *coincide.*

The disadvantage of formulation (20) is that its objective function does not have such a transparent meaning, as in (19). However, the linearity of the constraints makes problem (20) much more tractable compared to (19), whose constraint set contains the non-convex "lawn-mower" constraint (16).

Solution methods The choice of a particular numerical scheme to be deployed for solving the optimization problem (20) depends primarily on the properties of the temporary impact function $\delta_t(\cdot)$. As a rather general recourse, a DC[5] programming approach may be selected to solve (20). In our case, it would require the ability to represent each function $\delta_t(y)$ as a difference of two convex functions δ_t^+ and δ_t^-:

$$\delta_t(y) = \delta_t^+(y) - \delta_t^-(y), \quad \delta_t^{\pm}(y) \text{ convex on } [-1,\ 1]. \tag{21}$$

Replacing assumption (iv) for the function $\delta_t(\cdot)$ by a stronger assumption

(iv′) $\delta_t(y)$ *is Lipschitzian on* $[0,\ 1]$ *and has derivative* $\delta_t'(y)$ *of bounded variation*

we obtain that $\delta_t(\cdot)$ is a DC function on $[-1,\ 1]$, whereby the functions $\max\{0, \delta_t(\cdot)\}$ in the objective of (20) are also DC (Horst, Pardalos, and Thoai, 2000).

Although being quite general, the DC framework, we believe, can capture naturally the structure of impact functions $\delta_t(\cdot)$. In many applications the market trading costs have been represented by the so-called S-shaped functions (see, for instance, Fig. 2), for which a DC decomposition is readily available. For a survey of DC programming algorithms see, for example, Horst and Thoai (1999), Horst, Pardalos, and Thoai (2000), Tuy (1994, 2000), and references therein. Separability of the objective function (20) makes the DC algorithms that exploit this property (e.g., Konno and Wijayanayake (1999)) especially attractive.

Bounds by convex programming Here we focus on the case of a concave temporary impact function $\delta_t(\cdot)$. Concavity assumption means that larger trades trades lead to higher payoffs, but the marginal payoffs decrease as the transaction size increases. Concavity of the impact functions $\delta_t(\cdot)$ does not solve the non-concavity of the objective (20), but it allows for construction of lower and upper bounds for the optimal solution of (20) using the techniques of convex programming.

Lower bound. To obtain a lower bound for the solution of DC problem (20), we introduce a problem similar to (20), but with a concave objective:

$$\max_{\Xi} \quad \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\, \delta_t\big(u_t^j\big) \tag{22}$$

$$\begin{aligned}
\text{s. t.}\quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1,\dots,T, \quad j = 1,\dots,J,\\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, && t = 1,\dots,T, \quad j = 1,\dots,J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1,\dots,T, \quad j = 1,\dots,J,
\end{aligned}$$

which makes (22) a *convex* programming problem. Recall that the equivalent formulation (20) of the optimal closing problem (19) has been introduced to circumvent the non-convexity of the feasible region of (19) caused by the "lawn-mower" constraint (16). It can be thought as if the specially constructed objective of (20) "selects" from the convex feasible region of (20) an optimal solution that satisfies the non-convex "lawn-mower" constraint (16). Now, problem (20) is being approximated by problem (22) with the same feasible set but different

[5] Difference of convex functions.

objective. Thus, we have to make sure that optimal solution of (22) does also satisfy to the "lawn-mower" constraint (16).

Proposition 6. *The optimal solution of problem* (22) *satisfies the "lawn-mower" principle* (16).

Corollary 1. Since optimal solutions of (20) and (22) satisfy the "lawn-mower" constraint (16), and for all u_t^j the value of the objective function (20) is greater than that of the objective of (22),

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \, \delta_t(u_t^j) \le \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \, \max\left\{0, \delta_t\left(u_t^j\right)\right\},$$

an optimal solution of (22) represents a *lower bound* for the optimal solution of (20).

Upper bound. An upper bound for the solution of problem (20) can be constructed by solving the problem (19) without the "lawn-mower" constraint, i.e.,

$$\max_{\Xi} \quad \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \, \delta_t\left(\Delta\xi_t^j\right). \tag{23}$$

If functions $\delta_t(\cdot)$ are concave, (23) is also a convex programming problem. Recall that optimal solution of (23) is *anticipative*, which, nonetheless, does not preclude it from being used as an upper bound for the solution of problem (20).

The lower-bound solution (22) can be used to approximate the solution of problem (20). Although it appears to be difficult to assess the tightness of the lower bound (22) under the rather general assumptions adopted in this paper, our numerical experiments (see Section 3.1) show that the gap between the upper and lower bounds is quite tight, within 3% of the objective value. Also, it is possible to perform a numerical estimation of the goodness of the lower bound, at least for smaller-sized problems. In section 3.1 we compared the lower bound given by (22) with the solution of original problem (20) for instances of lower dimension (problem (20) was reduced to 0–1 programming problem using the statement of Proposition 4).

"Lawn-mower" strategy in the presence of non-zero market impact A substantial simplification of problem (19) can be achieved by utilizing the structure of optimal trading strategies under non-zero market impact. It has been shown by Bertsimas and Lo (1998) that under linear temporary market impact the optimal trading strategy exhibits little difference comparing to the so-called *naive* strategy, which consists of selling equal portions of the position at each time t,

$$\Delta\xi_t^{\text{naive}} = 1/T, \quad t = 1, \ldots, T, \tag{24}$$

provided that the time moments are equally spaced. Depending on the asset price dynamics and form of market impact, the optimal trading strategy may deviate from the naive strategy (24), yet the transaction size $\Delta\xi_t$ can be non-zero at each time step. The assumption of non-zero trades at each time moment rests on two prerequisites: (1) the time frame for liquidation is relatively short, and (2) the magnitude of market impact is significant enough. Then, the following simple observation can be used to eliminate the non-convexity in the feasible set of problem (19):

Proposition 7. *If* $\Delta\xi_t^j > 0$*, then the "lawn-mower" decision rule* $\xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}$ *is equivalent to* $\xi_t^j = x_t^{k(j,t)}$.

Corollary 2. *If the impact function* $\delta_t(\cdot)$ *is concave, and* $\Delta\xi_t^j > 0\ \forall t, j$*, the optimal liquidation problem with temporary market impact* (19) *can be reduced to solving a convex programming problem*

$$\max_{\Xi} \left\{ \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T} S_t^j \delta_t\big(\Delta\xi_t^j\big) \;\middle|\; \xi_t^j = x_t^{k(j,t)},\;\; k(j,t) \propto \pi(\mathfrak{S}) \right\}. \tag{25}$$

If optimal solution of (25) *satisfies condition* $\Delta\xi_t^{*j} > \varepsilon_0$*, for some* $\varepsilon_0 > 0$*, then it is also optimal for problem* (19) *with additional constraint* $\Delta\xi_t^j \geq \varepsilon_0$.

The simplification of Proposition 7 and Corollary 2 can be used when market impact is significant, e.g., the graph of function $\delta_t(\cdot)$ lies well below the diagonal line on Fig. 2, and the number T of time steps is small. If the market impact is weak (i.e., the graph of $\delta_t(\cdot)$ approaches the horizontal line on Fig. 2), or the number of time steps is large, then the statement of Corollary 2 is inapplicable. In such a case, solution of problem (20) can be approximated by the lower bound (22). In the case study reported in Section 3, we solved the simplified formulation (25) of the optimal liquidation problem with temporary market impact.

2.3 Optimal liquidation under permanent market impact

When the price changes caused by trader's activity have a permanent character, the problem of optimal position liquidation complicates considerably. In particular, the essential non-concavity and non-separability of the objective function that corresponds to permanent market impact (6) would not allow for transforming the optimal liquidation problem with "lawn-mower" constraints into an equivalent problem with a convex feasible set, as it has been done above.

Thus, we resort to Proposition 7, restricting ourselves to the forms of permanent market impact that lead to non-zero trades at each time period. Then the optimal closing problem under permanent market impact reads as

$$\begin{aligned} \max_{\Xi} \quad & \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \Delta\xi_t^j \gamma_t\big(\Delta\xi_t^j\big) - \theta_t\big(\Delta\xi_t^j\big) \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\} \\ \text{s. t.} \quad & \xi_t^j = x_t^{k(j,t)}, \qquad\qquad\qquad\qquad j = 1, \dots, J, \quad t = 1, \dots, T, \\ & 0 \leq x_t^{k(j,t)} \leq 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \;\; j = 1, \dots, J, \quad t = 1, \dots, T. \end{aligned} \tag{26}$$

Similarly to the general case of temporary market impact, problem (26) can be handled in the scope of DC programming framework.

Numerical efficiency of formulation (26) can be increased by directly substituting $x_t^{k(j,t)}$ for ξ_t^j in the objective (26), instead of having constraints $\xi_t^j = x_t^{k(j,t)}$. It dramatically reduces the scale of optimization problem, thus allowing one to solve the non-linear non-convex problem (26) more robustly.

2.4 Risk constraints

The purpose of this section is to demonstrate that the developed sample-path-based model for optimal position liquidation admits incorporation of various types of constraints in the trading strategy. One of the most important constraints yet to be incorporated in the optimal liquidation model developed so far, are the risk constraints. Indeed, it is very well known that market activity always involves risk, which in our case is associated with losses due to uncertain future prices of the security.

Here we consider risk constraints based on Conditional Value-at-Risk (CVaR) risk measure (Rockafellar and Uryasev, 2000, 2002). CVaR is a downside risk measure, which quantifies the risk in terms of percentiles of loss distribution. Perhaps, the most outstanding feature of Conditional Value-at-Risk is its convexity with respect to decision variables (provided that the loss function is also convex), which simplifies dramatically the problem of estimating and managing risks using the techniques of mathematical programming, and also makes these procedures much more robust compared to other risk measures, e.g., Value-at-Risk.

In developing CVaR risk constraints for the optimal liquidation problem we follow Rockafellar and Uryasev (2000, 2002). Given a loss function $\mathcal{L}(\xi, \eta)$, where η is a stochastic vector standing for market uncertainties, and ξ is the decision vector, Conditional Value-at-Risk $\phi_\alpha(\xi)$ with confidence level α can be described approximately as the conditional expectation of losses exceeding the α-quantile of the loss distribution, i.e., $(1-\alpha)\cdot 100\%$ of worst losses. A precise definition presents Conditional Value-at-Risk $\phi_\alpha(\xi)$ as a solution of a (convex) programming problem

$$\phi_\alpha(\xi) = \min_{\zeta\,\in\mathbb{R}} \Big\{ \zeta + (1-\alpha)^{-1} \mathsf{E}_\eta[\max\{0, \mathcal{L}(\xi,\eta) - \zeta\}] \Big\}.$$

According to Rockafellar and Uryasev (2002), in an optimization problem with multiple CVaR constraints

$$\max_{\xi\,\in\,\Xi} \{ g(\xi) \mid \phi_{\alpha_t}(\xi) \le \omega_t, \quad i = 1, \dots, T \}, \tag{27}$$

each constraint $\phi_{\alpha_t}(\xi) \le \omega_t$ can be replaced by a set of inequalities

$$\begin{aligned}
&\mathcal{L}(\xi, \eta_j) - \zeta_t \le w_t^j, \quad j = 1, \dots, J, \\
&\zeta_t + \frac{1}{1-\alpha_t} \frac{1}{J} \sum_{j=1}^{J} w_t^j \le \omega_t, \\
&\zeta_t \in \mathbb{R}, \quad w_t^j \ge 0, \quad j = 1, \dots, J,
\end{aligned} \tag{28}$$

where η_j, $j = 1, \dots, J$, are (equally probable) realizations of the random vector η. The set defined by inequalities (28) is convex provided that function $\mathcal{L}(\xi, \eta)$ is convex in ξ.

Risk constraints in the problem with temporary market impact We consider the loss function $\mathcal{L}_t(\xi, \cdot_j)$ as negative profit along j-th path $(\mathbf{S}_0, \mathbf{S}_1^j, \dots, \mathbf{S}_T^j)$ before time t

$$\mathcal{L}_t\big(\xi_\tau^j; \mathbf{S}_\tau^j \;\big|\; \tau \le t\big) = S_0 - \sum_{\tau=1}^{t} p_\tau\big(\xi_k^j; \mathbf{S}_k^j \;\big|\; k \le \tau\big). \tag{29}$$

To obtain risk constraints that will not jeopardize computational tractability of the developed optimal liquidation problems, we formulate risk constraints in the case of temporary market impact only, with additional assumption that impact function $\delta_t(\cdot)$ is concave. Under these conditions, the loss function (29) is convex,

$$\mathcal{L}_t\big(\xi_\tau^j\,; \mathbf{S}_\tau^j \;\big|\; \tau \le t\big) = S_0 - \sum_{\tau=1}^{t} S_\tau^j\, \delta_t\big(\Delta\xi_\tau^j\big), \tag{30}$$

ensuring the convexity of CVaR constraints (28). The optimal position liquidation problem with CVaR constraints then reads as

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\big\{0, \delta_t\big(u_t^j\big)\big\} && && (31)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1, \dots, T, && j = 1, \dots, J,\\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, && t = 1, \dots, T, && j = 1, \dots, J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1, \dots, T, && j = 1, \dots, J,\\
& \xi_t^j \in \Phi_t^{\alpha,\omega}, && t = 1, \dots, T, && j = 1, \dots, J.
\end{aligned}$$

Here $\Phi_t^{\alpha,\omega}$ stands for the feasible set (28) of CVaR constraint $\phi_\alpha(\xi) \le \omega_t$ based on loss function (30). Risk constraints $\xi_t^j \in \Phi_t^{\alpha,\omega}$ make sure that at each time t the average of $(1 - \alpha_t) \cdot 100\%$ worst losses, as compared to the initial wealth level S_0, does not exceed ω_t.

Next we show that our previous arguments on optimal liquidation with the "lawn-mower" decision rule remain valid if CVaR constraints are incorporated in the corresponding optimization problems.

Proposition 8. *Propositions 5, 6, and Corollary 2 remain valid with CVaR constraints*

$$\xi_t^j \in \Phi_t^{\alpha,\omega}, \quad t = 1, \dots, T, \quad j = 1, \dots, J,$$

imposed on the feasible regions of problems (19), (20), *and* (22), *where set* $\Phi_t^{\alpha,\omega}$ *is defined by* (28) *and* (30), *with function* $\delta_t(\cdot)$ *being concave.*

Remark 2. With CVaR constraints imposed, solution of problem (18) can be fractional. This reflects the general principle of *diversification of risk* (it is less risky to liquidate via many several small trades rather than one big trade).

3 Case study: Optimal closing of long position in a stock

The case study considered in this section was a part of a proprietary trading strategy that has been developed for trading in a set of stocks with similar characteristics. This trading strategy consisted of an opening algorithm that generated signals for opening positions in stocks, and a closing algorithm based on the presented above methodology. Here we discuss trading strategies developed for various market conditions, based on a particular test instance of data.

The case study concerns optimal liquidation of long position in a stock in the presence of different types of market impact. The time horizon for closing of the position spans 5 business days, with one trading opportunity per day ($T = 5$). Because of the short time frame, discounting was neglected.

The sample-path scenario set $\mathfrak{S}$ contains 5,000 paths ($J = 5{,}000$) that have been generated from historical data in the following fashion. The aforementioned opening algorithm has been employed to analyze historical data of a set of stocks and, upon recognition of special patterns in the trajectory of a stock, generate signals for opening and, subsequently, closing of a long position in that stock. The stock's 5-day historical trajectory following the closing signal was then included as a sample path in our scenario set.

More precisely, each entry $\mathbf{S}_t^j$ of j-th sample path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_5^j)$ contains daily closing price and daily trading volume in that stock: $\mathbf{S}_t^j = (S_t^j, V_t^j)$, with (S_0, V_0) being the price and volume on the day the closing signal was generated. Prices S_t of each sample path were normalized to make the starting price S_0 equal to 1. The sample paths in the collection $\mathfrak{S}$, although belonging to different stocks, possess similar characteristics as they were all selected by the opening algorithm in accordance to a certain criterion. This justifies the use of this sample paths collection as a scenario set for a security whose price process has the same properties.

Path grouping The path grouping (11) was introduced as a remedy for anticipativity of solutions, which is inherent to the "parallel" structure of the sample-path scenario model. In this case study, we partitioned the sample-paths into a constant number K of groups of equal size J/K in each period. The partitioning was performed with respect to the asset price, so that group G_t^k with larger index k contained paths with higher prices S_t^j:

$$\forall\, k_1 < k_2,\ j_1 \in G_t^{k_1},\ j_2 \in G_t^{k_2} \quad \Rightarrow \quad S_t^{j_1} \le S_t^{j_2}, \quad t = 1, \ldots, T.$$

In such a way, group G_t^1 contained J/K paths with the lowest prices, and group G_t^K contained J/K paths with the highest prices, at each time period t. The number of groups K per period has to be chosen such that non-anticipativity of the problem's solution is achieved (recall that $K = J$ leads to anticipative solution (9)).

To ensure that the selected path partitioning preserves the non-anticipativity in the model, we performed a sensitivity study of the optimal value of problem (22) in the case of zero market impact ($\delta_t(u) = u$) with the above path grouping by varying the number of groups K. In particular, we used $K =$10, 20, 50, 100, 250, 500, 1000, 2500, and 5000. Note that numbers in this sequence increase by factor 2.0 or 2.5, and the path grouping with $K = 5000$ is known to produce anticipative solution (9). The obtained optimal values of the objective (22) are presented in logarithmic scale in Fig. 5. Observe that the optimal objective value changes very slightly (within 0.26%) when the number of groups K varies from 10 to 500, and increases significantly when K runs from 1000 to 5000. Such rapid changes in the

Table 1 Lower-bound optimal trading strategy in frictionless market

No risk constraints						CVaR constrains (α_t =0.9, ω_t =0.1)					
k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	1	0	0	0	0	10	0	0	0	0	0
9	1	1	0	0	0	9	0	0	0	0	0
8	1	1	0	0	0	8	0	0	0	0	0
7	1	1	0	0	0	7	0	0	0	0	0
6	1	1	0	0	0	6	0	0	0	0	0
5	1	1	0	0	0	5	0.052	0	0	0	0
4	1	1	0	0	0	4	0.035	0	0	0	0
3	1	1	1	0	0	3	0.014	0	0	0	0
2	1	1	1	0	0	2	0	0	0	0	0
1	1	1	1	0	0	1	0	0	0	0	0

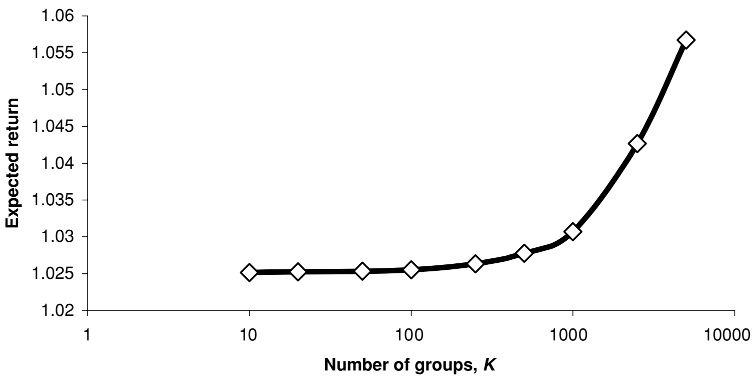


Fig. 5 Optimal solution of (22) in the zero-impact case depending on the number of groups K

optimal objective value represent the effect of anticipativity that is not eliminated by path partitioning with $K \geq 1000$. Hence, this sensitivity study suggests that given the sample-path set, the described path groping allows for non-anticipative solutions of optimal closing problem (22) when $K = 10, \ldots, 500$. This conclusion applies to optimal closing problems with all forms of market impact, because non-anticipativity is a property of the feasible set (achievable due to non-anticipativity constraints, implemented via path grouping in our case) and, as such, does not depend on the form of the objective function.

We present the optimal trading strategies in a tableau form (see Tables 1–6) by reporting the optimal values of variables x_t^k in the corresponding optimization problems ($K = 10$). We have considered the simple decision rule (12) and the "lawn-mower" principle (16). Recall that according to Proposition 2, the set of variables x_t^k defines uniquely the trading strategy in either case (16) or (12) of the group decision rule. All the tables report the optimal values of variables x_t^k for 10-group partition as described above. The results presented in Tables 1–6 have to be interpreted as follows: if at time t the observed price of the security falls into the price range of group G_t^k, execute the transaction in accordance to the specified group decision rule. For instance, in the case of the "lawn-mower" decision rule (16), reduce the position value to the level of variable x_t^k if the position exceeded x_t^k, or do nothing otherwise.

3.1 Optimal closing in frictionless market

Problem (18) with zero market impact has a 0–1 solution, which becomes fractional if risk constraints (in our case, based on the Conditional Value-at-Risk measure) are imposed. It implies that trading algorithm without risk constraints selects the most favorable moment to sell the position "in one shot" so as to gain the highest expected profit. Risk-averse trading strategy recommends closing the position by parts because waiting for such a favorable moment is risky. This corresponds to the general principle of *diversification of risk*.

When market impact is absent, (22) becomes a linear programming problem, and therefore can be solved fast and robustly. Problems (19) and (20), on the contrary, still remain non-convex programming problems, requiring significantly more computational efforts to solve. We used the lower bound given by formulation (22) as a solution of optimal closing problem in frictionless market.

Table 1 displays the liquidation strategies with and without control of risk, defined as solutions of problem (22) with $\delta_t(y) = y$ and the corresponding problem with CVaR constraints (28).

Parameters of CVaR constraints in (28) were chosen as $\alpha_t = 0.9$ and $\omega_t = 0.1$, meaning that in each time t the average loss in 10% of worst cases should not exceed 10% of the original dollar value of the position. Optimal strategy based on the solution with CVaR constraints tends to liquidate position early to reduce the risks of financial losses associated with holding the position for a longer period of time. The scenarios for CVaR constraints were constructed using past historical data, in order to avoid unnecessary complications in the case study, which just demonstrates the ability to incorporate risk constraints in the developed trading algorithm. If needed, other scenario sets can be employed for estimating the risk exposure during position liquidation.

The gap between the lower-bound solution (22) and the upper-bound solution (8) is tight: 3% of the objective value. Also, the quality of the lower bound produced by (22) with $\delta_t(y) = y$ was tested by comparing it with the optimal objective value of problem (18) on test instances of lower dimensions. The problem (18) was reformulated as a linear 0–1 programming problem, due to Proposition 4, and solved using CPLEX 8.1 solver. In particular, we solved problems with 100, 200, 500, and 1000 sample paths, which were selected at random from the original pool of 5000 paths. The number of groups K was kept the same, $K = 10$. The difference in values of the optimal objectives of problems (18) and (22) ranged from 1.15% to 1.56%.[6] This allows us to believe that the solution provided by lower-bound formulation (22) was also near-optimal when the number of sample paths in the problem was 5000. Figure 6 displays the dependence of the values of optimal solution, upper an lower bounds on the number of paths J in the problem.

"Tree" grouping and comparison with dynamic programming An interesting issue to address is the performance of the optimal closing problem (18) with a "tree" partitioning in comparison to the solution of the optimal closing problem obtained by the methods of dynamic programming and scenario trees. The dynamic programming approach in application to the optimal closing problem in frictionless market was considered by Butenko et al. (2003). The authors developed a technique for transforming a collection of sample paths into a scenario tree with a prescribed number of branches per node. The trading strategy, obtained as a solution of the corresponding dynamic programming problem, was tested on the historical data

[6] These numbers are relative differences in objective values, not the percents of expected return.

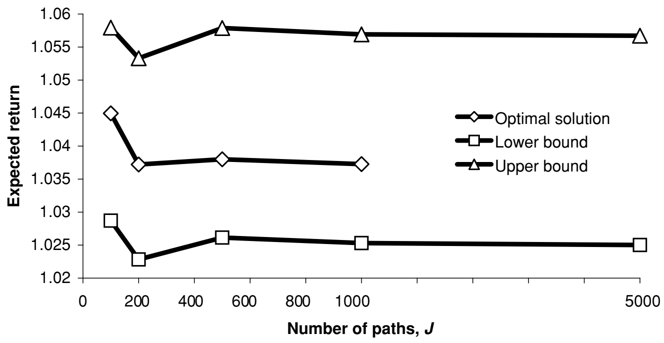


Fig. 6 Optimal solution (19), the upper (23) and lower (22) bounds in the zero-impact case

used in this case study. We compared their results for a 4-branch scenario tree, solution to which was presented among others, with the solution of problem (18) under a "tree" partition constructed by splitting each group into 4 groups of equal size at each time t. According to Proposition 1, "tree" partition reduces any decision rule (14) to the simple rule (12). We ran both algorithms on an instance of data containing 5120 paths, and the algorithm (18) slightly outperformed the dynamic programming approach, producing expected return of 1.0241 versus 1.02062 obtained by Butenko et al. (2003). The similarity of solutions obtained by different techniques validates the correctness of our approach and that presented in Butenko et al. (2003).

3.2 Optimal closing under temporary market impact

To study the optimal closing strategies under temporary market impact, we assumed that function $\delta_t(y)$ in the objective of problem (19) and (20) has the form

$$\delta_t(y) = y - \frac{1}{\beta c} y^\beta, \quad \beta > 1, \quad c \geq 1. \tag{32}$$

This function, evidently, satisfies assumptions (i)–(iv). In representation (32) $c = 1$ corresponds to the "most severe" market impact; when c increases, the impact function $\delta_t(\cdot)$ approaches the no-impact case $\delta_t(y) = y$. When $\beta = 2$, equality (32) describes the *linear* temporary market impact. By virtue of Proposition 7 and Corollary 2, in this case the optimization problem (19) was replaced with convex quadratic programming problem (25) with a diagonal matrix, which was solved using MINOS engine.

The correspondence between problems (19) and (25), established in Corollary 2, is not valid when market impact is weak enough to allow zero trades ($\Delta\xi_t^j = 0$) in optimal solution of (25). By varying parameter c in (32) we found that optimal solution of (25) violates condition $\Delta\xi_t^{*j} > 0$ for $c \geq 25$.

Linear temporary market impact ($\beta = 2$) Table 2 displays the optimal values of the decision variables x_t^k, obtained as solution of a quadratic programming problem (25) with function $\delta_t(\cdot)$ defined above for $\beta = 2$. The problem was solved for cases $c = 1$ ("severe market impact") and $c = 10$ ("weak impact").

Table 2 Optimal trading strategy under linear temporary market impact ($\beta = 2$)

c =1.0						c =10.0					
k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	0.804	0.608	0.407	0.205	0	10	0.869	0.701	0.490	0.262	0
9	0.805	0.609	0.409	0.204	0	9	0.867	0.713	0.516	0.255	0
8	0.805	0.610	0.408	0.203	0	8	0.864	0.723	0.506	0.245	0
7	0.804	0.608	0.409	0.203	0	7	0.856	0.702	0.516	0.248	0
6	0.806	0.607	0.408	0.204	0	6	0.885	0.697	0.508	0.259	0
5	0.804	0.607	0.410	0.206	0	5	0.860	0.697	0.523	0.280	0
4	0.805	0.609	0.409	0.206	0	4	0.862	0.711	0.518	0.276	0
3	0.804	0.608	0.409	0.206	0	3	0.851	0.707	0.518	0.273	0
2	0.804	0.609	0.408	0.205	0	2	0.857	0.711	0.505	0.263	0
1	0.806	0.607	0.409	0.205	0	1	0.874	0.692	0.519	0.263	0

First of all, observe that in the presence of "severe" market impact ($c = 1$), the optimal trading strategy becomes almost identical to the *naive* strategy

$$x_{\text{naive}} = \{0.8, \quad 0.6, \quad 0.4, \quad 0.2, \quad 0\}, \tag{33}$$

such that the position is sold in equal portions at all available time moments. This is consistent with the results by Bertsimas and Lo (1998) for linear-percentage price impact. As the effects of market impact diminish ($c \to \infty$, $\delta(y) \to y$), the solution starts to exhibit deviations from the *naive* strategy.

In the case of linear temporary market impact (32) with $c = 10$, the difference between the lower-bound solution (22) and the corresponding upper-bound solution (i.e., problem (22) without the constraint $\xi_t^j = x_t^{k(j,t)}$) does not exceed 2% of objective value. Also, the optimal objectives of lower-bound formulation (22) and the simplified formulation (25) coincide at the considered instances of optimal liquidation problem with temporary market impact.

Linear temporary market impact that depends on market parameters According to discussion in section 1.3, the market impact function $\delta(\cdot)$ depends on the market conditions at the moment of transaction. To accommodate this in our model, we consider function (32) with coefficient c that depends on the current volume V_t of trades in the market. In particular, we assumed the following expression for c:

$$c = 1 + e^{M\left(V_t^j - C\right)},$$

where M and C are some constants. This parametrization of the impact function $\delta_t(\cdot)$ reflects the presumption that the temporary market impact decreases ($c \to \infty$) if the current trading volume is high, and increases ($c \to 1$) when the volume is low. The optimal solution shows significant differences compared to the *naive* strategy (33) (see Table 3).

3.3 Optimal closing under permanent market impact

The problem of optimal closing in the presence of permanent market impact (26) was considered in two settings: in the assumption of linearity of the market impact functions $\gamma_t(\cdot)$ and

Table 3 Optimal trading strategy under linear temporary market impact that depends on price dynamics

k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	0.84	0.73	0.43	0.26	0
9	0.88	0.66	0.43	0.23	0
8	0.82	0.56	0.45	0.25	0
7	0.84	0.71	0.47	0.28	0
6	0.88	0.65	0.47	0.25	0
5	0.92	0.62	0.44	0.17	0
4	0.85	0.65	0.5	0.23	0
3	0.85	0.65	0.49	0.22	0
2	0.83	0.67	0.57	0.24	0
1	0.87	0.68	0.4	0.2	0

Table 4 Optimal trading strategy under linear permanent market impact

	$c=1.0$						$c=10.0$				
k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	0.913	0.804	0.659	0.440	0	10	0.929	0.836	0.694	0.472	0
9	0.914	0.805	0.661	0.440	0	9	0.931	0.839	0.704	0.467	0
8	0.913	0.805	0.659	0.439	0	8	0.930	0.842	0.700	0.461	0
7	0.913	0.804	0.661	0.439	0	7	0.928	0.836	0.704	0.463	0
6	0.914	0.804	0.659	0.439	0	6	0.934	0.835	0.701	0.470	0
5	0.913	0.804	0.660	0.441	0	5	0.929	0.835	0.707	0.484	0
4	0.913	0.805	0.660	0.441	0	4	0.929	0.839	0.704	0.481	0
3	0.913	0.804	0.660	0.440	0	3	0.927	0.838	0.705	0.479	0
2	0.913	0.805	0.659	0.440	0	2	0.928	0.839	0.700	0.473	0
1	0.914	0.804	0.660	0.440	0	1	0.932	0.834	0.705	0.473	0

$\theta_t(\cdot)$, and under more realistic assumption (7). Recall that in presence of permanent market impact, optimal liquidation problem reduces to nonlinear non-convex programming problem (26). However, as discussed above, the size of (26) can be reduced significantly, leaving only $x_t^{k(j,t)}$ as decision variables, which in case of $T=t$ and $K=10$, results in problem with 50 variables. This allowed us to use MINOS nonlinear solver to solve the corresponding non-linear problems.

Linear permanent market impact For simplicity, we assumed functions $\gamma_t(\cdot)$ and $\theta_t(\cdot)$ to be equal, and the temporary part of market impact in (26) is equal to (32) with $\beta=2$, i.e.,

$$\gamma_t(y)=\theta_t(y)=\frac{1}{2c}y.$$

In both cases of "severe" impact ($c=1$) and "light" impact ($c=10$) the optimal solution exhibits differences with the *naive* strategy (33) and the solution of the corresponding problem with temporary impact (see Table 4). Under permanent market impact, algorithm tends to sell smaller portions of position during earlier transactions, in order to reduce the losses due to permanent impact at later stages.

Linear permanent market impact with lag The main drawback of the linear permanent market impact model discussed above is that it allows for permanent changes in the security price to

Table 5 Optimal trading strategy under linear permanent market impact with lag

	c =1.0						c =10.0				
k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	0.885	0.755	0.594	0.382	0	10	0.892	0.791	0.608	0.422	0
9	0.885	0.753	0.593	0.380	0	9	0.899	0.758	0.589	0.403	0
8	0.885	0.753	0.595	0.380	0	8	0.891	0.754	0.621	0.398	0
7	0.885	0.755	0.595	0.379	0	7	0.891	0.789	0.625	0.391	0
6	0.886	0.753	0.595	0.380	0	6	0.9	0.751	0.627	0.404	0
5	0.886	0.753	0.593	0.377	0	5	0.9	0.759	0.594	0.355	0
4	0.886	0.753	0.595	0.379	0	4	0.9	0.761	0.622	0.388	0
3	0.886	0.754	0.595	0.379	0	3	0.9	0.771	0.624	0.384	0
2	0.886	0.754	0.597	0.381	0	2	0.899	0.778	0.651	0.415	0
1	0.887	0.754	0.594	0.378	0	1	0.9	0.766	0.608	0.370	0

Table 6 Optimal trading strategy under non-linear permanent market impact with lag

	c =1.0						c =10.0				
k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	k	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
10	0.815	0.642	0.424	0.227	0	10	0.896	0.896	0.534	0.368	0
9	0.821	0.620	0.416	0.221	0	9	0.966	0.706	0.451	0.324	0
8	0.815	0.618	0.432	0.220	0	8	0.896	0.677	0.599	0.313	0
7	0.813	0.641	0.434	0.217	0	7	0.896	0.896	0.617	0.294	0
6	0.830	0.616	0.434	0.221	0	6	0.993	0.675	0.625	0.324	0
5	0.830	0.620	0.419	0.206	0	5	0.993	0.697	0.487	0.214	0
4	0.823	0.625	0.432	0.216	0	4	0.993	0.725	0.602	0.286	0
3	0.823	0.629	0.433	0.215	0	3	0.999	0.769	0.613	0.278	0
2	0.821	0.635	0.447	0.224	0	2	0.968	0.806	0.675	0.351	0
1	0.837	0.626	0.425	0.210	0	1	0.993	0.740	0.540	0.236	0

be induced by infinitesimal trades. A more plausible model for the permanent impact function $\theta_t(\cdot)$ would be such that does not lead to permanent price changes if the trade is relatively small. Therefore, consider the functions $\gamma_t(\cdot)$ and $\theta_t(\cdot)$ in the form

$$\gamma_t(y) = \frac{1}{2c}y, \quad \theta_t(y) = \max\left\{0, \frac{1}{2c}(y-\lambda)\right\}. \tag{34}$$

Table 5 reports the solution for magnitude of lag $\lambda = 0.1$. The presence of permanent impact lag allows for selling larger portions of the position during earlier transactions. As a result, smaller portions of the position are left for liquidation at the final stage ($T = 3$).

Non-linear permanent market impact with lag Finally, let us consider the case of non-linear permanent market impact with lag. For this purpose, we select the price impact functions $\gamma_t(\cdot)$ and $\theta_t(\cdot)$ as follows:

$$\gamma_t(y) = \frac{1}{3c}y^2, \quad \theta_t(y) = \max\left\{0, \frac{1}{3c}\,\mathrm{sign}\,(y-\lambda)\,(y-\lambda)^2\right\}.$$

The form of function $\theta_t(\cdot)$ ensures its smoothness (continuous differentiability) in the "transition point" $y = \lambda$ that separates trades that do not cause permanent price changes ($y \leq \lambda$) and those that lead to permanent price drop ($y > \lambda$). As before, the value of λ is taken to be equal to $\lambda = 0.1$.

Solution in Table 6 shows less pronounced differences with the *naive* strategy due to the convexity of the non-linear impact functions, which implies smaller execution costs in comparison to the linear non-convex permanent market impact.

4 Conclusions

In this paper, we have developed an approach for optimal transaction execution based on the stochastic programming framework and a sample-path scenario model. The main contributions of our work with respect to the existing literature are as follows: (1) The developed trading strategy is truly dynamic, i.e. it allows for an adequate response to the observed market conditions at each time step; (2) The approach admits incorporation of various types of constraints in the trading strategy, such as institutional constraints, those expressing investor's preferences, etc. In particular, we considered risk-averse trading strategies, which control risk of financial losses during transactions using the Conditional Value-at-Risk measure; (3) The proposed approach can accommodate different models of temporary and permanent market impact, transaction costs, etc.; (4) A key feature of our approach is the sample-path scenario model, which does not impose restrictions on the price process of the security, and improves numerical tractability of the model.

To avoid anticipativity of the solutions, caused by specific properties of the sample-path scenario model, we introduced path partitioning with a new "lawn-mower" decision rule. Optimal liquidation with temporary market impact and this decision rule reduces to optimization problems with generally non-convex objective functions. In the case of concave temporary market impact, meaning that larger trades yield larger revenues, lower and upper bounds can be constructed by convex programming. On the other hand, the permanent market impact has essentially non-linear and non-convex nature, hence the optimal closing problem under permanent market impact is reducible to a nonlinear non-convex programming problem.

In frictionless market, the optimal trading strategy based on the "lawn-mower" constraint has 0–1 structure, i.e., it liquidates the entire position in one transaction. If the risk of financial losses during trades is controlled by CVaR constraints, the trading strategy becomes fractional. When "severe" temporary is present, the optimal trading strategy approaches the so-called "naive" strategy, which consists in selling equal portions of the position at each time step. Under "weaker" market impacts, the optimal trading strategy starts to deviate from the "naive" strategy. In presence of permanent market impact, the optimal trading strategy deviates significantly from the "naive" one by liquidating larger portions of the position at later stages (permanent price changes impel to sell less at earlier stages). It is shown that a "lag" in the permanent market impact function, which prevents small enough trades from inducing permanent price changes, allows for increasing the size of transactions at earlier stages.

Appendix

Proof of Proposition 1*:* Consider time $t = 1$. Then $\forall j \in G_1^k$: $\xi_1^j = f(x_1^{k(j,1)}, \xi_0^j) = f(x_1^{k(j,1)}, 1) = \tilde{x}_1^{k(j,1)}$. At the next step, $t = 2$, we have $\forall j \in G_2^k$: $\xi_2^j = f(x_2^{k(j,2)}, \xi_1^j, \xi_0^j) =$

$f(x_2^{k(j,2)}, \tilde{x}_1^{k(j,1)}, 1)$. Since $\tilde{x}_1^{k(j,1)}$ is constant $\forall j \in G_2^k$, then $f(x_2^{k(j,2)}, \tilde{x}_1^{k(j,1)}, 1) = \tilde{x}_2^{k(j,2)}$. Proceeding to $t = T$, in a finite number of steps we obtain that for any t and for all $j \in G_t^k$: $\xi_t^j = \tilde{x}_t^{k(j,t)}$. □

Proof of Proposition 2: Let us show that the "lawn-mower" principle (16) implies that the position variable ξ_t^j can be expressed as

$$\xi_t^j = \min\left\{x_1^{k(j,1)}, x_2^{k(j,2)}, \dots, x_t^{k(j,t)}\right\}, \quad j = 1, \dots, J, \quad t = 1, \dots, T. \tag{35}$$

Obviously, the assertion holds for $t = 1$: $\xi_1^j = \min\{1, x_1^{k(j,1)}\} = x_1^{k(j,1)} = \min\{x_1^{k(j,1)}\}$. Assume that (35) holds for some $t = \tau$, then

$$\begin{aligned}\xi_{\tau+1}^j &= \min\left\{\xi_\tau^j,\ x_{\tau+1}^{k(j,\tau+1)}\right\} = \min\left\{\min\left\{x_1^{k(j,1)}, \dots, x_\tau^{k(j,\tau)}\right\},\ x_{\tau+1}^{k(j,\tau+1)}\right\}\\ &= \min\left\{x_1^{k(j,1)}, \dots, x_{\tau+1}^{k(j,\tau+1)}\right\},\end{aligned}$$

which proves (35) by induction. □

Proof of Proposition 3: Consider $\xi_0^j = 1$, $\xi_t^j = 0$, $t = 1, \dots, T$, $j = 1, \dots, J$, and $x_t^{k(j,t)} = 0$, $t = 1, \dots, T$, $j = 1, \dots, J$. □

Proof of Proposition 4: First, observe that

$$\begin{aligned}\xi_t^j = \min\left\{\xi_{t-1}^j, x_t^{k(j,t)}\right\} &\Leftrightarrow \xi_{t-1}^j - \xi_t^j = \xi_{t-1}^j - \min\left\{\xi_{t-1}^j, x_t^{k(j,t)}\right\}\\ &\Leftrightarrow \xi_{t-1}^j - \xi_t^j = \max\left\{0, \xi_{t-1}^j - x_t^{k(j,t)}\right\},\end{aligned} \tag{36}$$

Taking into account form (36) of the "lawn-mower" principle and expression (35) for position variables ξ_t^j, problem (19) can be written as follows

$$\max \varphi(x) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\left\{0, \min\left\{x_1^{k(j,1)}, \dots, x_{t-1}^{k(j,t-1)}\right\} - x_t^{k(j,t)}\right\} \tag{37}$$

$$\text{s.t. } 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \quad t = 1, \dots, T, \quad j = 1, \dots, J.$$

Observe that $\varphi(x)$ is a piecewise-linear function of $x \in \mathbb{R}^K$, $K = \sum_{t=1}^{T} K_t$, and can be represented as

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\left\{0,\ x_{\tau_*}^{k(j,\tau_*)} - x_t^{k(j,t)}\right\}, \quad \text{where } \tau_* = \tau_*(j,t) = \underset{1 \le \tau \le t-1}{\arg\min}\left\{x_\tau^{k(j,\tau)}\right\}.$$

Using contradiction argument, we now show that there exists an optimal 0–1 solution to problem (37). For simplicity, let us consider the optimal solution of (37) with only one non-binary component $x_{t_0}^{k_0} \in (0, 1)$. Having all other elements of x fixed, consider $\varphi(\cdot)$ as a function of $x_{t_0}^{k_0}$ only:

$$\begin{aligned}\varphi\left(\ \cdot\ \middle|\ x_{t_0}^{k_0}\right) = \text{const}\ &+ \wp_1 \max\left\{0, 1 - x_{t_0}^{k_0}\right\}\\ &+ \wp_2 \max\left\{0, x_{t_0}^{k_0}\right\}\\ &+ \wp_3 \max\left\{0, -x_{t_0}^{k_0}\right\}\\ &+ \wp_4 \max\left\{0, x_{t_0}^{k_0} - 1\right\},\end{aligned} \tag{38}$$

where $\wp_1, \dots, \wp_4$ denote appropriate summations of S_t^j:

$$\wp_1 = \sum_{\left\{j \,\middle|\, k(j,t_0)=k_0,\ \forall\tau=1,\dots,t_0-1:\ x_\tau^{k(j,\tau)}=1\right\}} J^{-1} S_{t_0}^j\ ,$$

$$\wp_2 = \sum_{\left\{j,\ t>t_0 \,\middle|\, k(j,t_0)=k_0,\ \forall\ \tau\in\{1,\dots,t-1\}/\{t_0\}:\ x_\tau^{k(j,\tau)}=1,\ x_t^{k(j,t)}=0\right\}} J^{-1} S_t^j\ ,$$

$$\wp_3 = \sum_{\left\{j \,\middle|\, k(j,t_0)=k_0,\ \exists\ \tau\in\{1,\dots,t_0-1\}:\ x_\tau^{k(j,\tau)}=0\right\}} J^{-1} S_{t_0}^j\ ,$$

$$\wp_4 = \sum_{\left\{j,\ t>t_0 \,\middle|\, k(j,t_0)=k_0,\ \forall\ \tau\in\{1,\dots,t\}/\{t_0\}:\ x_\tau^{k(j,\tau)}=1\right\}} J^{-1} S_t^j\ .$$

Note that the last two terms in (38) equal to zero, so (38) takes the form

$$\varphi\left(\ \cdot\ \middle|\ x_{t_0}^{k_0}\right) = \text{const} + \wp_1(1 - x_{t_0}^{k_0}) + \wp_2 x_{t_0}^{k_0}. \tag{39}$$

If $\wp_1 \neq \wp_2$, function (39) can be improved by putting $x_{t_0}^{k_0} = 0$ or 1, which would mean that solution with a non-binary component cannot be optimal. If $\wp_1 = \wp_2$, the value of (39) does not change by selection $x_{t_0}^{k_0} = 0$ or 1, i.e., there exists an optimal 0–1 solution of (37).

The case with more than one fractional component in the optimal solution can be considered similarly, with the only difference that the variation of the objective function $\varphi(\cdot\,|\,\cdot)$ with respect to some variable with non-binary optimal value may have to be considered in a small ε-vicinity of an optimal point. □

Proof of Proposition 5: Using expression (36) for $\Delta\xi_t^j$, problem (19) can be rewritten as

$$\begin{aligned}\max_{\Xi}\quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\ \delta_t\left(\max\left\{0, \xi_{t-1}^j - x_t^{k(j,t)}\right\}\right)\\ \text{s. t.}\quad & \xi_t^j \le x_t^{k(j,t)}, \qquad t = 1, \dots, T, \quad j = 1, \dots, J,\\ & \xi_{t-1}^j - \xi_t^j = \max\left\{0,\ \xi_{t-1}^j - x_t^{k(j,t)}\right\}, \qquad t = 1, \dots, T, \quad j = 1, \dots, J,\\ & 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \qquad t = 1, \dots, T, \quad j = 1, \dots, J.\end{aligned}$$

Note that the first constraint $\xi_t^j \le x_t^{k(j,t)}$ follows form the "lawn-mower" principle $\xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}$. Therefore, this constraint does not change the feasible set. Taking into account the definition of variables u_t^j

$$u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, \quad t = 1, \dots, T, \quad j = 1, \dots, J, \tag{40}$$

and properties (i)–(v) of function $\delta_t(\cdot)$, the above formulation of problem (19) can be presented as

$$\max_{\Xi} \quad \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T} S_t^j \max\left\{0, \delta_t\left(u_t^j\right)\right\} \tag{41}$$

$$\begin{aligned}
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1, \dots, T, \quad j = 1, \dots, J, \\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)} && t = 1, \dots, T, \quad j = 1, \dots, J, \\
& \xi_{t-1}^j - \xi_t^j = \max\left\{0, u_t^j\right\}, && t = 1, \dots, T, \quad j = 1, \dots, J. \\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1, \dots, T, \quad j = 1, \dots, J.
\end{aligned}$$

Problem (41) differs from (20) only by the third constraint. Now we show that the optimal solution of the problem (20) does also satisfy the constraint

$$\xi_{t-1}^j - \xi_t^j = \max\left\{0, u_t^j\right\}, \tag{42}$$

which is an equivalent form of the lawn-mower principle (16), given the definition (40) of u_t^j.

Assume that the optimal solution of problem (20) does not satisfy the lawn-mower principle (16). Obviously, the optimal solution of (20) cannot satisfy inequality

$$\xi_t^j > \min\left\{\xi_{t-1}^j, x_t^{k(j,t)}\right\}$$

since it violates the first constraint of (20) and the definition of the set of trading strategies Ξ. Therefore, the only case left is when in the optimal solution of problem (20) we have for some t and j

$$\xi_t^j < \min\left\{\xi_{t-1}^j, x_t^{k(j,t)}\right\} \triangleq \chi_t^j.$$

Since $x_T^{k(j,T)} = 0$ by definition, there exists $t_1 > t$ such that $x_{t_1}^{k(j,t_1)} < \chi_t^j$. Let t_1 be the smallest possible, then

$$x_\tau^{k(j,\tau)} \ge \chi_t^j, \quad t \le \tau \le t_1 - 1.$$

Therefore, for all $\tau = t, \dots, t_1 - 1$, we can increase the values of position variables ξ_τ^j by positive amounts $\varepsilon_\tau^j = \chi_t^j - \xi_\tau^j > 0$ without violating feasibility:

$$\tilde{\xi}_\tau^j = \xi_\tau^j + \varepsilon_\tau^j = \chi_t^j, \quad t \le \tau \le t_1 - 1,$$

where $\tilde{\xi}_\tau^j$ are the new values of the position variables. As a result, variable $u_{t_1}^j$ increases by a positive amount: $\tilde{u}_{t_1}^j = \tilde{\xi}_{t_1-1}^j - x_{t_1}^{k(j,t_1)} = u_{t_1}^j + \varepsilon_{t_1-1}^j$. Positiveness of $\tilde{u}_{t_1}^j$,

$$\tilde{u}_{t_1}^j = \tilde{\xi}_{t_1-1}^j - x_{t_1}^{k(j,t_1)} = \chi_t^j - x_{t_1}^{k(j,t_1)} > 0,$$

and property (iii) of function $\delta_t(\cdot)$ imply that the objective of (20) has also been increased (the variable $u_{t_1}^j$ has either changed from negative to positive, or increased by positive amount $\varepsilon_{t_1-1}^j$).

This proves that the solution of (20) that does not satisfy the lawn-mower principle (16) is non-optimal. Consequently, the optimal solution of problem (20) does not change if the lawn-mower constraint (36) is added to the constraint set of (20), which would make the formulation of (20) identical to (41). Thus, we have shown that both problems (19) and (20) can be rewritten in the form (41), which implies that optimal values of their objectives, and their sets of optimal solutions in variables ξ_t^j and $x_t^{k(j,t)}$ coincide. □

Proof of Proposition 6: First, consider the case of monotonically non-increasing thresholds $x_{t-1}^{k(j,t-1)} \geq x_t^{k(j,t)}$, $t = 2, \ldots, T$. Assume that for some j and t in the optimal solution of (22) we have

$$\xi_t^j < x_t^{k(j,t)},$$

and let such t be the smallest possible for given path j. Evidently, in this case one can improve the objective term $p_{t+1}^j\, \delta_t(u_{t+1}^j)$ by increasing position variable ξ_t^j by a positive amount $\varepsilon_t^j = x_t^{k(j,t)} - \xi_t^j > 0$:

$$\tilde{\xi}_t^j = \xi_t^j + \varepsilon_t^j \quad \Rightarrow \quad \delta_t\big(\tilde{u}_{t+1}^j\big) = \delta_t\big(\tilde{\xi}_t^j - x_{t+1}^{k(j,t+1)}\big) = \delta_t\big(u_t^j + \varepsilon_t^j\big) > \delta_t\big(u_t^j\big),$$

where $\tilde{u}_{t+1}^j$ and $\tilde{\xi}_t^j$ are the new values of variables u_{t+1}^j and ξ_t^j. This contradicts our assumption that the solution with u_t^j, ξ_t^j is optimal. Thus, in the optimal solution with monotonic thresholds must be $\xi_t^j = x_t^{k(j,t)}$, implying that

$$\xi_{t-1}^j - \xi_t^j = \xi_{t-1}^j - x_t^{k(j,t)} = u_t^j \geq 0 \Rightarrow \xi_{t-1}^j - \xi_t^j = \max\left\{0, u_t^j\right\}$$
$$\Rightarrow u_t^j = \max\left\{0,\ \xi_{t-1}^j - x_t^{k(j,t)}\right\}.$$

Now suppose that thresholds $x_t^{k(j,t)}$ are non-monotonic. Let t be the smallest possible such that for some j we have

$$x_{t-1}^{k(j,t-1)} < x_t^{k(j,t)}.$$

If $\xi_t^j = \xi_{t-1}^j$ then $u_t^j < 0$ and the proposition holds. If, on the contrary, $\xi_t^j < \xi_{t-1}^j$, the solution (namely, objective variable u_{t+1}^j) can be improved by putting $\tilde{\xi}_t^j = \xi_{t-1}^j$, but u_t^j will remain negative as long as it is optimal to keep $x_{t-1}^{k(j,t-1)} < x_t^{k(j,t)}$. Therefore in the optimal solution of problem (22) the negative value of variable u_t^j implies that $\xi_t^j < x_t^{k(j,t)}$ and $\xi_t^j - \xi_{t-1}^j = 0 = \max\{0, u_t^j\}$. □

Proof of Proposition 7: According to (36),

$$\Delta\xi_t^j = \max\left\{0,\ \xi_{t-1}^j - x_t^{k(j,t)}\right\}.$$

If $\Delta\xi_t^j > 0$, then $\Delta\xi_t^j = \xi_{t-1}^j - x_t^{k(j,t)}$, and the result immediately follows. □

Proof of Proposition 8: It suffices to note that the CVaR-constrained set of feasible solutions is a subset of the original feasible region, so the arguments used in establishing Propositions 5, 6, and Corollary 2 remain valid. □

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