The Journal of Risk (45–70) Volume 15/Number 1, Fall 2012

Calibrating risk preferences with the generalized capital asset pricing model based on mixed conditional value-at-risk deviation

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The generalized capital asset pricing model based on mixed conditional value-at-risk (CVaR) deviation is used for calibrating the risk preferences of investors. Risk preferences are determined by coefficients in the mixed CVaR deviation. The corresponding new generalized beta is designed to capture the tail performance of S&P 500 returns. Calibration of the coefficients is done by extracting information about risk preferences from put-option prices on the S&P 500. Actual market option prices are matched with the estimated prices from the pricing equation based on the generalized beta. Calibration is done for 153 moments in time with intervals of approximately one month. Results demonstrate that the risk preferences of investors change over time, reflecting investors' concern about potential tail losses. A new index of fear is introduced, calculated as a sum of several coefficients in the mixed CVaR deviation.

The authors are grateful to Professor Michael Zabarankin (Stevens Institute of Technology) for valuable comments and suggestions regarding the theoretical part of the paper. They are also grateful to Professor Oleg Bondarenko (University of Illinois) for valuable comments and suggestions on conducting numerical experiments. They are specifically grateful for the suggestions regarding modeling index scenarios, and also for the advice on considering changes in the risk preferences of investors over time. The authors are also grateful to Professor Mark J. Flannery (University of Florida) for valuable general comments and suggestions.

1 INTRODUCTION

Since its foundation in the 1960s, the capital asset pricing model (CAPM) (see Sharpe (1964); Linther (1965); Mossin (1966); and Treynor (1961, 1962)) has became one of the most popular methodologies for estimating returns of securities and for explaining their combined behavior. The model assumes that all investors want to minimize their investments' risk levels, and that all investors measure risk by the standard deviation of return. The model implies that all optimal portfolios are mixtures of the market fund and a risk-free instrument. The market fund is commonly approximated by a stock market index such as the S&P 500.

An important practical use for the CAPM is for calculating hedged portfolios that are not correlated with the market. This is not the CAPM's ability, but rather a technique based on the CAPM. To reduce the risk of a portfolio, an investor can include additional securities and hedge market risk. The risk of the portfolio in terms of CAPM model is measured by "beta". The value of beta for every security or portfolio is proportional to the correlation between its return and the market return. This follows from the assumption that investors have risk attitudes expressed by standard deviation (volatility). The hedging is designed to reduce portfolio beta, with the aim of protecting the portfolio in case of a market downturn. However, beta is just a scaled correlation with the market, and there is no guarantee that hedges will cover losses during sharp downturns, because the protection only works on average for the frequently observed market movements. Recent credit crises have shown that hedges have a tendency to perform very poorly when they are most needed in extreme market conditions. The classical hedging procedures based on standard beta set up a defense around the mean of the loss distribution, but fail to do so in the tails. This deficiency has led to multiple attempts to improve the CAPM.

One approach to improving the CAPM is including additional factors in the model. For example, Kraus and Litzenberger (1976), Friend and Westerfield (1980) and Lim (1989) provide tests for the three-moment CAPM, including the co-skewness term. This model accounts for asymmetrical distribution of returns. Fama and French (1996) added two additional terms to the asset return linear regression model: the difference between the return on a portfolio of small stocks and the return on a portfolio of large stocks, and the difference between the return on a portfolio of high-book-to-market stocks and the return on a portfolio of low-book-to-market stocks. Recently, Barberis and Huang (2008) presented a CAPM extension based on prospect theory, which allows the pricing of a security's own skewness.

The second approach is to find alternative risk measures, which may more precisely represent the risk preferences of investors. For example, Konno and Yamazaki (1991) applied an $\mathcal{L}^1$ risk model (based on mean absolute deviation) to the portfolio optimization problem with Nikkei 225 stocks. Their approach led to linear programming instead of quadratic programming in the classical Markowitz model, but computational results were not significantly better. Further research has been focused on risk measures more correctly accounting for losses. For example, Estrada (2004) applied downside semideviation-based CAPM for estimating the returns of Internet company stocks during the Internet bubble crisis. Downside semideviation only calculates for the losses underperforming the mean of returns. Nevertheless, in a similar way to standard deviation, semideviation does not pay special attention to extreme losses associated with heavy tails. Sortino and Forsey (1996) also point out that downside deviation does not provide the complete information required to manage risk.

A much more advanced line of research is considered by Rockafellar *et al* (2006a,b, 2007). The assumption here is that there are different groups of investors with different risk preferences. The generalized capital asset pricing model (GCAPM) (see Rockafellar *et al* (2006a,b, 2007)) proposes that there is a collection of deviation measures representing the risk preferences of the corresponding groups of investors. These deviation measures are substitutes for the standard deviation of the classical theory. With the generalized pricing formula following from the GCAPM, one can estimate the deviation measure for a specific group of investors from market prices. This is done by considering parametric classes of deviation measures and calibrating parameters of these measures. The GCAPM provides an alternative to the classical CAPM measure of systematic risk, so-called generalized beta. In a similar way to classical beta, the generalized beta can be used in portfolio optimization for hedging purposes.

This paper considers the class of so-called mixed conditional value-at-risk (CVaR) deviations, which have several attractive properties. First, different terms in the mixed CVaR deviation give credit to different parts of the distribution. Therefore, by varying parameters (coefficients), one can approximate various structures of risk preferences. In particular, so-called tail beta, which accounts for heavy tail losses (eg, losses in the top 5% of the tail distribution), can be built. Second, mixed CVaR deviation is a "coherent" deviation measure, and it therefore satisfies a number of desired mathematical properties. Third, optimization of problems with mixed CVaR deviation can be done very efficiently. For example, for discrete distributions, the optimization problems can be reduced to linear programming.

This paper considers a setup with one group of investors (representative investors). We assume that these investors estimate risks with the mixed CVaR deviations having fixed quantile levels: 50%, 75%, 85%, 95% and 99% of the loss distribution. By definition, this mixed CVaR deviation is a weighted combination of average losses exceeding these quantile levels. The weights for CVaRs with the different quantile levels determine a specific instance of the risk measure. The generalized pricing formula and generalized beta for this class of deviation measures are used in this approach. With market option prices, the parameters of the deviation measure are calibrated, thus estimating risk preferences of investors.

Several numerical experiments calibrating risk preferences of investors at different moments in time were conducted. We have found that the deviation measure, representing investors' risk preferences, has the biggest weight on the $\mathrm{CVaR}_{50\%}$ term, which equals the average loss below median return. On average, about 11% of the weight is assigned to $\mathrm{CVaR}_{85\%}$, $\mathrm{CVaR}_{95\%}$ and $\mathrm{CVaR}_{99\%}$, which evaluate heavy-loss scenarios. Experiments also showed that risk preferences tend to change over time, reflecting investors' opinions about the state of the market.

This is not the first attempt to extract risk preferences from option prices. It is common knowledge that option prices convey a risk-neutral probability distribution. Studies such as Aït-Sahalia and Lo (2000), Jackwerth (2000) and Bliss and Panigirtzoglou (2004) contain various approaches to extracting risk preferences in the form of the utility function by comparing the objective (or statistical) probability density function with the risk-neutral probability density function, estimated from option prices. In our paper, risk preferences are expressed in the form of the deviation measure, thus making it impossible to compare results with previous studies. We believe, however, that a wide range of applicability of the generalized CAPM framework makes our results useful for a greater variety of applications in practical finance.

The paper is structured as follows. Section 2 recalls the necessary background, describes the assumptions of the model, provides the main definitions and statements, and presents the derivation of the generalized pricing formula. Section 3 contains a description of the case study. Section 4 presents the results of the case study. Section 5 concludes and provides several ideas for further research that can be performed in this area.

2 DESCRIPTION OF THE APPROACH

2.1 Generalized CAPM background

In classical Markovitz portfolio theory (Markovitz (1952)) all investors are mean–variance optimizers. Contrary to the classical approach, consider a group of investors who form their portfolios by solving optimization problems of the following type:

$$\min_{\substack{x_0 r_0 + x^{\mathrm{T}} E r \geqslant r_0 + \Delta \\ x_0 + x^{\mathrm{T}} e = 1}} \mathcal{D}(x_0 r_0 + x^{\mathrm{T}} r) \qquad (P(\Delta))$$

where $\mathcal{D}$ is some measure of deviation (not necessarily standard deviation), r_0 denotes the risk-free rate of return, r is a column vector of (uncertain) rates of return on available securities, and e is a column vector of 1s. Problem $(P(\Delta))$ minimizes deviation of the portfolio return subject to a constraint on its expected return and the budget constraint. Different investors within the considered group may demand different excess return Δ. Unlike classical theory, instead of variance or standard deviation, investors measure risk with their generalized deviation measure $\mathcal{D}$. According to the definition in Rockafellar *et al* (2006a), a functional $\mathcal{D}\colon \mathcal{L}^2 \to [0, \infty]$ is a deviation measure if it satisfies the following axioms.

(D_1) $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C.

(D_2) $\mathcal{D}(0) = 0$ and $\mathcal{D}(\lambda X) = \lambda\mathcal{D}(X)$ for all X and all $\lambda > 0$.

(D_3) $\mathcal{D}(X + Y) \leqslant \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y.

(D_4) $\mathcal{D}(X) \geqslant 0$ for all X with $\mathcal{D} > 0$ for nonconstant X.

Following Rockafellar *et al* (2006b), we can eliminate x_0, which is equal to $1 - x^{\mathrm{T}}e$:

$$\min_{x^{\mathrm{T}}(Er - r_0 e) \geqslant \Delta} \mathcal{D}(x^{\mathrm{T}}r) \qquad (P^0(\Delta))$$

A pair (x_0, x) is an optimal solution to $(P(\Delta))$ if and only if x is an optimal solution to $(P^0(\Delta))$ and $x_0 = 1 - x^{\mathrm{T}}e$. Theorem 1 of Rockafellar *et al* (2006c) shows that an optimal solution to $(P^0(\Delta))$ exists if deviation measure $\mathcal{D}$ satisfies the following property.

(D_5) $\{X \mid \mathcal{D}(X) \leqslant C\}$ is closed for every constant C.

A deviation measure $\mathcal{D}$ satisfying this property is called lower semicontinuous. For further results we will also require an additional property called lower-range dominance.

(D_6) $\mathcal{D}(X) \leqslant EX - \inf X$ for all X.

In this paper we consider only lower semicontinuous, lower-range-dominated deviation measures.

Rockafellar *et al* (2006b) show that if a group of investors solves problems $(P(\Delta))$, the optimal investment policy is characterized by the generalized one-fund theorem (Rockafellar *et al* (2006b, Theorem 2)). According to the result, the optimal portfolios have the following general structure:

$$x^{\Delta} = \Delta x^1, \qquad x_0^{\Delta} = 1 - \Delta(x^1)^{\mathrm{T}}e$$

where x_0^{Δ} is the investment in risk-free instrument, x^{Δ} is a vector of positions in risky instruments and (x_0^1, x^1) is an optimal solution to $(P(\Delta))$, with $\Delta = 1$. Portfolio (x_0^1, x^1) is called a basic fund. It is important to note that, in full generality, $(x^1)^{\mathrm{T}}Er$ could be positive, negative or equal to zero (threshold case), although for most situations the positive case should prevail.

According to Rockafellar *et al* (2006b), a portfolio $x^{\mathcal{D}}$ is called a master fund of positive (respectively, negative) type if $(x^{\mathcal{D}})^{\mathrm{T}}e = 1$ (respectively, $(x^{\mathcal{D}})^{\mathrm{T}}e = -1$), and $x^{\mathcal{D}}$ is a solution to $P^0(\Delta^*)$ for some $\Delta^* > 0$. From the definition, it follows that the master fund contains only risky securities, with no investment in risk-free security. With this definition, the generalized one-fund theorem can be reformulated in terms of the master fund. Below we present its formulation as it was given in Rockafellar *et al* (2006b).

THEOREM 2.1 (One-fund theorem in master fund form) *Suppose a master fund of positive (respectively, negative) type exists, furnished by an $x^{\mathcal{D}}$-portfolio that yields an expected return $r_0 + \Delta^*$ for some $\Delta^* > 0$. Then, for any $\Delta > 0$, the solution for the portfolio problem $(P(\Delta))$ is obtained by investing the positive amount Δ^*/Δ (negative amount $-\Delta^*/\Delta$) in the master fund, and the amount $1 - (\Delta^*/\Delta)$ (amount $1 + (\Delta^*/\Delta) > 1$) in the risk-free instrument.*

It follows from Theorem 2.1 that, for every investor in the considered group, the optimal portfolio can be expressed as a combination of investment in the master fund and investment in the risk-free security.

Rockafellar *et al* (2007) extends the framework to the case with multiple groups of investors. Every group of investors i, where $i = 1, \dots, I$, solves the problem $(P(\Delta))$ with their own deviation measure $\mathcal{D}_i$. It was shown that there exists a market equilibrium, and the optimal policy for every group of investors is defined by the generalized one-fund theorem. In this framework, investors from different groups may have different master funds. Hereafter, we assume that a generalized deviation measure represents risk preferences of a given group of investors.

Consider a particular group of investors with risk preferences defined by a generalized deviation $\mathcal{D}$. If their master fund is known, the corresponding GCAPM relations can be formulated. The exact relation depends on the type of the master fund. Let r_{M} denote the rate of return of the master fund. Then:

$$r_{\mathrm{M}} = (x^{\mathcal{D}})^{\mathrm{T}} r = \sum_{j=1}^{n} x_j^{\mathcal{D}} r_j$$

where the random variables r_j stand for rates of return on the securities in the considered economy, $x_j^{\mathcal{D}}$ are the corresponding weights of these securities in the master fund, and:

$$\sum_{j=1}^{n} x_j^{\mathcal{D}} = 1$$

The generalized beta of a security j, replacing the classical beta, is defined as follows:

$$\beta_j = \frac{\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}})} \tag{2.1}$$

In this formula $Q_{\mathrm{M}}^{\mathcal{D}}$ denotes the risk identifier corresponding to the master fund, taken from the risk envelope corresponding to the deviation measure $\mathcal{D}$. Examples of risk identifiers for specific deviation measures will be presented in the next subsection.

Rockafellar *et al* (2006c) derives optimality conditions for problems involving minimizing a generalized deviation of the return on a portfolio. The optimality conditions are applied to characterize three types of master funds. Theorem 5 of Rockafellar *et al* (2006c), presented below, formulates the optimality conditions in the form of CAPM-like relations.

THEOREM 2.2 *Let the deviation $\mathcal{D}$ be finite and continuous.*

(1) *An $x^{\mathcal{D}}$-portfolio with $x_1^{\mathcal{D}} + \cdots + x_n^{\mathcal{D}} = 1$ is a master fund of positive type if and only if $Er_{\mathrm{M}} > r_0$ and $Er_j - r_0 = \beta_j(Er_{\mathrm{M}} - r_0)$ for all j.*

(2) *An $x^{\mathcal{D}}$-portfolio with $x_1^{\mathcal{D}} + \cdots + x_n^{\mathcal{D}} = -1$ is a master fund of negative type if and only if $Er_{\mathrm{M}} > -r_0$ and $Er_j - r_0 = \beta_j(Er_{\mathrm{M}} + r_0)$ for all j.*

(3) *An $x^{\mathcal{D}}$-portfolio with $x_1^{\mathcal{D}} + \cdots + x_n^{\mathcal{D}} = 0$ is a master fund of threshold type if and only if $Er_{\mathrm{M}} > 0$ and $Er_j - r_0 = \beta_j Er_{\mathrm{M}}$ for all j.*

From now on, we call the conditions specified in Theorem 2.2 the GCAPM relations.

2.2 Pricing formulas in the GCAPM

Let $r_j = \zeta_j/\pi_j - 1$, where ζ_j is the payoff or the future price of security j, and π_j is the price of this security today.

In a similar way to classical theory, pricing formulas can be derived from the GCAPM relations, as was done in Sarykalin (2008). The following lemma presents these pricing formulas in both certainty-equivalent form and risk-adjusted form.

LEMMA 2.3

(1) *If the master fund is of positive type, then:*

$$\begin{aligned}\pi_j &= \frac{E\zeta_j}{1 + r_0 + \beta_j(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)} \\ &= \frac{1}{1 + r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)\right)\end{aligned}$$

(2) *If the master fund is of negative type, then:*

$$\begin{aligned}\pi_j &= \frac{E\zeta_j}{1 + r_0 + \beta_j(Er_{\mathrm{M}}^{\mathcal{D}} + r_0)} \\ &= \frac{1}{1 + r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} + r_0)\right)\end{aligned}$$

(3) *If the master fund is of threshold type, then:*

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j E r_{\mathrm{M}}^{\mathcal{D}}} = \frac{1}{1 + r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} E r_{\mathrm{M}}^{\mathcal{D}}\right)$$

See Appendix A for a proof.

Rockafellar *et al* (2007) proved the existence of equilibrium for multiple groups of investors optimizing their portfolios according to their individual risk preferences, and therefore the pricing formulas in Lemma 2.3 hold true for all groups of investors.

2.3 Mixed CVaR deviation and betas

Conditional value-at-risk has been studied by various researchers, sometimes under different names (eg, expected shortfall and tail VaR). We will use the notation from Rockafellar and Uryasev (2002). For more details on stochastic optimization with CVaR-type functions, see Uryasev (2000), Rockafellar and Uryasev (2000, 2002), Krokhmal *et al* (2002), Krokhmal *et al* (2006) and Sarykalin *et al* (2008).

Suppose that random variable X determines some financial outcome, future wealth or return on investment. By definition, value-at-risk at level α is the α-quantile of the distribution of $(-X)$:

$$\mathrm{VaR}_\alpha(X) = q_\alpha(-X) = -q_{1-\alpha}(X) = -\inf\{z \mid F_X(z) > 1 - \alpha\}$$

where F_X denotes the probability distribution function of random variable X.

Conditional value-at-risk for continuous distributions equals the expected loss exceeding VaR:

$$\mathrm{CVaR}_\alpha(X) = -E[X \mid X \leqslant -\mathrm{VaR}_\alpha(X)]$$

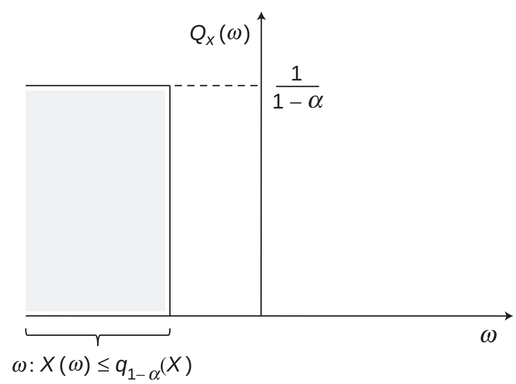
This formula underpins the understanding of CVaR as conditional expectation. For the general case, the definition is more complicated, and can be found, for example, in Rockafellar and Uryasev (2000). The CVaR deviation is defined as follows:

$$\mathrm{CVaR}_\alpha^\Delta(X) = \mathrm{CVaR}_\alpha(X - EX)$$

It follows from Rockafellar *et al* (2006a, Theorem 1) that there exists a one-to-one correspondence between lower-semicontinuous, lower-range-dominated deviation measures $\mathcal{D}$ and convex positive risk envelopes $\mathcal{Q}$:

$$\left.\begin{aligned} \mathcal{Q} &= \{Q \mid Q \geqslant 0,\ EQ = 1,\ EXQ \geqslant EX - \mathcal{D}(X) \text{ for all } X\} \\ \mathcal{D}(X) &= EX - \inf_{Q\in\mathcal{Q}} EXQ \end{aligned}\right\} \tag{2.2}$$

FIGURE 1 CVaR-type risk identifier for a given outcome variable X.



The random variable $Q_X \in \mathcal{Q}$, for which $\mathcal{D}(X) = EX - EXQ_X$, is called the risk identifier, associated via $\mathcal{D}$ with X.

For a given X and CVaR deviation, the risk identifier can be viewed as a step function, with a jump at the quantile point:

$$Q_X(\omega) = \frac{1}{1-\alpha}\mathbf{1}_{\{X(\omega)\leqslant q_{1-\alpha}(X)\}} \tag{2.3}$$

where ω denotes an elementary event on the probability space and $\mathbf{1}_{\{\text{condition}\}}$ is an indicator function, defined on the same probability space, which equals one if the condition is true and zero otherwise. Figure 1 illustrates the structure of the CVaR risk identifier, corresponding to some random outcome X. For simplicity, the probability space assumed in the figure is the space of values of the random variable X.

If the group of investors constructs its master fund by minimizing CVaR deviation, and all r_j are continuously distributed, beta for security j has the following expression, as derived in Rockafellar *et al* (2006c):

$$\begin{aligned} \beta_j &= \frac{\operatorname{cov}(-r_j, Q_{\mathrm{M}})}{\mathrm{CVaR}_\alpha^\Delta(r_{\mathrm{M}})} \\ &= \frac{E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_\alpha(r_{\mathrm{M}})]}{E[Er_{\mathrm{M}} - r_{\mathrm{M}} \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_\alpha(r_{\mathrm{M}})]} \end{aligned} \tag{2.4}$$

Classical beta is a scaled covariance between the security and the market. The new beta focuses on events corresponding to big losses in the master fund. For large α ($\alpha > 0.8$), this expression can be called tail beta.

The following two theorems lead to the definition of mixed CVaR deviation, which is used for the purposes of this paper.

THEOREM 2.4 *Let deviation measure $\mathcal{D}_l$ correspond to risk envelope $\mathcal{Q}_l$ for $l = 1, \dots, L$. If deviation measure $\mathcal{D}$ is a convex combination of the deviation measures $\mathcal{D}_l$:*

$$\mathcal{D} = \sum_{l=1}^{L} \lambda_l \mathcal{D}_l \quad \text{with } \lambda_l \geqslant 0, \quad \sum_{l=1}^{L} \lambda_l = 1$$

then $\mathcal{D}$ corresponds to risk envelope:

$$\mathcal{Q} = \sum_{l=1}^{L} \lambda_l \mathcal{Q}_l$$

See Appendix A for a proof.

The following theorem presents a formula for the beta corresponding to a deviation measure that is a convex combination of a finite number of deviation measures.

THEOREM 2.5 *If the master fund M, corresponding to the deviation measure $\mathcal{D}$, is known and if $\mathcal{D}$ is a convex combination of a finite number of deviation measures $\mathcal{D}_l$, $l = 1, \dots, L$:*

$$\mathcal{D} = \lambda_1 \mathcal{D}_1 + \cdots + \lambda_L \mathcal{D}_L$$

then:

$$\beta_j = \frac{\lambda_1 \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_1}) + \cdots + \lambda_L \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_{\mathrm{M}}) + \cdots + \lambda_L \mathcal{D}_L(r_{\mathrm{M}})}$$

where $Q_{\mathrm{M}}^{\mathcal{D}_l}$ is a risk identifier of master fund return corresponding to deviation measure $\mathcal{D}_l$.

See Appendix A for a proof.

For a given set of confidence levels $\alpha = (\alpha_1, \dots, \alpha_L)$ and coefficients $\lambda = (\lambda_1, \dots, \lambda_L)$ such that $\lambda_l \geqslant 0$ for all $l = 1, \dots, L$, and $\sum_{l=1}^{L} \lambda_l = 1$, mixed CVaR deviation $\mathrm{CVaR}_{\alpha;\lambda}^{\Delta}$ is defined as follows:

$$\mathrm{CVaR}_{\alpha,\lambda}^{\Delta}(X) = \lambda_1 \mathrm{CVaR}_{\alpha_1}^{\Delta}(X) + \cdots + \lambda_L \mathrm{CVaR}_{\alpha_L}^{\Delta}(X) \tag{2.5}$$

COROLLARY 2.6 *If:*

$$\mathcal{D} = \mathrm{CVaR}_{\alpha,\lambda}^{\Delta}$$

where $\alpha = (\alpha_1, \dots, \lambda_L)$ and $\lambda = (\lambda_1, \dots, \lambda_L)$, and the distribution of r_{M} is continuous, then:

$$\beta_j = \frac{\begin{array}{l}\lambda_1 E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_{\alpha_1}(r_{\mathrm{M}})] + \cdots \\ \qquad + \lambda_L E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_{\alpha_L}(r_{\mathrm{M}})]\end{array}}{\lambda_1 \mathrm{CVaR}_{\alpha_1}^{\Delta}(r_{\mathrm{M}}) + \cdots + \lambda_L \mathrm{CVaR}_{\alpha_L}^{\Delta}(r_{\mathrm{M}})} \tag{2.6}$$

See Appendix A for a proof.

2.4 The risk preferences of a representative investor

How can the risk preferences of investors be extracted from market prices?

According to the GCAPM, the risk preferences of a group of investors are represented by a deviation measure. This deviation measure determines the structure of a master fund. For a known deviation measure and a master fund, a risk identifier for the master fund can be specified. If a joint distribution of payoffs for securities is also known, then one can calculate the betas for securities, and then calculate GCAPM prices for these securities. Therefore, according to the GCAPM, the deviation measure and the distribution of payoff determine the price for each security. To estimate the deviation measure, having expected returns on securities and market prices, one can find a candidate deviation measure $\mathcal{D}$ for which the GCAPM prices are equal to the market prices.

In this and following sections, we consider a setup with one group of investors. In other words, all investors evaluate the risk of their investments according to the same deviation measure. Therefore, all further results can be referred to as describing a so-called representative investor. From market equilibrium, it follows that the master fund for a representative investor is known, and, therefore, can be approximated with a market index such as the S&P 500.

As an alternative to standard deviation, which measures the magnitude of possible price changes in both directions, CVaR deviation measures the average loss for the α worst-case scenarios. We assume that risk preferences can be expressed with a mixed CVaR deviation, defined by formula (2.5), which is a weighted combination of several CVaR deviations with appropriate weights, to capture different parts of the tail of the distribution.

Among the whole variety of securities traded in the market, in addition to the index fund itself, we consider S&P 500 put options with one month to maturity. By construction, put options' prices provide monetary evaluation of the tails of distribution, so they are expected to be the perfect candidate to calibrate coefficients in the mixed CVaR deviation.

To estimate the coefficients $\lambda_1, \dots, \lambda_L$, we will use GCAPM formulas, as presented in Theorem 2.2. Let P_K denote the market price of a put option with strike price K and one month to maturity, let ζ_K denote its (random) monthly return, and let $r_K = \zeta_K / P_K - 1$ denote its (random) return in one month. Let r_{M} be (random) return on the master fund, with its distribution at this moment assumed to be known; r_0 is the return on a risk-free security. If market prices are exactly equal to GCAPM prices and the deviation measure is a mixed CVaR deviation with fixed confidence levels $\alpha_1, \dots, \alpha_L$, then the set of coefficients $\lambda_1, \dots, \lambda_L$ is a solution to the following system of equations:

$$Er_K - r_0 = \beta_K(\lambda)(Er_{\mathrm{M}} - r_0), \qquad K = K_1, K_2, \dots, K_{J-1}, K_J \tag{2.7}$$

where:

$$\beta_K(\lambda) = \frac{\sum_{l=1}^{L} \lambda_l E[Er_K - r_K \mid r_M \leqslant -\operatorname{VaR}_{\alpha_l}(r_M)]}{\sum_{l=1}^{L} \lambda_l \operatorname{CVaR}^{\Delta}_{\alpha_l}(r_M)} \tag{2.8}$$

$$\sum_{l=1}^{L} \lambda_l = 1 \tag{2.9}$$

and:

$$\lambda_l \geqslant 0, \quad l = 1, \dots, L \tag{2.10}$$

Equations (2.7) are GCAPM formulas from Theorem 2.2, applied to market prices P_K of put options with strike prices $K = K_1, \dots, K_J$, and random payoffs ζ_K. Systematic risk measure $\beta(\lambda)$ is expressed through the coefficients λ_l according to Corollary 2.6.

By multiplying both sides of Equation (2.7) by $\sum_{l=1}^{L} \lambda_l \operatorname{CVaR}^{\Delta}_{\alpha_l}(r_M)$ and taking into account (2.8), we obtain:

$$\begin{aligned}(Er_K - r_0) &\sum_{l=1}^{L} \lambda_l \operatorname{CVaR}^{\Delta}_{\alpha_l}(r_M) \\ &= (Er_M - r_0) \sum_{l=1}^{L} \lambda_l E[Er_K - r_K \mid r_M \leqslant -\operatorname{VaR}_{\alpha_l}(r_M)], \quad K = K_1, \dots, K_J\end{aligned}$$

or, equivalently:

$$\sum_{l=1}^{L} ((Er_K - r_0) \operatorname{CVaR}^{\Delta}_{\alpha_l}(r_M) - (Er_M - r_0) E[Er_K - r_K \mid r_M \leqslant -\operatorname{VaR}_{\alpha_l}(r_M)])\lambda_l = 0 \tag{2.11}$$

for $K = K_1, \dots, K_J$. If the number of equations (options with different strike prices K) is greater than the number of variables, then the system of equations (2.11) may not have a solution. For this reason, we replace system (2.11) with alternative expressions with error terms e_K:

$$\sum_{l=1}^{L} ((Er_K - r_0) \operatorname{CVaR}^{\Delta}_{\alpha_l}(r_M) - (Er_M - r_0) E[Er_K - r_K \mid r_M \leqslant -\operatorname{VaR}_{\alpha_l}(r_M)])\lambda_l = e_K \tag{2.12}$$

for $K = K_1, \dots, K_J$. We estimate the coefficients $\lambda_1, \dots, \lambda_L$ as the optimal point to the following optimization problem minimizing a norm of vector $(e_{K_1}, \dots, e_{K_J})$:

$$\min_{\lambda_1, \dots, \lambda_L} \|(e_{K_1}, \dots, e_{K_J})\| \tag{2.13}$$

subject to:

$$\sum_{l=1}^{L}((Er_K - r_0)\,\mathrm{CVaR}^{\Delta}_{\alpha_l}(r_M) - (Er_M - r_0)E[Er_K - r_K \mid r_M \leqslant -\mathrm{VaR}_{\alpha_l}(r_M)])\lambda_l = e_K \tag{2.14}$$

for $K = K_1, \dots, K_J$, and:

$$\lambda_l \geqslant 0, \quad l = 1, \dots, L, \qquad \sum_{l=1}^{L} \lambda_l = 1 \tag{2.15}$$

In the above formulation, $\|\cdot\|$ is some norm. We now consider the $\mathcal{L}^1$-norm:

$$\|(e_{K_1}, \dots, e_{K_J})\|_1 = \frac{1}{J}\sum_{j=1}^{J}|e_{K_j}| \tag{2.16}$$

and the $\mathcal{L}^2$-norm:

$$\|(e_{K_1}, \dots, e_{K_J})\|_2 = \sqrt{\frac{1}{J}\sum_{j=1}^{J} e^2_{K_j}}$$

3 CASE STUDY DATA AND AN ALGORITHM

We carried out 153 experiments of estimating risk preferences, each for a separate date (henceforth termed the date of experiment), beginning with January 22, 1998. Dates were chosen with intervals of approximately one month in such a way that each date is one month prior to a next-month option expiration date. However, we present a detailed analysis for twelve dates with intervals of approximately six months starting with December 23, 2004. For every experiment we used a set of S&P 500 put options with strike prices $K_1, \dots, K_J$, where K_J is a strike price of the at-the-money option (option with strike price closest to the index value). We define option market price P_K as an average of bid and ask prices:

$$P_K = \tfrac{1}{2}(P_{\mathrm{ask},K} + P_{\mathrm{bid},K})$$

We chose K_1 as a minimum strike price, for which the following two conditions are satisfied. First, starting with the option K_1, prices P_{K_j} are strictly increasing, ie, $P_{K_{j+1}} > P_{K_j}$. Second, the open interest for all options in the range is greater than zero.

For every experiment we designed a set of scenarios of monthly index rates of return in the following way. Observing the historical values of the S&P 500 over the period from January 1, 1994 to October 1, 2010 for every trading day s, from historical observations we recorded the value $\tilde{r}_I^{(s)} = I_{s+21}/I_s - 1$, where I_s is the index value on day s.

We further calculate implied volatility σ of the at-the-money option (the option with strike price K_J) and the value:

$$\hat{\sigma} = \text{standard deviation}\,(\tilde{r}_I^{(s)})$$

Next, every scenario return was modified as follows:

$$r_I^{(s)} = \frac{\sigma}{\hat{\sigma}}(\tilde{r}_I^{(s)} - E\tilde{r}_I) + r_0 + \xi\sigma \tag{3.1}$$

where the value for the monthly risk-free rate of return r_0 was selected equal to 0.01%, and $\xi > 0$ is some parameter. The new scenarios will have volatility equal to the volatility σ of the at-the-money options and expected return equal to $r_0 + \xi\sigma$. In formula (3.1) the value of ξ was chosen such that expected returns on options are negative. We selected $\xi = \frac{1}{3}$. Numerical experiments show that results are not very sensitive to the selection of the parameter ξ.

Suppose, for modeling purposes, that the investors' preferences are described by a mixed CVaR deviation with confidence levels 50%, 75%, 85%, 95% and 99%:

$$\mathcal{D}(\lambda) = \sum_{l=1}^{L} \lambda_l \,\text{CVaR}_{\alpha_l}^{\Delta} \tag{3.2}$$

where:

$$L = 5, \quad \alpha_1 = 99\%, \quad \alpha_2 = 95\%, \quad \alpha_3 = 85\%, \quad \alpha_4 = 75\%, \quad \alpha_5 = 50\%$$

and:

$$\lambda_l \geqslant 0, \qquad \sum_{l=1}^{5} \lambda_l = 1 \tag{3.3}$$

The input data for the case study is listed in Table 1 on the facing page.

Multiple tests demonstrate that the results do not depend significantly on the choice of norm in the optimization problem (2.13). We present results obtained using the $\mathcal{L}^1$-norm later in the paper.

The following algorithm was used to estimate risk preferences from the option prices.

3.1 Algorithm

STEP 1 Calculate scenarios indexed by $s = 1, \dots, S$ for payoffs and net returns of put options according to the formula:

$$\zeta_K^{(s)} = \max(0, K - I_0(1 + r_I^{(s)})), \qquad r_K^{(s)} = \frac{\zeta_K^{(s)}}{P_K} - 1$$

TABLE 1 Case study data.

(a)

Date of experiment	Index value I_0	Lowest strike price $K_{\min}$	Highest strike price $K_{\max}$
12/23/2004	1210.13	1120	1210
6/16/2005	1210.96	1080	1210
12/22/2005	1268.12	1100	1270
6/22/2006	1245.60	1150	1245
12/21/2006	1418.30	1310	1420
6/21/2007	1522.19	1375	1520
12/20/2007	1460.12	1255	1460
6/19/2008	1342.83	1110	1345
12/18/2008	885.28	630	885
6/18/2009	918.37	735	920
12/17/2009	1096.08	900	1095
6/17/2010	1116.04	940	1115

(b)

Description	Notation	Value
Risk-free monthly interest rate (%)	r_0	0.4125
Number of terms in mixed CVaR deviation	L	5
Confidence level 1 (%)	α_1	99
Confidence level 2 (%)	α_2	95
Confidence level 3 (%)	α_3	85
Confidence level 4 (%)	α_4	75
Confidence level 5 (%)	α_5	50
Number of scenarios (days)	S	5443

where $K = K_1, \dots, K_J$ are the strike prices, and I_0 is the index value at the time of the experiment.

STEP 2 Calculate the following values:

$$E[Er_K - r_K \mid r_I \leqslant -\mathrm{VaR}_{\alpha_l}(r_I)] \quad \text{for all } K = K_1, \dots, K_J \text{ and } l = 1, \dots, L$$

and:

$$\mathrm{CVaR}^{\Delta}_{\alpha_l}(r_I) \quad \text{for all } l = 1, \dots, L$$

TABLE 2 Deviation measure calibration results.

Date of experiment	$\lambda_{99\%}$	$\lambda_{95\%}$	$\lambda_{85\%}$	$\lambda_{75\%}$	$\lambda_{50\%}$
12/23/2004	0.000	0.020	0.235	0.000	0.745
6/16/2005	0.036	0.016	0.000	0.000	0.948
12/22/2005	0.058	0.000	0.000	0.000	0.942
6/22/2006	0.071	0.033	0.000	0.000	0.895
12/21/2006	0.081	0.000	0.000	0.000	0.919
6/21/2007	0.055	0.040	0.000	0.000	0.905
12/20/2007	0.000	0.041	0.275	0.000	0.684
6/19/2008	0.000	0.055	0.181	0.000	0.765
12/18/2008	0.000	0.000	0.115	0.000	0.885
6/18/2009	0.015	0.014	0.168	0.000	0.803
12/17/2009	0.049	0.001	0.083	0.000	0.868
6/17/2010	0.041	0.048	0.023	0.000	0.889
Mean (12 dates)	0.034	0.022	0.090	0.000	0.854
Standard deviation (12 dates)	0.030	0.020	0.102	0.000	0.085
Mean (153 dates)	0.029	0.029	0.052	0.007	0.883
Standard deviation (153 dates)	0.028	0.033	0.077	0.047	0.078

Results for twelve experiments. Each experiment gives mixed CVaR deviation expressing risk preferences of a representative investor. Coefficient λ_{α_l} is a weight for $\mathrm{CVaR}^{\Delta}_{\alpha_l}(r_{\mathrm{M}})$ in mixed CVaR deviation.

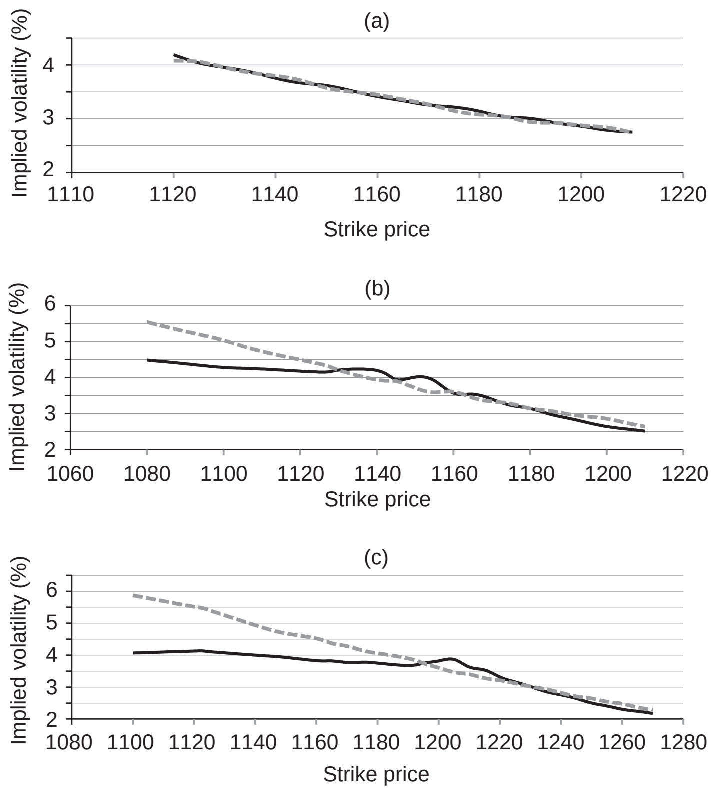
STEP 3 Build the design matrix for the constrained regression (2.13)–(2.15) with $r_{\mathrm{M}} = r_I$.

STEP 4 Find a set of coefficients λ_l by solving constrained regression (2.13)–(2.15) with $\mathcal{L}^1$ norm, given by Equation (2.16). The vector λ gives coefficients in mixed CVaR deviation.

4 CASE STUDY COMPUTATIONAL RESULTS

Computations were performed on a 64-bit WINDOWS 7 laptop with INTEL Core 2 Duo CPU P8800, 2.66GHz and 4GB RAM. The algorithm described in the previous section was programmed in MATLAB. Both optimization problems, the constrained regression and CVaR portfolio optimization, on each iteration of the algorithm were solved with the AORDA Portfolio Safeguard decision support tool (see American Optimal Decisions (2009)). For one date the computational time is around 15 seconds.

FIGURE 2 Calculated prices and market prices in the scale of implied volatilities, 2004–5.

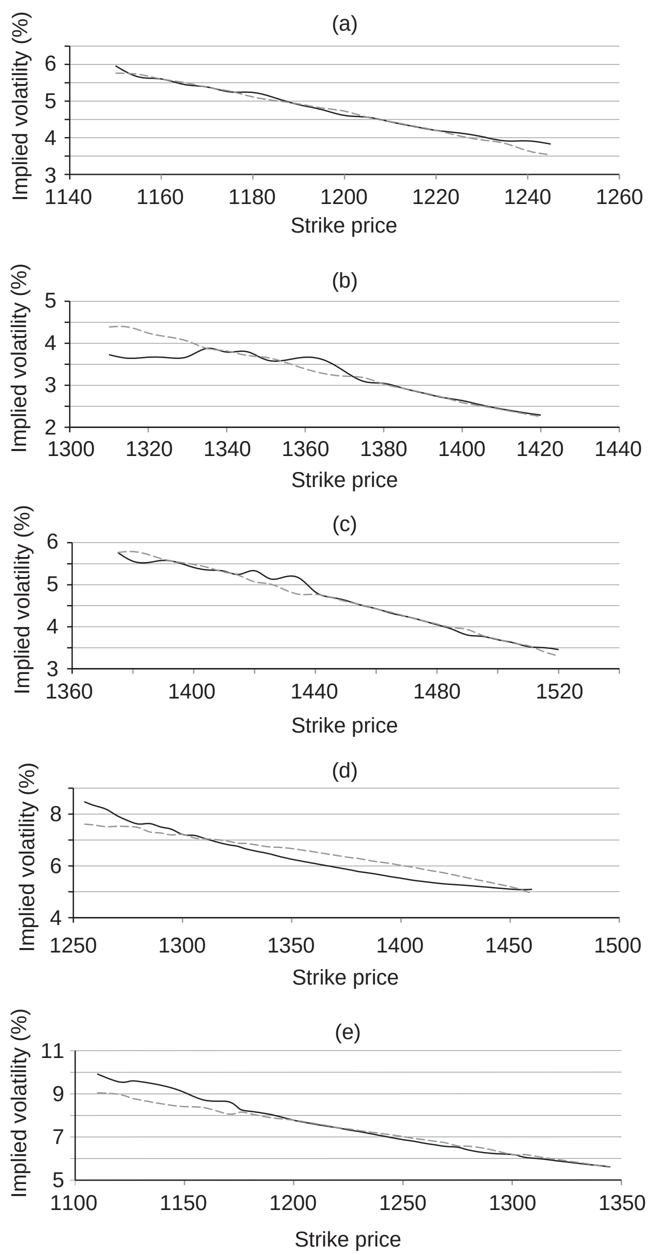


(a) December 23, 2004. (b) June 16, 2005. (c) December 22, 2005. The figures present the results of twelve experiments for different dates. We assume that the representative investor's risk preferences are expressed by mixed CVaR deviation. In each experiment, we use S&P 500 put-option market prices to calculate coefficients in the mixed CVaR deviation, and assume that the master fund equals the market portfolio (S&P 500 fund). We then use generalized pricing formulas to calculate option prices. Each graph presents market prices and calculated prices, mapped to the scale of monthly implied volatilities. This mapping is the inverse of the Black–Scholes formula.

The set of coefficients in the mixed CVaR deviation for every date is presented in Table 2 on the facing page. This table shows that, in all experiments, the obtained deviation measure has the biggest weight on $\mathrm{CVaR}_{50\%}$, and smaller weights on $\mathrm{CVaR}_{85\%}$, $\mathrm{CVaR}_{95\%}$ and $\mathrm{CVaR}_{99\%}$. This can be interpreted as representing the fact that investors are concerned with both the middle part of the loss distribution, expressed with $\mathrm{CVaR}_{50\%}$, and extreme losses, expressed with $\mathrm{CVaR}_{85\%}$, $\mathrm{CVaR}_{95\%}$ and $\mathrm{CVaR}_{99\%}$.

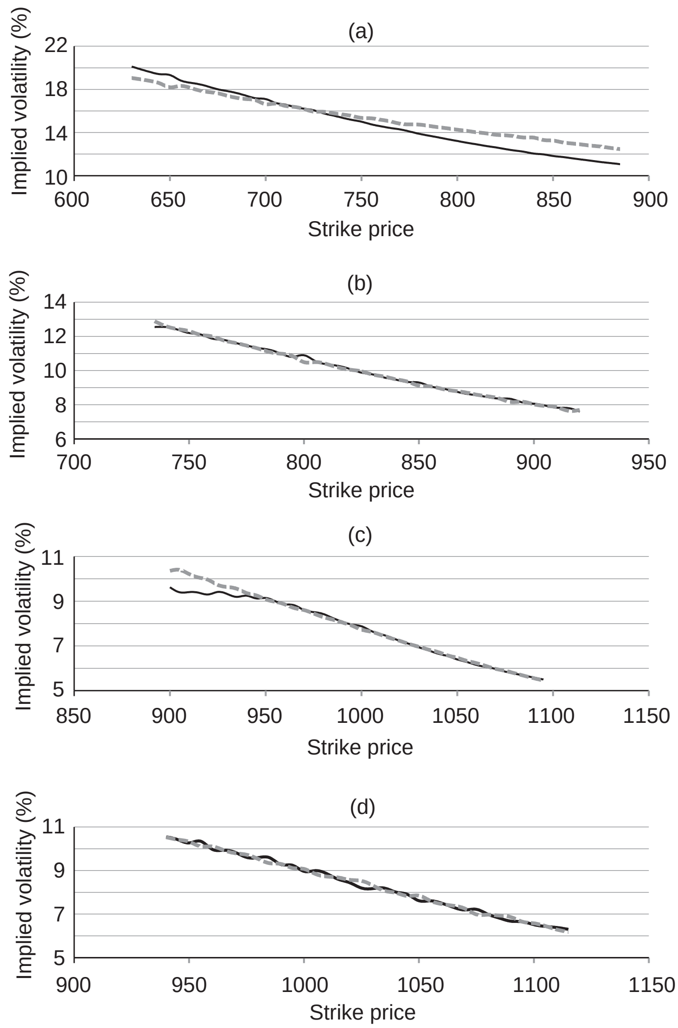
Let us denote by π_K the GCAPM option prices, calculated with pricing formulas in Lemma 2.3, using the calculated mixed CVaR deviation measure and the master fund. We mapped the obtained option prices π_K and the market prices P_K onto the

FIGURE 3 Calculated prices and market prices in the scale of implied volatilities, 2006–8.



(a) June 22, 2006. (b) December 21, 2006. (c) June 21, 2007. (d) December 20, 2007. (e) June 19, 2008.

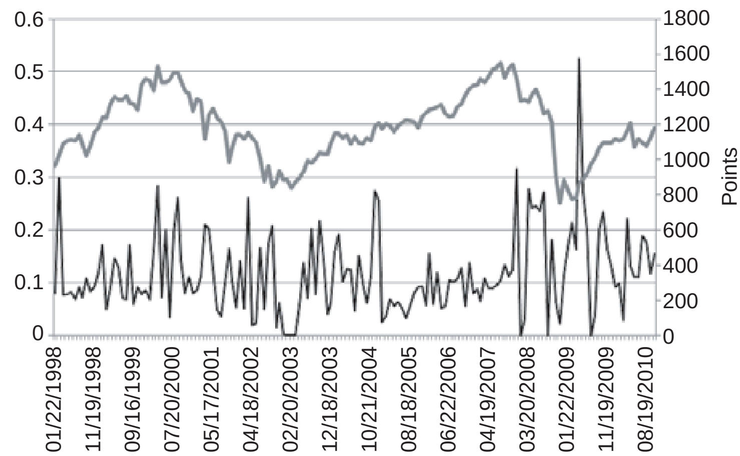
FIGURE 4 Calculated prices and market prices in the scale of implied volatilities, 2008–10.



(a) December 18, 2008. (b) June 18, 2009. (c) December 17, 2009. (d) June 17, 2010.

implied volatility scale. This mapping is defined by the Black–Scholes formula in implicit form. Figure 2 on page 61, Figure 3 on the facing page and Figure 4 present graphs of π_K and P_K for twelve dates in the scale of monthly implied volatilities. All the figures show that the GCAPM prices are close to market prices, except for parts (b) and (c) of Figure 2 on page 61.

FIGURE 5 S&P 500 value and risk-averseness dynamics.



Solid black line: $1-\lambda_{50\%}$. Solid gray line: S&P 500. The figure compares S&P 500 index dynamics with the changes of risk preferences of investors. We assume that representative investor's risk preferences are expressed by mixed CVaR deviation with confidence levels 50%, 75%, 85%, 95% and 99%. We conduct 153 experiments for different dates, and in each experiment we use market option prices to calculate coefficients in the mixed CVaR deviation. The figure contains two curves: the S&P 500 index and $1-\lambda_{50\%}$, or, equivalently, $\lambda_{75\%}+\lambda_{85\%}+\lambda_{95\%}+\lambda_{99\%}$. The curve $1-\lambda_{50\%}$ reflects investors' apprehension about potential tail losses, and their tendency to hedge the risk of extreme losses. Investors' concern regarding tail losses was increasing until the beginning of the market downturn, which demonstrates that the downturn was anticipated by the considered group of investors. After that, market participants poorly anticipated market trends. For example, in December 2008, the value of $1-\lambda_{50\%}$ was 0.115, which can be interpreted as investors' belief that the S&P 500 had reached its bottom, and there was no intention to hedge their investments in an index against losses. Nevertheless, at the beginning of 2009, the market fell even further.

Figure 5 compares the dynamics of the value $\eta = 1 - \lambda_{50\%}$ on 153 dates of the experiment with S&P 500 dynamics. High values of η indicate greater investor apprehension regarding potential tail losses and a greater inclination to hedge their investments in the S&P 500. It can be seen that risk preferences were relatively stable until 2008, when the distressed period began. It can also be seen that market participants did not always properly anticipate future market trends. In particular, in December 2008, the value of η was low (0.115), which indicated that the market wrongly anticipated that the index had reached its bottom and would not go up. Nevertheless, 2009 started with a further decline in the index.

5 CONCLUSION

We have described a new technique for expressing risk preferences with generalized deviation measures. We have presented a method for extracting risk preferences from market option prices using these formulas. We have conducted a case study for extracting risk preferences of a representative investor from put-option prices.

We extracted risk preferences for 153 dates with one-month intervals, and expressed them with mixed CVaR deviation. The results demonstrate that investors are concerned both with the middle part of the loss distribution, expressed with $\mathrm{CVaR}_{50\%}$, and with extreme losses, expressed with $\mathrm{CVaR}_{85\%}$, $\mathrm{CVaR}_{95\%}$ or $\mathrm{CVaR}_{99\%}$. Exact proportions vary, reflecting investors' anticipation of high or low returns.

An important application of the theory is that it provides an alternative, broader view of systematic risk, compared with the classical CAPM based on standard deviation. Similarly to the classical CAPM, we calculated new betas for securities that measure systematic risk in a different way, capturing the tail behavior of a master fund return. These betas can be used for hedging against tail losses, which occur in a down market.

Potential applications go beyond identifying risk preferences of considered investors. An investor can express risk attitudes in the form of a deviation measure and then recalculate betas for securities using this deviation measure. With these betas, the investor can build a portfolio hedged according to his risk preferences.

APPENDIX A

We now present proofs of the statements formulated in the rest of the paper. For the reader's convenience we repeat the formulations before every proof.

LEMMA A.1 (Lemma 2.3)

(1) *If the master fund is of positive type, then:*

$$\begin{aligned}\pi_j &= \frac{E\zeta_j}{1 + r_0 + \beta_j(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)}\\ &= \frac{1}{1+r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)\right)\end{aligned}$$

(2) *If the master fund is of negative type, then:*

$$\begin{aligned}\pi_j &= \frac{E\zeta_j}{1 + r_0 + \beta_j(Er_{\mathrm{M}}^{\mathcal{D}} + r_0)}\\ &= \frac{1}{1+r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} + r_0)\right)\end{aligned}$$

(3) *If the master fund is of threshold type, then:*

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j E r_{\mathrm{M}}^{\mathcal{D}}} = \frac{1}{1 + r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} E r_{\mathrm{M}}^{\mathcal{D}} \right)$$

PROOF OF LEMMA 2.3 Proofs for all three cases are similar, so we present the proof only for a master fund of positive type. According to the GCAPM relation specified in case (1):

$$E r_j - r_0 = \beta_j (E r_{\mathrm{M}}^{\mathcal{D}} - r_0)$$

Since $r_j = \zeta_j/\pi_j - 1$, then $E r_j = E\zeta_j/\pi_j - 1$, from which we obtain:

$$\frac{E\zeta_j}{\pi_j} - (1 + r_0) = \beta_j (E r_{\mathrm{M}}^{\mathcal{D}} - r_0) \tag{A.1}$$

This yields the generalized capital asset pricing formula in certainty-equivalent form:

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \beta_j (E r_{\mathrm{M}}^{\mathcal{D}} - r_0)} \tag{A.2}$$

Using the expression for beta (2.1), we can also write:

$$\pi_j = \frac{E\zeta_j}{1 + r_0 + \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}})/\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})(E r_{\mathrm{M}}^{\mathcal{D}} - r_0)} \tag{A.3}$$

By multiplying both sides of the equality (A.1) by π_j, we obtain:

$$E\zeta_j - \pi_j (r_0 + 1) = \pi_j \beta_j (E r_{\mathrm{M}}^{\mathcal{D}} - r_0) \tag{A.4}$$

With the expression for beta (2.1), we obtain:

$$\begin{aligned} \pi_j \beta_j &= \pi_j \frac{\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} \\ &= \frac{\operatorname{cov}(-\pi_j r_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} \\ &= \frac{\operatorname{cov}(-\pi_j (1 + r_j) + \pi_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} \\ &= \frac{\operatorname{cov}(-\pi_j (1 + r_j), Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} + \frac{\operatorname{cov}(\pi_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})} \end{aligned} \tag{A.5}$$

Here π_j is a constant and, consequently, the second term in the last sum equals zero. Therefore:

$$\pi_j \beta_j = \frac{\operatorname{cov}(-\pi_j (1 + r_j), Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}$$

Since $\pi_j(1+r_j) = \zeta_j$, then:

$$\pi_j \beta_j = -\frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}$$

Substituting the expression for $\pi_j \beta_j$ into (A.4) gives:

$$E\zeta_j - \pi_j(r_0 + 1) = -\frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)$$

The last equation implies the risk-adjusted form of the pricing formula:

$$\pi_j = \frac{1}{1+r_0}\left(E\zeta_j + \frac{\operatorname{cov}(\zeta_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}(r_{\mathrm{M}}^{\mathcal{D}})}(Er_{\mathrm{M}}^{\mathcal{D}} - r_0)\right) \tag{A.6}$$

□

THEOREM A.2 (Theorem 2.4) *Let deviation measure $\mathcal{D}_l$ correspond to risk envelope $\mathcal{Q}_l$ for $l = 1, \dots, L$. If deviation measure $\mathcal{D}$ is a convex combination of the deviation measures $\mathcal{D}_l$, then:*

$$\mathcal{D} = \sum_{l=1}^{L} \lambda_l \mathcal{D}_l \quad \text{with } \lambda_l \geqslant 0, \quad \sum_{l=1}^{L} \lambda_l = 1$$

then $\mathcal{D}$ corresponds to risk envelope $\mathcal{Q} = \sum_{l=1}^{L} \lambda_l \mathcal{Q}_l$.

PROOF OF THEOREM 2.4 With formula (2.2), we obtain:

$$\begin{aligned}
\mathcal{D}(X) &= \sum_{l=1}^{L} \lambda_l \mathcal{D}_l(X) \\
&= EX - \sum_{l=1}^{L} \lambda_l \inf_{Q \in \mathcal{Q}_l} EXQ \\
&= EX - \inf_{(Q_1,\dots,Q_L) \in (\mathcal{Q}_1,\dots,\mathcal{Q}_L)} EX\left(\sum_{l=1}^{L} \lambda_l Q_l\right) \\
&= EX - \inf_{Q \in \sum_{l=1}^{L} \lambda_l \mathcal{Q}_l} EXQ
\end{aligned} \tag{A.7}$$

□

THEOREM A.3 (Theorem 2.5) *If the master fund M, corresponding to the deviation measure $\mathcal{D}$, is known, and $\mathcal{D}$ is a convex combination of a finite number of deviation measures $\mathcal{D}_l$, $l = 1, \dots, L$:*

$$\mathcal{D} = \lambda_1 \mathcal{D}_1 + \cdots + \lambda_L \mathcal{D}_L$$

then:

$$\beta_j = \frac{\lambda_1 \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_1}) + \cdots + \lambda_L \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_{\mathrm{M}}) + \cdots + \lambda_L \mathcal{D}_L(r_{\mathrm{M}})}$$

where $Q_{\mathrm{M}}^{\mathcal{D}_l}$ *is a risk identifier of master fund return, corresponding to deviation measure* $\mathcal{D}_l$*.*

PROOF OF THEOREM 2.5 From Theorem 2.4, it follows that:

$$\begin{aligned}\beta_j &= \frac{\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}})}{\mathcal{D}} \\ &= \frac{\operatorname{cov}(-r_j, \lambda_1 Q_{\mathrm{M}}^{\mathcal{D}_1} + \cdots + \lambda_L Q_{\mathrm{M}}^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_{\mathrm{M}}) + \cdots + \lambda_L \mathcal{D}_L(r_{\mathrm{M}})} \\ &= \frac{\lambda_1 \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_1}) + \cdots + \lambda_L \operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_L})}{\lambda_1 \mathcal{D}_1(r_{\mathrm{M}}) + \cdots + \lambda_L \mathcal{D}_L(r_{\mathrm{M}})} \end{aligned} \tag{A.8}$$

Next:

$$\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_l}) = E(Er_j - r_j)(Q_{\mathrm{M}}^{\mathcal{D}_l} - EQ_{\mathrm{M}}^{\mathcal{D}_l}) \tag{A.9}$$

According to the definition of the risk envelope, $EQ_{\mathrm{M}}^{\mathcal{D}_l} = 1$. Therefore, from (A.9) we have:

$$\begin{aligned}\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_l}) &= E(Er_j - r_j)(Q_{\mathrm{M}}^{\mathcal{D}_l} - 1) \\ &= E(Er_j - r_j)Q_{\mathrm{M}}^{\mathcal{D}_l} - E(Er_j - r_j) \\ &= E(Er_j - r_j)Q_{\mathrm{M}}^{\mathcal{D}_l}\end{aligned}$$

□

COROLLARY A.4 (Corollary 2.6) *If* $\mathcal{D} = \mathrm{CVaR}_{\alpha,\lambda}^{\Delta}$, *where* $\alpha = (\alpha_1, \dots, \lambda_L)$ *and* $\lambda = (\lambda_1, \dots, \lambda_L)$*, and distribution of* r_{M} *is continuous, then:*

$$\beta_j = \frac{\begin{array}{l}\lambda_1 E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_{\alpha_1}(r_{\mathrm{M}})] + \cdots \\ \qquad\qquad + \lambda_L E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_{\alpha_L}(r_{\mathrm{M}})]\end{array}}{\lambda_1 \mathrm{CVaR}_{\alpha_1}^{\Delta}(r_{\mathrm{M}}) + \cdots + \lambda_L \mathrm{CVaR}_{\alpha_L}^{\Delta}(r_{\mathrm{M}})} \tag{A.10}$$

PROOF OF COROLLARY 2.6 For $\mathcal{D}_l = \mathrm{CVaR}_{\alpha_l}^{\Delta}$, according to (2.3):

$$Q_{\mathrm{M}}^{\mathcal{D}_l}(\omega) = \frac{1}{1-\alpha_l} \mathbf{1}_{\{r_{\mathrm{M}}(\omega) \leqslant -\mathrm{VaR}_{\alpha_l}(r_{\mathrm{M}})\}}$$

Then:

$$\begin{aligned}\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_l}) &= E(Er_j - r_j)\frac{1}{1-\alpha_l} \mathbf{1}_{\{r_{\mathrm{M}}(\omega) \leqslant -\mathrm{VaR}_{\alpha_l}(r_{\mathrm{M}})\}} \\ &= E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\mathrm{VaR}_{\alpha_l}(r_{\mathrm{M}})] \end{aligned} \tag{A.11}$$

Substituting the expression for $\operatorname{cov}(-r_j, Q_{\mathrm{M}}^{\mathcal{D}_l})$ and the expression for mixed CVaR deviation (2.5) into (A.8) gives:

$$\beta_j = \frac{\lambda_1 E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\operatorname{VaR}_{\alpha_1}(r_{\mathrm{M}})] + \cdots + \lambda_L E[Er_j - r_j \mid r_{\mathrm{M}} \leqslant -\operatorname{VaR}_{\alpha_L}(r_{\mathrm{M}})]}{\lambda_1 \operatorname{CVaR}_{\alpha_1}^{\Delta}(r_{\mathrm{M}}) + \cdots + \lambda_L \operatorname{CVaR}_{\alpha_L}^{\Delta}(r_{\mathrm{M}})}$$

□

REFERENCES

Aït-Sahalia, Y., and Lo, A. (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics* **94**, 9–51.

American Optimal Decisions (2009). *Portfolio Safeguard (PSG)*. URL: www.aorda.com.

Barberis, N., and Huang, M. (2008). Stocks as lotteries: the implications of probability weighting for security prices. *American Economic Review* **85**(5), 2066–2100.

Bliss, R. R., and Panigirtzoglou, N. (2004). Option-implied risk aversion estimates. *Journal of Finance* **49**(1), 407–446.

Estrada, J. (2004). The cost of equity of internet stocks: a downside risk approach. *European Journal of Finance* **10**, 239–254.

Fama, E. F., and French, K. R. (1996). Multifactor explanations of asset pricing anomalies. *Journal of Finance* **51**(1), 55–84.

Friend, I., and Westerfield, R. (1980). Co-skewness and capital asset pricing. *Journal of Finance* **35**, 897–913.

Jackwerth, J. C. (2000). Recovering risk aversion from option prices and realized returns. *Review of Financial Studies* **13**, 433–467.

Konno, H., and Yamazaki, H. (1991). Mean absolute deviation portfolio optimization model and its application to the Tokyo stock market. *Management Science* **37**, 519–531.

Kraus, A., and Litzenberger, R. H. (1976). Skewness preference and the valuation of risk assets. *Journal of Finance* **31**, 1085–1100.

Krokhmal, P., Palmquist, J., and Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *The Journal of Risk* **4**(2), 43–68.

Krokhmal, P., Rockafellar, R. T., and Uryasev, S. (2006). Risk management and optimization in finance. *Journal of Banking and Finance* **30**(2), 315.

Lim, K.-G. (1989). A new test of the three-moment CAPM. *Journal of Financial and Quantitative Analyis* **24**, 205–216.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* **47**(1), 13–37.

Markowitz, H. M. (1952). Portfolio selection. *Journal of Finance* **7**(1), 77–91.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica* **34**(4), 768–783.

Rockafellar, R. T., and Uryasev, S. (2000). Optimization of conditional value-at-risk. *The Journal of Risk* **2**(3), 21–42.

Rockafellar, R. T., and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance* **26**, 1443–1471.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2006a). Generalized deviations in risk analysis. *Finance and Stochastics* **10**, 51–74.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2006b). Master funds in portfolio analysis with general deviation measures. *Journal of Banking and Finance* **30**(2), 743–778.

Rockafellar, R. T., Uryasev, S. and Zabarankin, M. (2006c). Optimality conditions in portfolio analysis with generalized deviation measures. *Mathematical Programming* **108**(2), 515–540.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2007). Equilibrium with investors using a diversity of deviation measures. *Journal of Banking and Finance* **31**(11), 3251–3268.

Sarykalin, S. (2008). Calibration of general deviation measures from market data. Doctoral Thesis, University of Florida.

Sarykalin, S., Serraino, G., and Uryasev, S. (2008). VaR vs CVaR in risk management and optimization. *Tutorials in Operations Research: INFORMS* **2008**, 270–294.

Sharpe, W. F. (1964). Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance* **19**, 425–442.

Sortino, F. A., and Forsey, H. J. (1996). On the use and misuse of downside risk. *Journal of Portfolio Management* **22**(2), 35–42.

Treynor, J. L. (1961). Market value, time, and risk. Unpublished manuscript.

Treynor, J. L. (1962). Toward a theory of market value of risky assets. Unpublished manuscript.

Uryasev, S. (2000). Conditional value-at-risk: optimization algorithms and applications. *Financial Engineering News* **14**, 1–5.