CASE STUDY: Portfolio Optimization with Probabilistic Constraint and Fixed and Proportional Transaction Costs (pr_pen, cardn_pos, avg)

Background

This case study considers a probabilistic portfolio optimization problem with the variance objective function, probability constraint, expected returns constraint and a budget constraint with proportional linear transaction cost and the fixed transaction cost upon the total dollar value of the bought/sold assets. The standard stochastic programming problem (SPM) with fixed transaction cost is mixed integer problem that involves Boolean variables. This case study uses cardinality function for assigning fixed transaction cost for instruments with non-zero positions (non-zero positions are defined by exceeding some small threshold). With cardinality functions we do not need to introduce additional discrete variables (but cardinality function is not convex, and numerically the problem has the same complexity as problem with integer variables). There are two slightly different problem statements in this case study: 1) Problem 1, where probability constraint does not include transaction costs, 2) Problem 2 with transaction costs included in the probability constraint.

The considered problems are similar to the problem discussed in Bonami and Lejeune (2009). However, the problem statement in Bonami and Lejeune (2009) does not include constraint on expected return. This may lead to very risky portfolio which may "blow up" under certain market conditions. For instance, the "selling naked short options" strategy satisfies the probabilistic constraint requiring that majority of time moments are profitable; but the strategy loses all accumulated value, when market sharply goes down.

120 scenarios are built for each asset. Scenarioa are monthly returns of securities. The problem is solved for datasets containing 100, 200, 500, 1000 and 2000 assets

For more details on portfolio optimization problem with transactions costs see, for instance Lobo et al. (2007). For more detail on optimization problems with stochastic and integer constraints see, for instance, Bonami and Lejeune (2009).

References

- Bonami, P., Lejeune, M.A. (2009): *Portfolio Optimization Problems Under Stochastic and Integer Constraints*, Operations Research 57(3), pp. 650–670.
- Lobo, M., Fazel, M. and Boyd, S. (2007): *Portfolio optimization with linear and fixed transaction costs*, Annals of Operations Research 152(1), pp. 341-365.

Notation

I= number of instruments;

J=number of scenarios;

$\vec{v} = (v_1, \dots .. v_I)$ = decision vector defining positions in instruments;

u = variable representing the upper bound for the number of instrument included in the portfolio;

θ_{ij} = rate of return of the *i*-th instrument under scenario *j*;

$\vec{\theta} = (\theta_1, \dots, \theta_I)$ = random vector of rates of returns of instruments, *i=1,…,I*;

$\overrightarrow{\theta_J} = (\theta_{1j}, \dots, \theta_{Ij})$ = vector of rates of returns of instruments, *i=1,…,I*, under scenarios *j*;

$L(\vec{v}, \overrightarrow{\theta_J}) = -\sum_{i=1}^{I} \theta_{ij} v_i$ = Loss under scenario *j*;

$\vec{h} = (h_1, \dots, h_I)$ = vector of upper bounds on positions for asset, *i=1,…,I*;

$\vec{l} = (l_1, \dots, l_I)$ = vector of lower bounds on positions for asset, *i=1,…,I*;

U = lower bound on the expected return of the portfolio;

c = proportional linear transaction cost for instruments in portfolio;

d = value of fixed transaction cost for instruments in portfolio;

α = confidence level for Probability Exceeding Penalty in probability constraint;

R = constant threshold for Probability Exceeding Penalty in probability constraint;

$pr\_pen_R\{L(\vec{v}, \vec{\theta})\}$ = Probability Exceeding Penalty for Loss, defined as follows:

$$pr\_pen_R\{L(\vec{v}, \vec{\theta})\} = \sum_{j=1}^{J} p_j h\,(L(\vec{v}, \vec{\theta}_j), R)\ ,$$

where

$$h(y, R) = \begin{cases} 1, if\ y \geq R \\ 0, otherwise \end{cases} .$$

s = constant (small) threshold for Cardinality Positive function. The threshold identifies the level below which the position is considered to be zero (*s* = 10^{-6} in this case study).

$card\ \_pos(\vec{v}, s)$ = Cardinality Positive function, defined as follows:

$$\text{card } \_pos(\vec{v}, s) = \sum_{j=1}^{J} g\,(v_j, s)$$

where

$$g(y, t) = \begin{cases} 1, if\ y \geq t \\ 0, otherwise \end{cases} .$$

$\text{avg } \_\text{g}\left(L(\vec{v}, \vec{\theta})\right) = \frac{1}{J} \sum_{j=1}^{J} \left(-L(\vec{v}, \vec{\theta}_j)\right)$ = Average Gain as a function of positions in instruments.

The following ***Problem (CS1)*** assumes that transaction costs do not affect the distribution function of losses. The problem statement does not have boolean variables and uses Cardinality Positive function for counting non-zero variables (with precision *s*).

Problem 1 (CS1)

Minimizing variance of estimated return

$$min_v \, variance \left[L(\vec{v}, \vec{\theta})\right] \tag{CS1.1}$$

subject to

Probability constraint

$$pr\_pen_R\{L(\vec{v}, \vec{\theta})\} \leq 1 - \alpha \tag{CS1.2}$$

Budget constraint

$$\sum_{i=1}^{I} v_i + c\sum_{i=1}^{I} v_i + d\, card\,\_\mathrm{pos}(\vec{v}, s) \leq 1 \tag{CS1.3}$$

Expected return constraint

$$\mathrm{avg}\,\_\mathrm{g}\left(L(\vec{v}, \vec{\theta})\right) - c\sum_{i=1}^{I} v_i - d\, card\,\_\mathrm{pos}(\vec{v}, s) \geq \mathrm{U} \tag{CS1.4}$$

Bounds on positions

$$l_i \leq v_i \leq h_i, \;\; i = 1, \dots, I \tag{CS1.5}$$

The following ***Problem 2 (CS2)*** uses additional continuous variable, *u*, which bounds the number of non-zero decision variables (actually, variables exceeding some small threshold *s*). Fixed and proportional transaction costs are included in the probabilistic constraint.

Problem 2 (CS2)

Minimizing variance of estimated return

$$min_v \, variance \left[L(\vec{v}, \vec{\theta})\right] \tag{CS2.1}$$

subject to

Probability constraint

$$pr\_pen_R\{L(\vec{v}\,,\vec{\theta}) + c\sum_{i=1}^{I} v_i + d\,u\} \leq 1-\alpha \tag{CS2.2}$$

Budget constraint

$$\sum_{i=1}^{I} v_i + c\sum_{i=1}^{I} v_i + d\;card\;\_\mathrm{pos}(\vec{v}, s) \leq 1 \tag{CS2.3}$$

Expected return constraint

$$\mathrm{avg}\;\_\mathrm{g}\left(L(\vec{v}\,,\vec{\theta})\right) - c\sum_{i=1}^{I} v_i - d\;card\;\_\mathrm{pos}(\vec{v}, s) \geq \mathrm{U} \tag{CS2.4}$$

Bounds on positions

$$card\;\_\mathrm{pos}(\vec{v}, s) \leq u \tag{CS2.5}$$

$$l_i \leq v_i \leq h_i,\;\; i = 1, \ldots, I \tag{CS2.6}$$

$$u \geq 0 \tag{CS2.7}$$