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Optimal structuring of collateralized debt obligation contracts: an optimization approach

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The objective of this paper is to help a bank originator of a collateralized debt obligation (CDO) to build a maximally profitable CDO. We consider an optimization framework for structuring CDOs. The objective is to select attachment/detachment points and underlying instruments in the CDO pool. In addition to "standard" CDOs we study so-called "step-up" CDOs. In a standard CDO contract the attachment/detachment points are constant over the life of a CDO. In a step-up CDO the attachment/detachment points may change over time. We show that step-up CDOs can save about 25–35% of tranche spread payments (ie, profitability of CDOs can be boosted by about 25–35%). Several optimization models are developed from the bank originator perspective. We consider a synthetic CDO where the goal is to minimize payments for the credit risk protection (premium leg), while maintaining a specific credit rating (assuring the credit spread) of each tranche and maintaining the total incoming credit default swap spread payments. The case study is based on the time-to-default scenarios for obligors (instruments) generated by the Standard & Poor's CDO Evaluator. The Portfolio Safeguard package by AORDA was used to optimize the performance of several CDOs based on example data.

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1 INTRODUCTION

The market of credit risk derivatives was booming before the recent financial crisis. Collateralized debt obligations (CDOs) accounted for a significant fraction of this market. The appeal of CDOs was their high profit margins. They offered returns that were sometimes 2–3% higher than corporate bonds with the same credit rating. The recession seems to be over now and banks keep searching for new opportunities with credit risk derivatives. Optimal structuring techniques may help to increase the profitability of CDOs and other similar derivatives. A CDO is based on so-called "credit tranching", where the losses of the portfolio of bonds, loans or other securities are repackaged. The paper considers synthetic CDOs in which the underlying credit exposures are taken with credit default swaps (CDSs) rather than with physical assets. The CDO is split into different risk classes or tranches. For instance, a CDO may have four tranches (senior, mezzanine, subordinate and equity). Losses are applied to the later classes of debt before earlier ones. A range of products is created from the underlying pool of instruments, varying from a very risky equity debt to a relatively riskless senior debt. Each tranche is specified by its attachment and detachment points as the percentages of the total collateral. The lower tranche boundary is called the attachment point, while the upper tranche boundary is called the detachment point. The CDO tranche loss occurs when the cumulative collateral loss exceeds the tranche attachment point.

The tranche spread is defined as a fraction of the total collateral. The amount of money that the originating bank should pay per year (payments are usually made quarterly) to have this tranche "insured" is the spread times the tranche size. In a standard CDO contract the attachment and detachment points for each tranche are the same for the whole contract period. Therefore, the bank originator should make the same payments every period (if the tranche has not defaulted).

This paper also considers step-up CDOs, where the attachment/detachment points may vary across the life of the CDO (typically increasing each time period). A specific risk exposure can be built in each time period.

Approaches for structuring credit risk have been well-studied in the literature (see, for example, Choudhry (2010), Das (2005), Lancaster *et al* (2008) and Rajan *et al* (2007)). However, the main focus of the suggested methodologies is on modeling the default events, rather than on building optimal (from a risk–return perspective) credit derivative structures. Works using optimization to calibrate copulas in CDOs to match market prices include Hull and White (2010), Halperin (2009), Jewan *et al* (2009) and Rosen and Saunders (2009).

In practice, CDOs are typically structured with a brute-force trial-and-error approach involving the following steps:

(1) a selection of securities included in a CDO;

(2) structuring of CDO and setting of attachment/detachment points;

(3) evaluation of the suggested CDO and the estimation of credit ratings of CDO tranches.

If the structure does not satisfy the desired goals, then the process is repeated. For instance, the attachment/detachment points are adjusted and default probabilities (and desired credit ratings) are again calculated. This process is time consuming and usually gives suboptimal solutions.

We are not aware of publications related to CDO structuring (adjusting attachment/datachment levels and selecting securities for CDO portfolio) from an optimization point of view, except for Jewan *et al* (2009), which has a section on using optimization for CDO structuring. Jewan *et al* (2009) applied a genetic optimization algorithm for finding an optimal structure of a bespoke CDO. Genetic algorithms are very powerful and can be applied to a wide range of problems, but they may perform poorly for high-dimensional problems, especially when the calculation of performance functions requires a lot of time.

We apply an advanced optimization approach that may improve the structure profitability by up to 35% for some cases. We focus on problem formulations rather than on the development of optimization algorithms for such problems. Optimization problems are formulated with standard nonlinear functions which are precoded in nonlinear programming software packages. In particular, we use the Portfolio Safeguard (PSG) optimization package, which contains an extensive library of precoded nonlinear functions, including the partial moment function, denoted by "pm_pen", and the probability that a system of linear constraints with random coefficients is satisfied, denoted by "prmulti_pen" (see AORDA (2008)). A full list of PSG functions and their mathematical descriptions is available.[1] With PSG, the problem solving involves three main stages.

Mathematical formulation of a problem with a metacode using PSG nonlinear functions. Typically, a problem formulation involves five to ten operators of a metacode (see, for example, Appendix A, with metacode for optimization problem (4.10)–(4.15)).

Preparation of data for the PSG functions in an appropriate format. For instance, the standard deviation function is defined on a covariance matrix or a matrix of loss scenarios. One of those matrices should be prepared if we use this function in the problem statement.

[1] See www.ise.ufl.edu/uryasev/files/2011/12/definitions_of_functions.pdf.

Solving the optimization problem with PSG using the predefined problem statement and data for PSG functions. The problem can be solved in several PSG environments, such as a MATLAB environment and run-file (text) environment.

In the first CDO optimization problem discussed in this paper, we only changed attachment/detachment points in a CDO: the goal was to minimize payments for the credit risk protection (premium leg), while maintaining the specific credit ratings of tranches. In this case, the pool of instruments and income spreads for credit default swaps is fixed. We considered several variants of the problem statement with various assumptions and simplifications.

In the second optimization problem we bounded from below the total income spread payments and simultaneously optimized the set of instruments in a CDO pool and the attachment/detachment points. Low outgoing spread payments were ensured by maintaining the credit ratings of tranches.

In the third optimization problem we minimized the total cost, which is defined as the difference between the total outcome and income spreads. We simultaneously optimized the set of instruments in a CDO pool and the attachment/detachment points while maintaining the specific credit rating of tranches.

The case study solved several problems with different credit ratings and other constraints. It is based on the time-to-default scenarios for obligors (instruments) generated by the Standard & Poor's CDO Evaluator for some example data.

The results show that, compared with a standard CDO with constant attachment/detachment points, a step-up CDO can save about 25–35% of outgoing tranche spread payments for a bank originator.

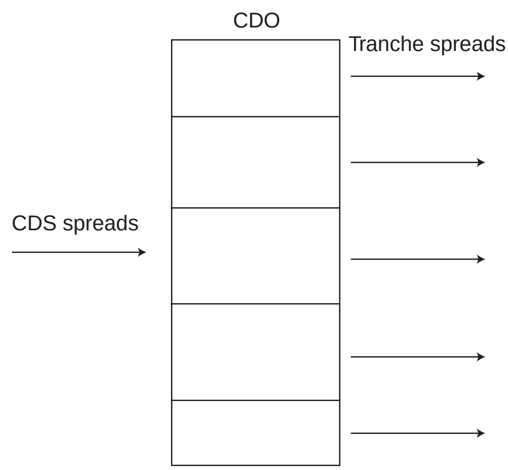
The paper proceeds as follows. Section 2 provides a brief description of CDOs and discusses the general ideas involved in CDO structuring. Section 3 describes optimization models. It provides formal optimization problem statements and optimality conditions. Section 4 provides a case study with calculation results.

2 BACKGROUND TO COLLATERALIZED DEBT OBLIGATIONS

This section provides a brief description of CDOs and the ideas involved in CDO structuring.

A CDO is a complex credit risk derivative product. This paper considers so-called synthetic CDOs. A synthetic CDO consists of a portfolio of CDSs. A CDS is a credit risk derivative with a bond as an underlying asset. It can be viewed as an insurance against possible bond losses due to credit default events. A CDS buyer pays a certain cashflow (CDS spread) during the life of the bond. If this bond incurs credit default losses, the CDS buyer is compensated for that loss. Typically, the higher the rating of the underlying bond, the smaller the spread of the CDS. It should be noted that a CDS buyer does not need to hold an underlying bond in its portfolio.

FIGURE 1 Synthetic CDO spreads flow structure.



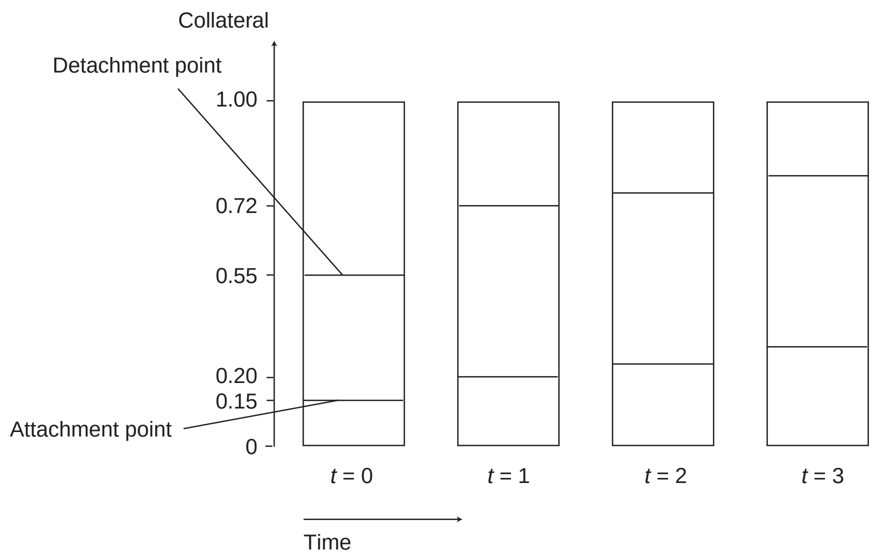
The CDO receives payments (CDS spreads) from each CDS and covers credit risk losses in case of default. Therefore, this portfolio covers possible losses up to the total collateral amount. A CDO originator repackages possible credit risk losses to "credit tranches". Losses are applied to the later classes of debt before earlier ones. Therefore, a range of products is created from the basket of CDSs, varying from a very risky equity debt to a relatively riskless senior debt. The methodology considered in this paper is quite general and can be applied to a CDO with any number of tranches.

To "insure" credit losses in a tranche, a CDO should pay the (per year) spread times the tranche size. Usually payments are made on a quarterly basis. The spread of a tranche is mostly determined by its credit rating, which is based on the default probability of this tranche.

Figure 1 shows the structure of CDO cashflows. The bank originator sells the CDSs. Then the bank repackages losses and buys an "insurance" (credit protection) for each tranche. If the sum of spreads of CDSs in a CDO pool is greater than the sum of tranche spreads, the CDO originator locks in an arbitrage.

Each tranche in a CDO contract can be given a rating (eg, AAA, AA, A, BBB, etc, in the Standard & Poor's classification). A tranche rating corresponds to a probability of default estimated by a credit agency. For example, a tranche has the AAA Standard & Poor's rating if the probability that the loss will exceed the attachment point during the contract period is less than 0.12% (this corresponds to the settings of Standard & Poor's CDO Evaluator).

FIGURE 2 CDO attachment and detachment points structure.



The next section discusses optimization models for minimizing the sum of tranche spreads on the condition that the pool of CDSs is fixed. These models are then extended to the case when the bank originator simultaneously chooses CDSs for the pool and adjusts the attachment/detachment points of tranches. We consider a CDO with attachment/detachment points that may increase over time (see Figure 2). Such a CDO creates a desirable risk exposure in each time period. In a standard CDO with constant attachment/detachment points, the losses are cumulated over time. Therefore, the probability that a loss will hit a tranche attachment point in the first period is much smaller than the probability that a loss will hit it in the last period. We will show that by changing tranche's attachment points over time, we will maintain the tranche's credit ratings, thereby significantly decreasing the cumulative amount of spread payments from the bank originator.

3 OPTIMIZATION MODELS

This section presents several optimization models for CDO structuring, ie, the selection of CDO underlying instruments and attachment/detachment points. The objective is to maximize profits for the bank originator.

3.1 Optimization of attachment/detachment points (with a fixed pool of assets)

First we consider a structuring problem for a CDO with a fixed pool of assets. We select optimal attachment/detachment points for tranches. Consider a CDO with a contract period T and a fixed number of tranches M. Note that there are only $M - 1$ attachment/detachment points to be determined, since the attachment point for the first tranche is fixed and it is equal to zero.

- Let s_m denote the tranche m spread.
- Let L^t denote the cumulative collateral loss by period $t + 1$.
- Let x_m^t denote the attachment point of tranche m in period t.

All the values above are measured as a fraction of the total collateral. Furthermore, we always assume that $x_1^t = 0$, and $x_{M+1}^t = 1$ for all $t = 1, \dots, T$. The CDO is usually structured so that each tranche has a particular credit rating. Here we assume that each tranche spread is fully determined by its credit rating. In other words, the vector of spreads $(s_1, \dots, s_M)$ is fixed if we ensure appropriate ratings for the tranches. Note that $s_1 > s_2 > \cdots > s_M$, since the higher the tranche number, the higher its credit rating and the lower its spread. During the CDO contract period, the losses accumulate and a bank originator only pays for the remaining amount of the total collateral. For instance, if the size of the tranche M (super-senior) is 70% of the size of the total collateral, then the bank originator should pay $(100\% - \max(30\%, L_t)) \times s_M$ in the period t to have this tranche (or its remaining part) "insured". Then the total payment for all tranches in the period t is:

$$\sum_{m=1}^{M} (x_{m+1}^t - \max(x_m^t, L^t))^+ s_m$$

where the function $(\cdot)^+$ is defined as:

$$x^+ = \begin{cases} x, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

The vector of losses $L = (L_1, \dots, L_T)$ is a random vector. In our case study, we use Standard & Poor's CDO Evaluator to generate the time-to-default scenarios for obligors (instruments) and calculate the vector $L^s = (L_1^s, \dots, L_T^s)$ for a particular scenario $s = 1, \dots, S$.

We want to find the attachment points $\{x_m^t\}_{2,\dots,M}^{1,\dots,T}$ in order to minimize the present value of the expected spread payments over all periods for all tranches. We impose constraints on default probabilities of tranches (to ensure that they have a credit rating) and some constraints on attachment points. We also define $m = 2, \dots, M$, since the attachment point of the lowest tranche is fixed ($x_1^t = 0$).

Let us define

- p_m^{rating} to be the upper bound on default probability of tranche m corresponding to its credit rating;
- $p_m^t(x_m^1, \dots, x_m^t)$ to be the default probability of tranche m up to time moment $t+1$ (ie, the probability that the cumulative collateral loss exceeds the tranche attachment point at least once in periods $1, \dots, t$), calculated from the scenarios $s = 1, \dots, S$:

$$p_m^t(x_m^1, \dots, x_m^t) = 1 - \Pr\{L_1 \leqslant x_m^1, \dots, L_t \leqslant x_m^t\}$$
$$= 1 - \frac{1}{S}\sum_{s=1}^{S} \mathbf{1}_{\{L_1^s \leqslant x_m^1, \dots, L_t^s \leqslant x_m^t\}} \tag{3.1}$$

  where:

$$\mathbf{1}_{\{L_1^s \leqslant x_m^1, \dots, L_t^s \leqslant x_m^t\}} = \begin{cases} 1 & \text{if } L_1^s \leqslant x_m^1, \dots, L_t^s \leqslant x_m^t \\ 0 & \text{otherwise} \end{cases}$$

- $p_m^T(x_m^1, \dots, x_m^T)$ to be the default probability of tranche m, the special case of $p_m^t(x_m^1, \dots, x_m^t)$ for $t = T$;
- q_m^t to be the upper bound for the default probability $p_m^t(x_m^1, \dots, x_m^t)$; and
- r to be the one-period interest rate.

The probability function $p_m^t(x_m^1, \dots, x_m^t)$ is denoted by "prmulti_pen" and is pre-coded in PSG software, which we use to solve optimization problems.

The first optimization problem is formulated as follows.

Problem A

Minimize the present value of expected spread payments:

$$\min_{\{x_m^t\}_{m=2,\dots,M}^{t=1,\dots,T}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} E[(x_{m+1}^t - \max(x_m^t, L^t))^+ s_m] \tag{3.2}$$

subject to rating constraints:

$$p_m^T(x_m^1, \dots, x_m^T) \leqslant p_m^{\text{rating}}, \quad m = 2, \dots, M \tag{3.3}$$

default probability constraints:

$$p_m^t(x_m^1, \dots, x_m^t) \leqslant q_m^t, \quad m = 2, \dots, M,\ t = 1, \dots, T-1 \tag{3.4}$$

attachment point monotonicity constraints:

$$x_m^t > x_{m-1}^t, \quad m = 3, \dots, M,\ t = 1, \dots, T \tag{3.5}$$

and box constraints for the attachment point:

$$0 \leqslant x_m^t \leqslant 1, \quad m = 2, \dots, M,\ t = 1, \dots, T \tag{3.6}$$

Note that constraint (3.3) maintains the credit ratings of tranches. In contrast, constraint (3.4) gives additional flexibility to a decision maker to control default probabilities at a specific period. It might be driven by the bank originator requirements or some other considerations. Since a collateral loss is cumulative, it is reasonable to set upper bounds that are monotonically increasing with time for the cumulative default probabilities:

$$q_m^1 \leqslant q_m^2 \leqslant \cdots \leqslant q_m^n$$

The expected values in the objective function are taken over all simulated losses L^s, $s = 1, \dots, S$. A typical CDO contract with attachment points that are constant over time can be defined by the linear constraints:

$$x_m^t = x_m^{t-1}, \quad m = 2, \dots, M,\ t = 1, \dots, T \tag{3.7}$$

To solve Problem A we derive an equivalent representation of the objective function.

THEOREM 3.1 (Equivalent representation of objective) *Let* $\Delta s_m = s_m - s_{m+1}$, $m = 1, \dots, M-1$ *and* $\Delta s_M = s_M$. *Then the following equality holds:*

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} E[(x_{m+1}^t - \max(x_m^t, L^t))^+ s_m] = \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m E[(x_{m+1}^t - L^t)^+]$$

PROOF Let us prove the equation implying the statement of the theorem:

$$\sum_{m=1}^{M} (x_{m+1}^t - \max(x_m^t, L^t))^+ s_m = \sum_{m=1}^{M} (x_{m+1}^t - L^t)^+ \Delta s_m \tag{3.8}$$

Consider the right-hand side of (3.8):

$$
\begin{aligned}
&\sum_{m=1}^{M}(x_{m+1}^t - L^t)^+ \Delta s_m \\
&\quad = \sum_{m=1}^{M-1}(x_{m+1}^t - L^t)^+(s_m - s_{m+1}) + (x_{M+1}^t - L^t)^+ s_M \\
&\quad = \sum_{m=1}^{M-1}(x_{m+1}^t - L^t)^+ s_m + (x_{M+1}^t - L^t)^+ s_M - \sum_{m=1}^{M-1}(x_{m+1}^t - L^t)^+ s_{m+1} \\
&\quad = \sum_{m=1}^{M}(x_{m+1}^t - L^t)^+ s_m - \sum_{m=2}^{M}(x_m^t - L^t)^+ s_m \\
&\quad = \sum_{m=1}^{M}\{(x_{m+1}^t - L^t)^+ - (x_m^t - L^t)^+\}^+ s_m + (x_1^t - L^t)^+ s_1 \qquad (3.9)
\end{aligned}
$$

The following inequality holds:

$$
\begin{aligned}
\{(x_{m+1}^t - L^t)^+ - (x_m^t - L^t)^+\}^+ &= (x_{m+1}^t - \max(x_m^t, L^t))^+ \\
&= \begin{cases} x_{m+1}^t - x_m^t, & L^t < x_m^t \\ x_{m+1}^t - L^t, & x_m^t \leqslant L^t \leqslant x_{m+1}^t \\ 0, & L_t > x_{m+1}^t \end{cases} \qquad (3.10)
\end{aligned}
$$

The term is zero since $x_1^t = 0$. Therefore, (3.9) and (3.10) imply (3.8). After taking expectation over both sides of the equation and summing them up over time t, we obtain the statement of the theorem. □

With Theorem 3.1 we write an equivalent formulation for Problem A.

Problem A (equivalent formulation)

Minimize the present value of expected spread payments:

$$
\min_{\{x_m^t\}_{m=2,\dots,M}^{t=1,\dots,T}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m E[(x_{m+1}^t - L^t)^+]
$$

subject to constraints (3.3)–(3.6).

3.2 Simultaneous optimization of CDO pool and credit tranching

This section considers two problems of selecting both the assets in a CDO pool and CDO attachment points. The first problem in this section minimizes the total expected payment of CDO tranches while bounding the total income spread from below. The second problem in this section minimizes the total cost, which is equal to the difference between the total expected CDO tranches payment and the total income spread payment of CDSs in the CDO pool. I is the number of CDSs available for adding to the CDO pool. The pool composition is defined by the vector $y = (y_1, \dots, y_I)$, where y_i is the weight of CDS i (ie, CDS i). Each y_i is bounded by value u. A list of definitions follows.

- c_i denotes the income spread payment for CDS i.
- $-\theta_i^t$ denotes the random cumulative loss of CDS i by period $t + 1$.[2]
- $L^t(\theta, y) = -\sum_{i=1}^{I} \theta_i^t y_i$ denotes random cumulative loss of the portfolio by period $t + 1$.
- $p_m^T(x_m^1, \dots, x_m^t, y_1, \dots, y_I)$ denotes the default probability of tranche m up to time moment $t+1$ (ie, the probability that the cumulative collateral loss exceeds the tranche attachment point at least once in periods $1, \dots, t$), calculated from the scenarios $s = 1, \dots, S$:

$$\begin{aligned} p_m^t(x_m^1, \dots, x_m^t, y_1, \dots, y_I) &= 1 - \Pr\{L_1(\theta, y) \leqslant x_m^1, \dots, L_t(\theta, y) \leqslant x_m^t\} \\ &= 1 - \frac{1}{S} \sum_{s=1}^{S} \mathbf{1}_{\{L_1^s(\theta,y) \leqslant x_m^1, \dots, L_t^s(\theta,y) \leqslant x_m^t\}} \end{aligned} \tag{3.11}$$

  where:

$$\mathbf{1}_{\{L_1^s(\theta,y) \leqslant x_m^1, \dots, L_t^s(\theta,y) \leqslant x_m^t\}} = \begin{cases} 1 & \text{if } L_1^s(\theta, y) \leqslant x_m^1, \dots, L_t^s(\theta, y) \leqslant x_m^t \\ 0 & \text{otherwise} \end{cases}$$

- l is the lower bound on CDO spread income payments.
- u is the upper bound on the weight of any CDS in the CDO pool.

We now formulate the first optimization problem in this section.

Problem B

Minimize the present value of expected spread payments:

$$\min_{\{y, \{x_m^t\}_{m=1,\dots,M}^{t=1,\dots,T}\}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=2}^{M} \Delta s_m E[(x_{m+1}^t - L^t(\theta, y))^+]$$

[2] The CDS loss occurs when the obligor of an underlying asset defaults. The loss amount is calculated as total amount of collateral × (1 − recovery rate), where the recovery rate is a random number.

subject to rating constraints:

$$p_m^T(x_m^1, \dots, x_m^T, y_1, \dots, y_I) \leqslant p_m^{\text{rating}}, \quad m = 2, \dots, M \tag{3.12}$$

default probability constraints:

$$p_m^t(x_m^1, \dots, x_m^t, y_1, \dots, y_I) \leqslant q_m^t, \quad m = 2, \dots, M,\ t = 1, \dots, T-1 \tag{3.13}$$

income spread payments constraint:

$$\sum_{i=1}^{I} c_i y_i \geqslant l \tag{3.14}$$

budget constraint:

$$\sum_{i=1}^{I} y_i = 1 \tag{3.15}$$

box constraints for weights:

$$0 \leqslant y_i \leqslant u, \quad i = 1, \dots, I \tag{3.16}$$

attachment point monotonicity constraints:

$$x_m^t \geqslant x_{m-1}^t, \quad m = 3, \dots, M,\ t = 1, \dots, T \tag{3.17}$$

and box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad m = 2, \dots, M,\ t = 1, \dots, T \tag{3.18}$$

With the income spread constraint (3.14) we bound from below the income payments to the CDO. This constraint is active and the CDO incoming payments are defined by the average spread l. With the fixed incoming payments we minimize the expected present value of the outcoming payments. We can solve many instances of Problem B with different values of the average spread l and select the most profitable CDO.

The third problem in this section is formulated as follows.

Problem C

Minimize the total cost:

$$\min_{\{y, \{x_m^t\}_{m=1,\dots,M}^{t=1,\dots,T}\}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=2}^{M} \Delta s_m E[(x_{m+1}^t - L^t(\theta, y))^+] - \sum_{i=1}^{I} c_i y_i$$

subject to rating constraints:

$$p_m^T(x_m^1, \ldots, x_m^T, y_1, \ldots, y_I) \leqslant p_m^{\text{rating}}, \quad m = 2, \ldots, M \tag{3.19}$$

budget constraint:

$$\sum_{i=1}^{I} y_i = 1 \tag{3.20}$$

box constraints for weights:

$$0 \leqslant y_i \leqslant u, \quad i = 1, \ldots, I \tag{3.21}$$

attachment point monotonicity constraints:

$$x_m^t \geqslant x_{m-1}^t, \quad m = 3, \ldots, M,\ t = 1, \ldots, T \tag{3.22}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad m = 2, \ldots, M,\ t = 1, \ldots, T \tag{3.23}$$

Note that in Problem C (in contrast to Problem B) the objective is the total cost and there is no income spread constraint.

3.3 Simplification 1: problem decomposition for large-scale problems

For a moderate number of scenarios (eg, 50 000) and a moderate number of instruments (eg, 200), Problem A can be easily solved with the proposed formulations using AORDA (2008). However, for solving a larger-size Problem A (eg, 500 000 scenarios), we decompose the problem into $M - 1$ separate subproblems, ie, we find the optimal attachment points for each tranche separately and then combine the solutions. We also show (see the proof of Theorem 3.2 in this section) that, under certain conditions, inequality (3.24) becomes an equality. The minimum of the sum is always greater than the sum of the minima of its parts. Therefore:

$$\begin{aligned}
\min_{\{x_m^t\}_{m=2,\ldots,M}^{t=1,\ldots,T}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m E[(x_{m+1}^t - L^t)^+] \\
\geqslant \sum_{m=2}^{M} \min_{\{x_m^t\}_{t=1,\ldots,T}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \Delta s_{m-1} E[(x_m^t - L^t)^+] \\
+ \sum_{t=1}^{T} \frac{1}{(1+r)^t} \Delta s_M E[1 - L^t]
\end{aligned} \tag{3.24}$$

In inequality (3.24) we use the fact that $x_{M+1}^t = 1$. The left-hand side is the objective of Problem A. To solve Problem A we can solve $M - 1$ following problems for each $m = 2, \ldots, M$.

Problem A_m

Minimize the present value of expected size of tranche m:

$$\min_{\{x_m^t\}_{t=1,\dots,T}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} E[(x_m^t - L^t)^+] \tag{3.25}$$

subject to rating constraint:

$$p_m^T(x_m^1, \dots, x_m^T) \leqslant p_m^{\text{rating}} \tag{3.26}$$

default probability constraints:

$$p_m^t(x_m^1, \dots, x_m^t) \leqslant q_m^t, \quad t = 1, \dots, T-1 \tag{3.27}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad t = 1, \dots, T \tag{3.28}$$

Note that we omit the term Δs_m in the objective (3.25) since it is a fixed nonnegative number, and it does not affect the optimal solution point.

Here is the formal proof that Problem A can be decomposed to Problems A_m, for $m = 2, \dots, M$.

THEOREM 3.2 (Decomposition theorem) *If optimal solutions for Problem A_m for $m = 2, \dots, M$ satisfy inequalities:*

$$x_m^t \geqslant x_{m-1}^t, \quad m = 3, \dots, M, \ t = 1, \dots, T$$

then, taken together, these optimal solutions are the optimal solution of the corresponding Problem A.

PROOF Denote the optimal objective values for Problem A_m by A_m, and the optimal objective value for the corresponding Problem A (with the same parameters and data) by $\bar{A}$. We can rewrite (3.24) as:

$$\bar{A} \geqslant \sum_{m=2}^{M} A_m \Delta s_m + \sum_{t=1}^{T} \frac{1}{(1+r)^t} \Delta s_{M+1} E[1 - L^t]$$

The optimal solutions of Problem A_m satisfy (3.5). Hence, these optimal solutions satisfy all the constraints (3.3)–(3.6). Therefore, these optimal solutions taken together form a feasible point of Problem A. Hence:

$$\bar{A} \leqslant \sum_{m=2}^{M} A_m \Delta s_m + \sum_{t=1}^{T} \frac{1}{(1+r)^t} \Delta s_{M+1} E[1 - L^t]$$

Consequently:

$$\bar{A} = \sum_{m=2}^{M} A_m \Delta s_m + \sum_{t=1}^{T} \frac{1}{(1+r)^t} \Delta s_{M+1} E[1 - L^t]$$

and the theorem is proved. $\square$

3.4 Simplification 2: lower and upper bounds minimization

This section considers the problem of minimizing upper and lower bounds of an objective function in Problem A_m. It shows that the problems of minimizing the lower and upper bounds are equivalent. Using the fact that the cumulative losses L^t are always nonnegative, we can write:

$$x_m^t \geqslant E[(x_m^t - L^t)^+] \geqslant E[x_m^t - L^t] = x_m^t - E[L^t]$$

for any $m = 2, \dots, M$, $t = 1, \dots, T$.

Thus, the objective in Problem A_m can be bounded by:

$$\begin{aligned} \sum_{t=1}^{T} \frac{x_m^t}{(1+r)^t} &\geqslant \sum_{t=1}^{T} \frac{1}{(1+r)^t} E[(x_m^t - L^t)^+] \\ &\geqslant \sum_{t=1}^{T} \frac{x_m^t}{(1+r)^t} - \sum_{t=1}^{T} \frac{E[L^t]}{(1+r)^t} \end{aligned} \tag{3.29}$$

Since $E[L^t]$ does not depend on x_m^t, the problems of minimizing an upper and lower bounds in (3.29) are equivalent in the sense that they give the same optimal vectors (although optimal objective values are different).

The objective function can be written as:

$$\min_{\{x_m^t\}, t=1,\dots,T} \sum_{t=1}^{T} \frac{1}{(1+r)^t} x_m^t \tag{3.30}$$

We can optimize this objective for either a fixed pool of assets with constraints (3.3)–(3.6), or a nonfixed pool of assets with constraints (3.12)–(3.18). Our numerical experiments show that there is no significant difference between the optimal solutions of Problem A_m and the optimal solutions of an upper bound minimization (3.30). Such a small difference can be explained by the fact that the cumulative CDO losses L^t are usually fairly small compared with x_m^t. The higher the tranche number, the closer $(x_m^t - L^t)^+$ and x_m^t become. The advantage of such simplification is that the nonlinear objective function of the simplified problem (3.30) is a linear function of x_m^t and requires much less time to solve compared with the problem with the objective

(3.25). The objective (3.25) includes the partial moment function $E[(x_m^t - L^t)^+]$, which can be linearized for a discrete number of scenarios. However, this linearization will lead to a problem of much higher dimension than the problem with simplified objective (3.30).

In addition to mathematical expressions we provide a further intuitive explanation that leads to the same simplification. The higher the rating of the tranche, the lower its spread; therefore, the larger the size of the highest tranche, the lower the cost for the total "insurance". Thus, we may start by finding the "best" attachment point for the super-senior tranche, while maintaining its credit rating. When the attachment point for the super-senior tranche is found, we can proceed with the next lower tranche (suppose it is senior) by solving the same optimization problem for the new rating. Every attachment point that is found serves as a detachment point for the lower tranche. Thus, the attachment points can be obtained recursively for all tranches. Optimizing the single tranche is an independent problem with the same objective function (3.30), as soon as higher tranches are fixed.

The problem formulation with the objective (3.30) is very simple and its solution may be a good starting point for the deeper analysis. Therefore, the use of it may provide a preliminary analysis on the assets one might want to include in the portfolio.

4 CASE STUDY

This section reports numerical results for several problems described in the previous section. In particular, we considered Problem A_m with objective (3.25) and simplified objective (3.30), and Problem B. We solved optimization problems with PSG. There are several documented case studies on CDO structuring in the standard version of the PSG package.[3] The interested reader is referred to the standard PSG installation for other case studies.

Consider a CDO with $T = 5$ years, and $M = 5$ (number of tranches). The times of the adjustments in attachment points are:

$$t_1 = 1, \qquad t_2 = 2, \qquad t_3 = 3, \qquad t_4 = 4, \qquad t_5 = 5$$

The spread payments are usually made quarterly. For simplicity we assume that, during the period i, all the spread payments are made in the middle of one yearly period, so we discount payments with the coefficient $1/(1 + r)^{t_i - 0.5}$. We set interest rate $r = 7\%$. The credit ratings (Standard & Poor's) of the tranches are BBB, A, AA and AAA. These ratings correspond to the maximum default probabilities:

$$p_{\text{AAA}} = 0.12\%, \qquad p_{\text{AA}} = 0.36\%, \qquad p_{\text{A}} = 0.71\%, \qquad p_{\text{BBB}} = 2.81\%$$

[3] CDO case studies, data and codes are available in the regular edition of PSG. A free trial of PSG that allows the download of data and results is available at www.aorda.com/aod/psg.action.

as defined in Standard & Poor's CDO Evaluator. The attachment point of the lowest tranche is fixed ($x_1^t = 0$). The number of assets in the pool is $I = 53$. We use Standard & Poor's CDO Evaluator to generate the time-to-default scenarios for the CDSs (instruments).

Case 1a

In this case we optimized Problem A_m with rating constraint for each tranche m ($m = 2, \ldots, M$) separately.

Optimization Problem 1a (corresponds to Problem A_m with the rating constraint)

Minimize the present value of expected size of tranche m:

$$\min_{\{x_m^t\}_{t=1,\ldots,5}} \sum_{t=1}^{5} \frac{1}{(1+r)^{t-0.5}} E[(x_m^t - L^t)^+] \tag{4.1}$$

subject to rating constraint:

$$p_m^5(x_m^1, \ldots, x_m^5) \leqslant p_m^{\text{rating}} \tag{4.2}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad t = 1, \ldots, 5 \tag{4.3}$$

For this case we use 10 000 time-to-default scenarios.

Case 1b

To investigate the difference between the step-up CDO with attachment points changing over time and the standard CDO, we added a constraint assuring the constancy of attachment/detachment points over the time.

Optimization Problem 1b (corresponds to Problem A_m with rating constraint and attachment point constraints)

Minimize the present value of expected size of tranche m:

$$\min_{\{x_m^t\}_{t=1,\ldots,5}} \sum_{t=1}^{5} \frac{1}{(1+r)^{t-0.5}} E[(x_m^t - L^t)^+]$$

subject to rating constraint:

$$p_m^5(x_m^1, \ldots, x_m^5) \leqslant p_m^{\text{rating}} \tag{4.4}$$

constancy of attachment points constraint:

$$x_m^t = x_m^{t-1}, \quad t = 2, \dots, 5 \tag{4.5}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad t = 1, \dots, 5 \tag{4.6}$$

Case 2

Data, codes and calculation results for case 2 with a standard CDO (ie, with attachment points constant over time for the BBB tranche) can be downloaded from the AORDA website.[4]

This case considers the difference between the solution of Problem A_m with the original objective function (3.25) and with the simplified objective function (3.30).

Optimization Problem 2 (corresponds to Problem A_m with rating constraint and simplified objective function)

Minimize the simplified objective function of Problem A_m:

$$\min_{\{x_m^t\}, t=1,\dots,5} \sum_{t=1}^{T} \frac{1}{(1+r)^{t-0.5}} x_m^t \tag{4.7}$$

subject to rating constraint:

$$p_m^5(x_m^1, \dots, x_m^5) \leqslant p_m^{\text{rating}} \tag{4.8}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad t = 1, \dots, 5 \tag{4.9}$$

Table 1 on the facing page gives the computational results for Problems 1a, 1b and 2. There is a substantial difference between the optimal points and the optimal objectives for optimization Problems 1a and 1b. The difference between the optimal objectives for different tranches is around 25–35% (for instance, for tranche BBB the objective equals 0.5797 for Problem 1a and 0.7818 for Problem 1b). There is a significant difference between spread payments for each tranche in the step-up CDO and in the standard CDO. The results show that the step-up CDO allows the bank originator to save a substantial amount of money. Note that there is a slight difference between the solutions of Problems 1a and 2 for tranches BBB and A,

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[4] See www.aorda.com/aod/casestudy/CS_Structuring_Step-up_CDO_Optimization_I/problem_example_1_case_2__BBB.

TABLE 1 Optimization problems 1a, 1b and 2: optimal attachment points of a five-period CDO contract.

(a) Problem 1a

Period	Tranche rating BBB	A	AA	AAA
1	0.1491	0.1768	0.1877	0.2258
2	0.1837	0.2147	0.2239	0.2554
3	0.2213	0.2611	0.2722	0.2942
4	0.2604	0.2947	0.3141	0.3307
5	0.2910	0.3342	0.3502	0.3671
Objective	0.5797	0.7261	0.7814	0.8908

(b) Problem 1b

Period	Tranche rating BBB	A	AA	AAA
1–5	0.2639	0.3043	0.3212	0.3500
Objective	0.7818	0.9524	1.0240	1.1460

(c) Problem 2

Period	Tranche rating BBB	A	AA	AAA
1	0.1513	0.1768	0.1877	0.2290
2	0.1816	0.2099	0.2239	0.2554
3	0.2213	0.2652	0.2722	0.2923
4	0.2598	0.3011	0.3141	0.3307
5	0.2910	0.3299	0.3502	0.3671

while solutions for the tranches AA and AAA coincide. This observation has an intuitive interpretation. Simplified objective function (4.7) in Problem 2 expresses the bank payment if the given tranche does not default. For senior tranches AA and AAA, the default probability is relatively small (less than 0.5%). It is therefore quite reasonable that solutions of Problems 1a and 2 for tranches AA and AAA coincide. This observation justifies the proposed simplified objective (3.30).

Case 3

Data, codes and calculation results for the BBB tranche with $l = 0.91\%$ can be downloaded from the AORDA website.[5] Finally, we simultaneously optimized the CDO portfolio and attachment/detachment points with simplified objective (3.30). For each tranche $m = 2, \dots, M$, we optimized the following problem.

Optimization Problem 3 (corresponds to Problem B with rating constraint and simplified objective)

Minimize the present value of size of tranche m:

$$\min_{\{x_m^t, t=1,\dots,5; y_1,\dots,y_{53}\}} \sum_{t=1}^{T} \frac{1}{(1+r)^{t-0.5}} x_m^t \tag{4.10}$$

subject to income spread payments constraint:

$$\sum_{i=1}^{53} c_i y_i \geqslant l \tag{4.11}$$

rating constraints:

$$p_m^5(x_m^1, \dots, x_m^5, y_1, \dots, y_{53}) \leqslant p_m^{\text{rating}}, \quad m = 2, \dots, M \tag{4.12}$$

budget constraint:

$$\sum_{i=1}^{53} y_i = 1 \tag{4.13}$$

box constraints for weights:

$$0 \leqslant y_i \leqslant u, \quad i = 1, \dots, 53 \tag{4.14}$$

box constraints for attachment points:

$$0 \leqslant x_m^t \leqslant 1, \quad t = 1, \dots, 5 \tag{4.15}$$

The upper bound u was set to $u = 2.5\%$. We considered two different income spread constraints: $l = 0.93\%$ and $l = 0.97\%$. Table 2 on the facing page shows the results. There is a significant difference between the solutions with different income spread payment constraints.

An "efficient frontier" can be created by varying the spread payment constraint l. Then the parameter l can be selected by maximizing the difference between incoming spread payments l and outcoming spread payments described by the objectives of optimization problems. PSG metacode for Optimization Problem 3 can be found in Appendix A.

[5] See www.aorda.com/aod/casestudy/CS_Structuring_Step-up_CDO_Optimization_II/problem_example_4_case_1__BBB_91/.

TABLE 2 Optimization Problem 3: optimal attachment points of a five-period CDO contract, for $u = 2.5\%$, $l = 0.93\%$ and 0.97%.

(a) $l = 0.93$

Period	Tranche rating BBB	A	AA	AAA
1	0.1184	0.1388	0.1398	0.1460
2	0.1509	0.1722	0.1829	0.1925
3	0.1896	0.2146	0.2299	0.2445
4	0.2221	0.2545	0.2665	0.2865
5	0.2581	0.2870	0.2938	0.3238

(b) $l = 0.97$

Period	Tranche rating BBB	A	AA	AAA
1	0.1616	0.1916	0.2000	0.1989
2	0.1991	0.2416	0.2500	0.2489
3	0.2491	0.2783	0.3000	0.3229
4	0.2866	0.3283	0.3416	0.3729
5	0.3241	0.3616	0.3666	0.4229

5 CONCLUSION

This paper studies an optimization framework for structuring CDOs. Three optimization models were developed from a bank originator perspective. With the first model we optimized only attachment/detachment points in a CDO: the goal was to minimize payments for the credit risk protection (premium leg), while maintaining the credit rating of tranches. In this case the pool of CDSs and income spreads was fixed. With the second model we bounded from below the total income spread payments and optimized the set of CDSs in a CDO pool and the attachment/detachment points. With the third model, we minimized the difference between the total outcome and income spreads. We simultaneously optimized the set of CDSs in a CDO pool and the attachment/detachment points, while maintaining specific credit rating of tranches.

Section 4 presented numerical results for Problem A_m and Problem B. We compared results for the step-up CDO (Problem 1a) and the standard CDO (Problem 1b). The difference between the expected payments for tranches was around 25–35%. We also investigated a step-up CDO with original and simplified objectives (Problem 2). We observed that there is only a slight difference between optimal points corresponding to the step-up CDO with the original objective (Problem 1a) and simplified objective (Problem 2). Finally, we simultaneously optimized the CDO pool and attachment/detachment points for two different income spread payment constraints (Problem 3). There was a significant difference in solutions with different income spread payment constraints.

APPENDIX A. PORTFOLIO SAFEGUARD EXAMPLE CODE

This appendix presents the PSG metacode for solving Optimization Problem 3 (see formulas (4.10)–(4.15)). Metacode, data and solutions can be downloaded online.[6]

A.1 Metacode for Optimization Problem 3

(1) Problem: problem_example_4_case_1__BBB_91, type = **minimize**

(2) **Objective:** objective_linear_sum_of_x_5, linearize = 0

(3) **linear**_sum_of_x_5 **(matrix_sum_of_x_5)**

(4) **Constraint:** constraint_mult_prob_BBB, **upper_bound = 0.0281**

(5) **prmulti_pen**_5_periods **(0,matrix_4_1,matrix_4_2,matrix_4_3, matrix_4_4,matrix_4_5)**

(6) **Constraint:** constraint_spread, **lower_bound = 91**, linearize = 0

(7) **linear**_spread **(matrix_spread)**

(8) **Constraint:** constraint_budget, lower_bound = 1, upper_bound = 1, linearize = 0

(9) **linear**_budget **(matrix_budget)**

(10) **Box_of_Variables:** lowerbounds = point_lowerbounds, upperbounds = point_upperbounds

(11) Solver: VAN, precision = 2, stages = 6

Here we give a brief description of the presented metacode. The important parts of the code are shown in bold. The keyword “minimize” tells a solver that Problem 3 is a minimization problem. To define an objective function the keyword “Objective” is used. The linear objective function (4.10) that is the present value of the expected size

[6] See Problem 6, Data set 2 at www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/structuring-step-up-cdo/.

of a tranche is defined in lines (2) and (3) with the keyword "linear" and the data matrix located in the file matrix_sum_of_x_5.txt. Each constraint starts from the keyword "Constraint". The rating constraint (4.12) is defined in lines (4) and (5). In PSG, the keyword "prmulti_pen" denotes the probability that a system of linear equations with random coefficients is satisfied (see the mathematical definition of function "prmulti_pen" in the document[7]). The random coefficients for four linear inequalities are given by four matrices of scenarios in the files matrix_4_1.txt,…,matrix_4_4.txt. The income spread payment constraint (4.11) and budget constraint (4.13) are defined with linear functions in lines (6) and (7), and (7) and (8), respectively, similar to the objective function.

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[7] See www.ise.ufl.edu/uryasev/files/2011/12/definitions_of_functions.pdf, page 104.