We will prove that there exists a vector $\binom{f_1}{f_2}$ with entries f_1, f_2 in disc algebra C_A such that its inner and outer parts do not belong to C_A. Here C_A denotes the set of all functions f analytic in the open unit disk $\mathbf{D}$ and continuous in its closure $\overline{\mathbf{D}}$, or equivalently, the set of all continuous functions on the unit circle $\mathbf{T} = \partial\mathbf{D}$ with vanishing negative Fourier coefficients, $C_A = H^2 \cap C(\mathbf{T})$. Clearly the standard conformal mapping $z = (w-1)/(w+1)$ of the right-half plane onto the unit disk generates bijection between C_A in the half plane and C_A in the unit disk, so we use the same notation for both.

In the general case, it is difficult to find inner–outer factorization for a matrix-valued function. But in our case of a vector-valued function, it is sufficiently easy to do. Namely, it is well known that outer part of vector $\binom{f_1}{f_2}$ is usual (scalar valued) outer function f such that $|f|^2 = |f_1|^2 + |f_2|^2$ a.e. on $\mathbf{T} = \partial\mathbf{D}$, and inner part of this function equals $\binom{f_1/f}{f_2/f}$. Therefore, it is sufficient to construct a pair of functions f_1, f_2 in C_A such that the outer function f defined by $|f|^2 = |f_1|^2 + |f_2|^2$ is not continuous. Recall that an outer function f is defined by its modulus uniquely up to a constant unimodular factor, see [2]

$$f(z) = \exp\,(\log|f(z)| + i\mathcal{H}(\log|f(z)|)), \qquad z \in \mathbf{T}$$

where $\mathcal{H}$ denotes Hilbert transform

$$(\mathcal{H}g)(e^{i\varphi}) = \frac{1}{2\pi}\lim_{\epsilon\to 0+}\int_{\substack{\vartheta\in[-\pi,\,\pi)\\ |e^{i\vartheta}-e^{i\varphi}|>\epsilon}} \cot\left(\frac{\varphi-\vartheta}{2}\right) g(e^{i\vartheta})\,d\vartheta, \qquad \varphi \in [-\pi,\,\pi). \quad (1)$$

The general result by Globevnik says that any continuous nonnegative function w satisfying $\int_{\mathbf{T}} \log w > -\infty$ can be represented as $w = (|f_1|^2 + |f_2|^2)^{1/2}$, $f_{1,2} \in C_A$, see [3] (case $w > 0$), [4] (general case). Since not all positive continuous functions have continuous Hilbert transform, our counterexample can be obtained immediately from that result.

We would like to present an elementary construction of the counterexample not requiring an advanced technique of [3], [4].

Let us consider a vector $\binom{f_1}{f_2}$, $f_1 \equiv 1$ and f_2 be an outer function with modulus F, where F is C^1-smooth positive function on $\mathbf{T}$ except the point $z = 1$. Let us suppose that in the point $z = 1$ our function F has the following asymptotic

$$F(e^{ix}) = \begin{cases} \frac{1}{(\log\,(1/x))^{1/2}} + o(x^2), & x > 0 \\ o(x), & x \le 0 \end{cases}.$$

It is evident that $f_2 \in C_A$. Indeed, function F is continuous on $\mathbf{T}$. Since $\log F$ is C^1-smooth on the set $\mathbf{T}\backslash\{1\}$, the function $\mathcal{H}(\log F)$ (and therefore the functions $e^{i\mathcal{H}(\log F)}$ and $f_2 = F \cdot e^{i\mathcal{H}(\log F)}$) are continuous on this set. But $\lim_{z\to 1}|f_2(z)| = \lim_{z\to 1}F(z) = 0 = |f_2(1)|$ and hence f_2 is continuous also at the point $z = 1$. (Note that it does not matter whether $\mathcal{H}(\log F)$ is continuous at this point or not.) By definition $f_2 \in \mathcal{H}^2$ and hence $f_2 \in C_A$.

Now let f be an outer function satisfying $|f|^2 = |f_1|^2 + |f_2|^2$. Note that the function $\log|f| = (1/2)\log(|f_1|^2 + |f_2|^2) = (1/2)\log(1 + |f_2|^2) = (1/2)\log(1 + F^2)$ is C^1-smooth on the set $\mathbf{T}\backslash\{1\}$ and has near the point $z = 1$ the following asymptotic

$$\log|f(e^{ix})| = \begin{cases} \frac{1}{2\cdot\log\,(1/x)} \cdot (1 + o(1)), & x > 0 \\ o(x^2), & x \le 0 \end{cases}.$$

See Fig. 1. Since the kernel $\cot((\varphi - \vartheta)/2)$ in (1) has for ϑ near φ the following singularity

$$\cot\left(\frac{\varphi-\vartheta}{2}\right) = \frac{2}{\varphi-\vartheta}\cdot(1 + o(\varphi-\vartheta))$$

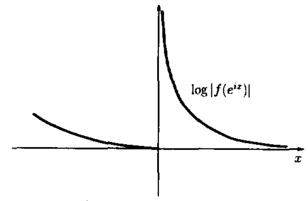

Fig. 1.

it is an easy exercise to verify that the Hilbert transform $\mathcal{H}(\log|f|)$ of $\log|f|$ is not continuous at the point $z = 1$ (tends to ∞ when $z \to 1$). Because $\log|f|$ is C^2-smooth on $\mathbf{T}\backslash\{1\}$ the function $\mathcal{H}(|\log f|)$ is continuous on this set. Therefore, the function $e^{i\mathcal{H}(\log|f|)}$ is not continuous at the point $z = 1$. Hence, because $e^{\log|f(1)|} \neq 0$ the function $f = e^{\log|f|} \cdot e^{i\mathcal{H}(\log|f|)}$ is also not continuous at this point. □

Remark: In the above construction it was essential that one of the functions has a zero on the unit circle $\mathbf{T}$. Nevertheless it is possible using the technique developed in [3], [4] to construct a counterexample such that both functions f_1 and f_2 have no zeros on $\mathbf{T}$ (and, moreover on the unit disc $\mathbf{D}$).

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On Relaxation Algorithms in Computation of Noncooperative Equilibria

Stanislav Uryas'ev and Reuven Y. Rubinstein

I. Introduction

This paper considers a special class of numerical algorithms, the so-called relaxation algorithm, for Nash equilibrium points in noncooperative games. The relaxation algorithms are studied in papers [2], [3], [5] for the deterministic case. Convergence conditions of this algorithm are based on fixed point theorems. For example,

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S. Uryas'ev was with the International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria and is now with Brookhaven National Laboratory, Department of Advanced Technology, Upton, NY 11973 USA.

R. Rubinstein is with the Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel.

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Başar [2] and Li, Başar [5] have proved its convergence for a two-player game via the contraction mapping theorem. For the quadratic case these conditions can be easily checked. For other nonlinear payoff functions it is sometimes difficult to check these convergence conditions. In this paper, we propose an alternative approach using the residual terms of the Nikaido–Isoda function. The convergence theorem is proved for nonsmooth weakly convex–concave Nikaido–Isoda functions. The family of weakly convex–concave functions is broad enough for applications, since it includes the family of smooth functions. When the payoff functions are twice continuously differentiable, the condition for the residual terms is reduced to strict positiveness of a matrix representing the difference of the Hessians of the Nikaido–Isoda function with respect to the first and second groups of variables. An analogous condition was used to prove convergence of the gradient-type algorithm for the Nash equilibrium problem [9]. In this paper we discuss only deterministic case; nevertheless this approach can be generalized for the stochastic Nash equilibrium problems with uncertainties in parameters (see, statement of the stochastic problem and approach for the solution in [8]).

II. Nash Equilibria for Noncooperative n-Person Games

Let X be a convex closed subset of the product $\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n} = \mathbb{R}^m$ of Euclidean spaces $\mathbb{R}^{m_i}$, $i = 1,\cdots,n$. A point $x_i \in \mathbb{R}^{m_i}$ is a strategy of the ith player $i = 1,\cdots,n$ and $\phi_i(x) = \phi_i(x_1,\cdots,x_n)$ is his payoff function. The element $(x_1,\cdots,x_{i-1}y_i, x_{i+1},\cdots,x_i)$ is denoted by $(y_i \mid x)$. The point $x^* = (x_1^*,\cdots,x_n^*) \in X$ is referred to as the Nash equilibrium of an n-person game on X if for $i = 1,\cdots,n$

$$\phi_i(x^*) = \max_{y_i}\{\phi_i(y_i \mid x^*): (y_i \mid x^*) \in X\}.$$

Let us introduce the Nikaido–Isoda [6] function $\Psi(x, y)$

$$\Psi(x, y) = \sum_{i=1}^{n}[\phi_i(y_i \mid x) - \phi_i(x)], \qquad y = (y_1,\cdots,y_n). \quad (2.1)$$

It is not difficult to see that $\Psi(x, x) \equiv 0$, $x \in X$. We suppose that the functions $\phi_i(x)$, $i = 1,\cdots,n$ are continuous on X. A point $x^* \in X$ is defined as a Nash normalized equilibrium point if

$$\max_{y\in X}\Psi(x^*, y) = 0. \quad (2.2)$$

Lemma 2.1 [1]: A Nash normalized equilibrium point is also a Nash equilibrium point; the reverse is true if

$$X = X_1 \times\cdots\times X_n; \qquad X_i \subset \mathbb{R}^{m_i}, i = 1,\cdots,n.$$

III. Algorithm with Relaxation

Let us define an optimum response function $Z(x)$

$$Z(x) = \operatorname{Arg}\max_{y\in X}\Psi(x, y). \quad (3.1)$$

We consider the following algorithm with relaxation

$$x^{s+1} = (1-\alpha_s)x^s + \alpha_s z^s, \quad z^s \in Z(x^s), \quad 0 < \alpha_s \le 1, \quad s = 0, 1, \cdots. \quad (3.2)$$

Convergence of this algorithm was proved in [2] with $\alpha^s = \alpha =$ const and in [3] for the convex–concave function

$$\Phi(x, y) = \sum_{i=1}^{n}\phi_i(y_i \mid x)$$

where

$$\sum_{i=1}^{n}\phi_i(x) = 0, \quad \text{for} \quad x \in X \quad (3.3)$$

with step sizes satisfying conditions

$$\sum_{s=0}^{\infty}\alpha_s = \infty, \qquad \alpha_s \to 0 \quad \text{as} \quad s \to \infty. \quad (3.4)$$

IV. Convergence of Algorithm with Relaxation

Here we prove the convergence of relaxation algorithm (3.2) for weakly convex–concave [7] functions $\Psi(x, y)$. The family of weakly convex functions includes smooth and convex functions and is closed with respect to the summation and pointwise maximum. We use the definition of weakly convex functions proposed in [9].

Definition 4.1: Let X be a convex subset of the Euclidean space $\mathbb{R}^m$. A continuous function $f: X \to \mathbb{R}$ is called weakly convex on X if for all $x \in X$, $y \in X$, $0 \le \alpha \le 1$ the following inequality holds

$$\alpha f(x) + (1-\alpha)f(y) \ge f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)r(x, y)$$

where the remainder $r: X \times X \to \mathbb{R}$ satisfies

$$\frac{r(x, y)}{\| x - y \|} \to 0, \quad \text{as} \quad x \to z, y \to z \quad (4.1)$$

for all $z \in X$.

Let $\Psi: X \times X \to \mathbb{R}$ be a function defined on a product $X \times X$, where X is a convex closed subset of the Euclidean space $\mathbb{R}^m$. Further, we consider that $\Psi(x, y)$ is weakly convex on X with respect to the first argument, i.e.,

$$\alpha_1\Psi(x, z)+\alpha_2\Psi(y, z) \ge \Psi(\alpha_1 x+\alpha_2 y, z)+\alpha_1\alpha_2 r_z(x, y) \quad (4.2)$$

for all x, y, $z \in X$; $\alpha_1 + \alpha_2 = 1$; $\alpha_1\alpha_2 \ge 0$ and

$$\frac{r_z(x, y)}{\| x - y \|} \to 0, \quad \text{as} \quad \| x - y \| \to 0, \qquad \text{for all } x \in X.$$

We suppose that the function $\Psi(x, y)$ is weakly concave with respect to the second argument on X, i.e.,

$$\alpha_1\Psi(z, x)+\alpha_2\Psi(z, y) \le \Psi(z, \alpha_1 x+\alpha_2 y)+\alpha_1\alpha_2\mu_z(x, y) \quad (4.3)$$

for all x, y, $z \in X$; $\alpha_1 + \alpha_2 = 1$; α_1, $\alpha_2 \ge 0$ and also

$$\frac{\mu_z(x, y)}{\| x - y \|} \to 0 \quad \text{as} \quad \| x - y \| \to 0, \qquad \text{for all } z \in X.$$

We say that the function $\Psi(x, y)$ is weakly convex–concave, if it satisfies conditions (4.2) and (4.3).

Theorem 4.1: Let

1) the set X be a convex compact subset of the Euclidean space $\mathbb{R}^m$;
2) the function $\Psi: X \times X \to \mathbb{R}$ be a continuous weakly convex–concave function and

$$\Psi(x, x) = 0, \qquad x \in X; \quad (4.4)$$

3) the optimum response function

$$Z(x) = \operatorname{Arg}\max_{y\in X}\Psi(x, y)$$

be single valued for any $x \in X$ and the function $Z(x)$ be continuous on X;
4) the residual term $r_z(x, y)$ be uniformly continuous on X with respect to the variable z for x, $y \in X$;
5) the residual terms satisfy the inequality

$$r_y(x, y) - \mu_x(y, x) \ge \beta(\| x - y \|), \qquad x, y \in X \quad (4.5)$$

where β is a strictly monotonically increasing function (i.e., $\beta(t_2) > \beta(t_1)$ if $t_2 > t_1$), and $\beta(0) = 0$;

6) the step sizes satisfy the conditions

$$\alpha_s > 0, s = 0, 1, \cdots; \sum_{s=0}^{\infty} \alpha_s = \infty; \tag{4.6}$$

$$\alpha_s \to 0 \quad \text{as} \quad s \to \infty. \tag{4.7}$$

Then there exists a unique Nash normalized equilibrium point x^* to which the relaxation algorithm

$$x^{s+1} = (1 - \alpha_s)x^s + \alpha_s Z(x^s), \qquad s = 0, 1, \cdots \tag{4.8}$$

converges.

Proof of Theorem 4.1: Existence of a normalized equilibrium point x^* follows from the Kakutani's theorem (see, for example, [3]).

Let us denote by $G(x) = \partial_y \Psi(x, y) \mid_{y=x}$ the differential [9] of the function $\Psi(x, y)$ with respect to the second argument at point (x, x). The following inequality was proved in [9]

$$\langle g(x) - g(y), y - x\rangle \geq r_y(x, y) - \mu_x(y, x),$$

$$x, y \in X, g(x) \in G(x), g(y) \in G(y).$$

Condition 5 of the theorem implies strict anti-monotonicity (see [9]) of the multivalued map $G(x)$ and, consequently, uniqueness of the normalized equilibrium point x^*. The normalized equilibrium point is also a Nash equilibrium point (see Lemma 2.1).

To prove convergence of algorithm (4.8) we use the following Lyapunov function

$$V(x) = \Psi(x, Z(x)).$$

The function $V(x)$ equals zero at the normalized equilibrium point x^*

$$V(x^*) = 0$$

and

$$V(x^*) > 0, x \in X, x \neq x^*.$$

Weak convexity of the function $\Psi(x, y)$ with respect to x implies

$$\begin{aligned} V(x^{s+1}) &= \Psi(x^{s+1}, Z(x^{s+1})) = \Psi((1-\alpha_s)x^s \\ &\quad + \alpha_s Z(x^s), Z(x^{s+1})) \\ &\leq (1-\alpha_s)\Psi(x^s, Z(x^{s+1})) + \alpha_s \Psi(Z(x^s), Z(x^{s+1})) \\ &\quad - (1-\alpha_s)\alpha_s r_{Z(x^{s+1})}(x^s, Z(x^s)). \end{aligned}$$

Since

$$\Psi(x^s, Z(x^{s+1})) \leq \Psi(x^s, Z(x^s))$$

and

$$\Psi(x^s, x^s) = 0$$

then

$$\begin{aligned} V(x^{s+1}) &\leq (1-\alpha_s)\Psi(x^s, Z(x^s)) + \alpha_s \Psi(x^s, x^s) \\ &\quad + \alpha_s \Psi(Z(x^s), Z(x^{s+1})) \\ &\quad - (1-\alpha_s)\alpha_s r_{Z(x^{s+1})}(x^s, Z(x^s)). \end{aligned} \tag{4.9}$$

Weak concavity of the function $\Psi(x, y)$ with respect to the variable y implies

$$\begin{aligned} &(1-\alpha_s)\Psi(x^s, Z(x^s)) + \alpha_s \Psi(x^s, x^s) \\ &\quad \leq \Psi(x^s, (1-\alpha_s)Z(x^s) + \alpha_s x^s) \\ &\quad + (1-\alpha_s)\alpha_s \mu_{x^s}(Z(x^s), x^s) \leq V(x^s) \\ &\quad + (1-\alpha_s)\alpha_s \mu_{x^s}(Z(x^s), x^s). \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) we have

$$\begin{aligned} V(x^{s+1}) &\leq V(x^s) + (1-\alpha_s)\alpha_s[\mu_{x^s}(Z(x^s), x^s) \\ &\quad - r_{Z(x^{s+1})}(x^s, Z(x^s))] \\ &\quad + \alpha_s \Psi(Z(x^s), Z(x^{s+1})) \\ &= V(x^s) + (1-\alpha_s)\alpha_s[\mu_{x^s}(Z(x^s), x^s) \\ &\quad - r_{Z(x^s)}(x^s, Z(x^s))] \\ &\quad + (1-\alpha_s)\alpha_s[r_{Z(x^s)}(x^s, Z(x^s)) \\ &\quad - r_{Z(x^{s+1})}(x^s, Z(x^s))] \\ &\quad + \alpha_s \Psi(Z(x^s), Z(x^{s+1})). \end{aligned} \tag{4.11}$$

Further, in view of (4.5)

$$\begin{aligned} V(x^{s+1}) &\leq V(x^s) - (1-\alpha_s)\alpha_s \beta(\|x^s - Z(x^s)\|) \\ &\quad + (1-\alpha_s)\alpha_s[r_{Z(x^s)}(x^s, Z(x^s)) \\ &\quad - r_{Z(x^{s+1})}(x^s, Z(x^s))] + \alpha_s \Psi(Z(x^s), Z(x^{s+1})). \end{aligned} \tag{4.12}$$

Before proceeding to prove the theorem, let us prove a lemma about convergence of some subsequence of the sequence x^s.

Lemma 4.1: There exists a subsequence x^{s_k} of the sequence x^s, such that

$$x^{s_k} \to x^*, \qquad k \to \infty.$$

Proof: Suppose the lemma is not valid and there exists $\epsilon > 0$ and $\bar{s}$ such that

$$\|x^s - Z(x^s)\| \geq \epsilon > 0, \qquad s > \bar{s}. \tag{4.13}$$

It follows from (4.8) that

$$\| x^{s+1} - x^s \| = \|(1-\alpha_s)x^s + \alpha_s Z(x^s) - x^s\| = \alpha_s \|Z(x^s) - x^s\|.$$

The set X is compact and the value $\| x - Z(x) \|$ is bounded on X. Since $\alpha_s \to 0$ (see condition (4.7) of the theorem), then

$$\| x^{s+1} - x^s \| \to 0, \qquad s \to \infty. \tag{4.14}$$

Thus, the continuity of the function $Z(x)$ (condition 3) implies

$$\|Z(x^{s+1}) - Z(x^s)\| \to 0, \qquad s \to \infty \tag{4.15}$$

and, consequently, condition 2 implies

$$\Psi(Z(x^{s+1}), Z(x^s)) \to 0, \qquad s \to \infty. \tag{4.16}$$

Since the residual term $r_z(x, y)$ is uniformly continuous with respect to z, then with limit equation (4.15)

$$[r_{Z(x^s)}(x^s, Z(x^s)) - r_{Z(x^{s+1})}(x^s, Z(x^s))] \to 0, \quad \text{as} \quad s \to \infty. \tag{4.17}$$

If the number $\bar{s}$ is sufficiently large, then (4.7), (4.16), and (4.17) imply

$$\begin{aligned} &[r_{Z(x^s)}(x^s, Z(x^s)) - r_{Z(x^{s+1})}(x^s, Z(x^s))] \\ &\qquad + (1-\alpha_s)^{-1}\Psi(Z(x^s), Z(x^{s+1})) \leq 2^{-1}\beta(\epsilon) \end{aligned} \tag{4.18}$$

for $s > \bar{s}$. Further, combining (4.12), (4.13), and (4.18)

$$\begin{aligned} V(x^{s+1}) &\leq V(x^s) - (1-\alpha_s)\alpha_s \beta(\|x^s - Z(x^s)\|) \\ &\quad + (1-\alpha_s)\alpha_s[r_{Z(x^s)}(x^s, Z(x^s)) \\ &\quad - r_{Z(x^{s+1})}(x^s, Z(x^s)) \\ &\quad + (1-\alpha_s)^{-1}\Psi(Z(x^s), Z(x^{s+1}))] \\ &\leq V(x^s) - (1-\alpha_s)\alpha_s\beta(\epsilon) + (1-\alpha_s)\alpha_s[2^{-1}\beta(\epsilon)] \\ &\leq V(x^s) - 2^{-1}(1-\alpha_s)\alpha_s\beta(\epsilon) \leq V(x^{\bar{s}}) \\ &\quad - 2^{-1}\beta(\epsilon)\sum_{k=\bar{s}}^{s}(1-\alpha_k)\alpha_k. \end{aligned}$$

Thus,

$$0 \le V(x^{s+1}) \le V(x^{\bar{s}}) - 2^{-1}\beta(\epsilon)\sum_{k=\bar{s}}^{s}(1-\alpha_k)\alpha_k. \tag{4.19}$$

In view of condition 6

$$\sum_{k=\bar{s}}^{s}(1-\alpha_k)\alpha_k \to +\infty \quad \text{as} \quad s \to \infty.$$

The last limit equation contradicts inequality (4.19), which proves the lemma. □

Now, let us prove Theorem 4.1. Suppose its statement is not valid. It follows from Lemma 4.1 and (4.14) that there exists a subsequence s_l and $\delta > 0$ such that

$$V(x^{s_l+1}) > V(x^{s_l}) \quad \text{and} \quad \|x^{s_l} - Z(x^{s_l})\| \ge \delta > 0.$$

On the other hand, inequality (4.19) implies

$$V(x^{s_l+1}) \le V(x^{s_l}) - 2^{-1}(1-\alpha_{s_l})\alpha_{s_l}\beta(\delta)$$

for sufficiently large numbers l. This contradiction proves the theorem. □

V. Examples

We consider two examples with twice differentiable function $\Psi(x, y)$ with respect to both arguments. Denote by $A(x, y) = \Psi_{xx}(x, y)$ the Hessian with respect to the first argument, and by $B(x, y) = \Psi_{yy}(x, y)$ the Hessian with respect to the second.

Lemma 5.1 [9]: If the function $\Psi(x, y)$ is twice continuously differentiable with respect to both arguments on the set $X \times X$ then (see (4.2), (4.3)) the reminders $r_z(x, y)$, $\mu_z(x, y)$ are given by

$$r_y(x, y) = \frac{1}{2}\langle A(x, x)(x-y), x-y\rangle + o_1(\| x-y \|^2),$$

$$\mu_x(y, x) = \frac{1}{2}\langle B(x, x)(x-y), x-y\rangle + o_2(\| x-y \|^2)$$

where

$$\frac{o_i(\| x-y \|^2)}{\| x-y \|^2} \to 0 \quad \text{for} \quad \| x-y \| \to 0, \qquad i = 1, 2.$$

Moreover, if the function $\Psi(x, y)$ is convex with respect to x, then $o_1(\| x-y \|^2) = 0$, and if the function $\Psi(x, y)$ is concave with respect to y, then $o_2(\| x-y \|^2) = 0$.

Denote $Q(x, x) = A(x, x) - B(x, x)$, then by Lemma 5.1

$$r_y(x, y) - \mu_x(y, x) = \frac{1}{2}\langle Q(x, x)(x-y), x-y\rangle + o(\| x-y \|^2).$$

Thus, to prove the key condition 5 of Theorem 4.1 it suffices to prove that

$$\frac{1}{2}\langle Q(x, x)(x-y), x-y\rangle + o(\| x-y \|^2) \ge \beta(\| x-y \|), \qquad x, y \in X. \tag{5.20}$$

If the function $\Psi(x, y)$ is convex–concave, then

$$o(\| x-y \|^2) = 0$$

and condition (5.20) means strict positiveness of the matrix $Q(x, x)$ on X.

Often, convex–concavity of the function $\Psi(x, y)$ is a rather restrictive condition. If $\| X \|$ is sufficiently small, i.e., the equilibrium point is known with good precision, then strict positiveness of the matrix $Q(x, x)$ implies (5.20) without that restriction. Indeed, if

$$\langle Q(x, x)(x-y), x-y\rangle \ge \nu \| x-y \|, \qquad \nu > 0 \tag{5.21}$$

then for sufficiently small values $\| x-y \|$

$$\frac{1}{2}\langle Q(x, x)(x-y), x-y\rangle + o(\| x-y \|^2) \ge \frac{1}{3}\nu \| x-y \|. \tag{5.22}$$

Therefore, local convergence of Algorithm 3.2 can be proved if the matrix $Q(x, x)$ is strictly positive. We explain this approach with two illustrative examples.

Example 1—An Optimization Problem: Let us consider the following two-player game

$$n = 2, X = X_1 \times X_2 \subset \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \tag{5.23}$$

$$\phi_1(x_1, x_2) = -l(x_1, x_2), \phi_2(x_1, x_2) = -l(x_1, x_2). \tag{5.24}$$

In this case, the set of Nash equilibrium points coincides with the solution set of the optimization problem

$$\min_{(x_1, x_2) \in X_1 \times X_2} l(x_1, x_2)$$

for a smooth function $l(x_1, x_2)$.

The function $\Psi(x, y)$ (see (2.1)) is given by

$$\Psi(x, y) = -l(x_1, y_2) - l(y_1, x_2) + 2l(x_1, x_2).$$

In this case

$$\begin{aligned}(x_1^{s+1}, x_2^{s+1}) &\in \operatorname{Arg}\max_{y\in X} \Psi((x_1^s, x_2^s), y)\\ &= \operatorname{Arg}\max_{y\in X}\sum_{i=1}^{2}[\phi_i(y_i \mid (x_1^s, x_2^s)) - \phi_i(x_1^s, x_2^s)]\\ &= \operatorname{Arg}\max_{(x_1, x_2)\in X_1\times X_2}[-l(x_1, x_2^s) - l(x_1^s, x_2)\\ &\quad + 2l(x_1^s, x_2^s)]\\ &= \operatorname{Arg}\min_{(x_1, x_2)\in X_1\times X_2}[l(x_1, x_2^s) + l(x_1^s, x_2)].\end{aligned}$$

It is easy to see that

$$\begin{aligned}A(x, x) &= \begin{pmatrix}\Psi_{x_1x_1}(x, y)|_{y=x} & \Psi_{x_1x_2}(x, y)|_{y=x}\\ \Psi_{x_2x_1}(x, y)|_{y=x} & \Psi_{x_2x_2}(x, y)|_{y=x}\end{pmatrix}\\ &= \begin{pmatrix} l_{x_1x_1}(x_1, x_2) & 2l_{x_1x_2}(x_1, x_2)\\ 2l_{x_2x_1}(x_1, x_2) & l_{x_2x_2}(x_1, x_2)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}B(y, y) &= \begin{pmatrix}\Psi_{y_1y_1}(x, y)|_{x=y} & \Psi_{y_1y_2}(x, y)|_{x=y}\\ \Psi_{y_2y_1}(x, y)|_{x=y} & \Psi_{y_2y_2}(x, y)|_{x=y}\end{pmatrix}\\ &= \begin{pmatrix}-l_{y_1y_1}(y_1, y_2) & 0\\ 0 & -l_{y_2y_2}(y_1, y_2)\end{pmatrix}\end{aligned}$$

thus

$$\begin{aligned}Q(x, x) &= A(x, x) - B(x, x)\\ &= \begin{pmatrix}2l_{x_1x_1}(x_1, x_2) & 2l_{x_1x_2}(x_1, x_2)\\ 2l_{x_2x_1}(x_1, x_2) & 2l_{x_2x_2}(x_1, x_2)\end{pmatrix} = 2l_{xx}(x).\end{aligned}$$

Therefore, the matrix $Q(x, x)$ is positive if the Hessian $l_{xx}(x)$ is strictly positive, i.e., the function $l(x)$ is strictly convex.

Example 2—The Model for International Oil Trade [4]: There is a market for a single homogeneous product which consists of some sellers (exporters) and a single buyer (importer). Let $i = 1, \cdots, n$ be the exporters, $f^i(z)$ the marginal cost at which any exporter i produces the amount z of the product for marketing and $r(z)$ the price at which the importer would agree to buy that amount; with x_i denoting the amount of the product sold by exporter i, his revenue $\phi_i(x)$ can be expressed as

$$\phi_i(x) = r(x_1 + \cdots + x_n)x_i - \int_0^{x_i} f^i(z)\,dz.$$

It follows from the sense of the problem that $x_i \geq 0$, $f^i(z) \geq 0$, $r(z) \geq 0$, $i = 1, \cdots, n$. We assume also that exporter i is able to sell not more than μ_i of the product. Supposing that each exporter will choose the amount x_i to maximize his revenue in any market situation characterized by a vector $x = (x_1, \cdots, x_n)$, then the problem will read as follows.

Find an equilibrium point $x^* = (x_1^*, \cdots, x_n^*)$ such that

$$\phi_i(x^*) = \max_{0 \leq y_i \leq \mu_i} \phi_i(x_1^*, \cdots, x_{i-1}^*, y_i, x_{i+1}^*, \cdots, x_n^*),$$

$$i = 1, \cdots, n.$$

The feasible set

$$X = \{x \in \mathbf{R}^n : 0 \leq x_i \leq \mu_i, i = 1, \cdots, n\}$$

is convex and compact. The function $\Psi(x, y)$ (see (2.1)) is given by

$$\begin{aligned}\Psi(x, y) = &\sum_{i=1}^{n} r(x_1 + \cdots + x_{i-1} + y_i + x_{i+1} + \cdots + x_n) y_i \\ &- \sum_{i=1}^{n} \int_0^{y_i} f^i(z)\, dz \\ &- \sum_{i=1}^{n} r(x_1 + \cdots + x_n) x_i + \sum_{i=1}^{n} \int_0^{x_i} f^i(z)\, dz.\end{aligned}$$

We assume that the functions $f^i(z)$, $i = 1, \cdots, n$ are continuously differentiable and the function $r(z)$ is twice continuously differentiable. Denoting

$$A(x, x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad B(x, x) = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} \end{pmatrix},$$

$$Q(x, x) = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}, \quad z = \sum_{k=1}^{n} x_i$$

it is not difficult to find

$$a_{ii} = -r_{zz}(z) x_i - 2r_z(z) + f_{x_i}^i(x_i), \quad i = 1, \cdots, n;$$

$$a_{ij} = -r_{zz}(z)(x_i + x_j) - 2r_z(z), \quad i \neq j; i, j = 1, \cdots, n;$$

$$b_{ii} = r_{zz}(z) x_i + 2r_z(z) - f_{x_i}^i(x_i), \quad i = 1, \cdots, n;$$

$$b_{ij} = 0, \quad i \neq j; i, j = 1, \cdots, n;$$

$$q_{ii} = -2r_{zz}(z) x_i - 4r_z(z) + 2f_{x_i}^i(x_i), \quad i = 1, \cdots, n;$$

$$q_{ij} = -r_{zz}(z)(x_i + x_j) - 2r_z(z), \quad i \neq j; i, j = 1, \cdots, n;$$

$$\begin{aligned}Q(x, x) = &-r_{zz}(z) \begin{pmatrix} x_1 x_1 & \cdots & x_1 \\ x_2 x_2 & \cdots & x_2 \\ \cdot & \cdot & \cdot \\ x_n x_n & \cdots & x_n \end{pmatrix} \\ &- r_{zz}(z) \begin{pmatrix} x_1 x_2 & \cdots & x_n \\ x_1 x_2 & \cdots & x_n \\ \cdot & \cdot & \cdot \\ x_1 x_2 & \cdots & x_n \end{pmatrix} \\ &- 2r_z(z) \begin{pmatrix} 11 & \cdots & 1 \\ 11 & \cdots & 1 \\ \cdot & \cdot & \cdot \\ 11 & \cdots & 1 \end{pmatrix} \\ &+ 2 \begin{pmatrix} f_{x_1}^1(x_1) - r_z(z) & & 0 \\ & \ddots & \\ 0 & & f_{x_n}^n(x_n) - r_z(z) \end{pmatrix}.\end{aligned}$$

Let us see under what conditions the matrix $Q(x, x)$ satisfies (5.21). Let $e = (1, \cdots, 1)$ be the n-dimensional vector, and

$$\rho(x) = \min_{1 \leq 1 \leq n} (f_{x_i}^i(x_i) - r_z(z)).$$

Assume also that

$$r_z(z) \leq 0, \; r_{zz}(z) \geq 0 \quad \text{for} \quad x \in X. \tag{5.25}$$

The following inequality is valid

$$\begin{aligned}\langle Q(x, x)h, h\rangle &\geq -2r_{zz}(z)\langle x, h\rangle\langle e, h\rangle \\ &\quad - 2r_z(z)\langle e, h\rangle^2 + 2\rho(x) \| h \|^2 \\ &\geq -2r_{zz}(z) \| x \| \sqrt{n} \| h \|^2 + 2\rho(x) \| h \|^2 \\ &\geq \left(-2r_{zz}(z)\sqrt{n\sum_{i=1}^{n}\mu_i^2} + 2\rho(x)\right) \| h \|^2 .\end{aligned}$$

Hence, the inequality

$$\left(-2r_{zz}(z)\sqrt{n\sum_{i=1}^{n}\mu_i^2} + 2\rho(x)\right) > \nu$$

implies (5.21). It is valid if

$$-r_{zz}(z)\sqrt{n\sum_{i=1}^{n}\mu_i^2} + f_{x_i}^i(x_i) - r_z(z) \geq \frac{1}{2}\nu > 0 \tag{5.26}$$

for all $x \in X$, $1 \leq i \leq n$. Consequently, (5.25) and (5.26) imply (5.21), and we can use Lemma 5.1 to verify condition 5 of Theorem 4.1.

In view of (5.25) and (5.26), the function $\Psi(x, y)$ is strongly concave with respect to y since the matrix $B(x, y)$ is negative definite. Consequently, the optimum response function $Z(x)$ is single valued (see condition 3 of Theorem 4.1) and a unique Nash equilibrium point exists [6].

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