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Calibrating probability distributions with convex-concave-convex functions: application to CDO pricing

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Abstract This paper considers a class of functions referred to as convex-concave-convex (CCC) functions to calibrate unimodal or multimodal probability distributions. In discrete case, this class of functions can be expressed by a system of linear constraints and incorporated into an optimization problem. We use CCC functions for calibrating a risk-neutral probability distribution of obligors default intensities (hazard rates) in collateral debt obligations (CDO). The optimal distribution is calculated by maximizing the entropy function with no-arbitrage constraints given by bid and ask prices of CDO tranches. Such distribution reflects the views of market participants

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on the future market environments. We provide an explanation of why CCC functions may be applicable for capturing a non-data information about the considered distribution. The numerical experiments conducted on market quotes for the iTraxx index with different maturities and starting dates support our ideas and demonstrate that the proposed approach has stable performance. Distribution generalizations with multiple humps and their applications in credit risk are also discussed.

Keywords OR banking · Convex optimization · Convex-concave-convex probability distribution · Implied copula · CDO pricing

Mathematics Subject Classification 90 (Operations Research, Mathematical Programming)

1 Introduction

The problem of recovering a probability distribution using limited information about the value of interest is considered in a variety of applications. Calibrated distributions provide more informative picture about an underlying parameter than just two commonly used statistical measures: *mean* and *standard deviation*. Although the methodology is very general and can be used in any area, this paper focuses on financial applications. For instance, in financial markets, the evolution of risk-neutral probability distributions around events related to monetary policy actions over time can be used by policy makers to analyze how market participants respond to implemented policies and measure its effectiveness Bahra (1997). The prices of financial derivatives give a broad picture of market expectations as there might be many products associated with a single asset with various terms such as different strike prices and time to maturity. Therefore, the prices of financial derivatives may reflect market views on different parts of probability distributions and can be used for calibration. A substantial body of work has been done on recovering risk-neutral probability distributions of underlying asset prices (Bahra 1997; Bu and Hadri 2007; Jackwerth and Rubinstein 1996; Monteiro et al. 2008) or other uncertain parameters, such as exchange rates (Campa et al. 1998; Malz 1997) from option prices [see Jackwerth (1999) for review].

This paper focuses on methods for estimating such probabilistic distributions. In particular, we propose a new class of probabilistic distributions so-called convex-concave-convex (CCC), which improves the stability of estimation procedures to noise in data. CCC is a wide class of distributions including *normal, log-normal, gamma*, and *F* distributions. By definition, the PDF of a CCC distribution is a convex function from the beginning to some point, then it is concave to some further point, and then it is again convex to the end. For discrete distributions, we describe CCC distributions by a system of linear constraints. The class of CCC distributions is quite general and it can be used in various applications, including those being calibrated from the prices of financial derivatives. We also demonstrate how CCC class can be used to model distributions with multiple humps by allowing distributions to satisfy CCC constraints on different intervals.

The credit risk derivatives market is an area where the efficient techniques for calibrating probability distributions are quite important. High profit margins has led to a growing appeal for Collateralized Debt Obligations (CDOs) or the bespoke-CDO baskets of instruments from many market participants. A CDO is a credit risk derivative, which is based on so-called "credit tranching", where the losses of the portfolio of bonds, loans or other securities are repackaged and traded on the market. The losses are applied to the later classes of debt before earlier ones. A range of products can be created from the underlying pool of instruments, varying from a very risky *equity* debt to a relatively riskless *senior* debt. It allows investors to invest in instruments with almost any risk which satisfies their preferences and the view on creditworthiness of underlying companies. The pricing of CDOs is a difficult quantitative problem faced by credit risk markets. The main issue is the uncertainty about obligors default risk in the corresponding pool of assets and its tranched structure. A large amount of studies has been done by academic researchers and market participants on analysis and development of different CDO pricing models (Andersen and Sidenius 2004; Arnsdorf and Halperin 2007; Burtschell et al. 2005; Dempster et al. 2007; Halperin 2009; Hull and White 2010, 2006; Laurent and Gregory 2003; Nedeljkovic et al. 2010; Rosen and Saunders 2009). This paper uses the implied copula model proposed by Hull and White (Hull and White 2010, 2006) since it requires an efficient calibration techniques for probability distribution function of market states. In the simplest version of the standard implied copula approach the time to default of each obligor is assumed to be an exponential random variable with a hazard rate (same for all obligors) depending on a market state. Using the current CDO prices one can recover the risk-neutral probability distribution of market states which fits no-arbitrage assumptions. A number of recent papers have proposed methods for implying risk-neutral distributions based on observed CDO tranche prices. These include, for example, the implied copula model (Hull and White 2006), and its parametric variant (Hull and White 2010) as well as the approaches based on minimum entropy, including (Meyer-Dautrich and Wagner 2007; Dempster et al. 2007; Nedeljkovic et al. 2010). Since the amount of data for calibrating distribution is usually quite small, the efficient noise-reduction techniques may be very useful.

This paper applies an "entropy" approach to the implied copula model motivated by the results of papers which study the *maximum entropy principle* for asset pricing (Avellaneda 1998; Avellaneda et al. 2001; Meyer-Dautrich and Wagner 2007; Dempster et al. 2007; Nedeljkovic et al. 2010) and the references therein. As it is stated in (Dempster et al. 2007), the minimum entropy principle is well-suited to the estimation of copulas in portfolio credit risk modeling. By maximizing entropy we find the most uncertain distribution consistent with given information and do not assume anything else. The observed prices may not fully reflect the information about the distribution of market states. In other words, other *non-data* information may need to be embedded into the entropy maximization problem such as the shape of distribution, smoothness, bounds, etc. Intuitively, CCC functions may help to incorporate an additional information about the nature of the distribution of market states.

We use Monte Carlo method to simulate the expected tranche payoffs in each market state and identify the optimal probability distribution of market states by maximizing the entropy with no-arbitrage constraints given by bid and ask prices of CDO tranches. In our computational experiments, we compare the model proposed in (Hull and White 2006) and our approach based on CCC distributions. We use December, 2006 iTraxx tranche quotes from (Halperin 2009) containing the bid and ask qoutes as well as more recent data from 2007–2008 years where the market was in an unstable condition. We also demonstrate how our model can be generalized to distributions with multiple humps. The case study is implemented using portfolio safeguard (PSG) package (MATLAB and Run-File Text Environments), see (Portfolio safeguard 2009). The computational experiments show that the proposed approach has stable performance. The MATLAB and Text codes used for conducting numerical experiments are provided[1].

The paper proceeds as follows: Sect. 2 summarizes the implied copula model introduced in (Hull and White 2006). Section 3 describes the usage of maximum entropy principle for calibrating probability distributions. Section 4 proposes the CCC distribution and describes how to calibrate it with the entropy approach. It provides the formal optimization problem statements and a heuristic algorithm for finding probability distribution. Section 5 discusses the case study. Section 6 concludes.

2 Implied copula CDO pricing model: background

This section briefly describes the simplest version of implied copula model proposed in (Hull and White 2006), which we use to test the performance of proposed calibration techniques. More details on fundamentals of CDO pricing models using copulas and implied copulas can be found in (Andersen and Sidenius 2004; Dempster et al. 2007; Hull and White 2010; Laurent and Gregory 2003; Li 2000) and references therein. For the sake of readability, we also omit the formal description of CDO contracts, specifics of tranching, and the details on their payment structure. We only mention that a CDO has K instruments (obligors), T time to maturity, and J tranches, whose net payoffs (difference between expected present value of premium leg payments and default leg payments) are completely determined by the time to default of each obligor and tranche prices (more details on CDO prices given, for example, in Table 1 are presented in the case study section). The default time of each obligor is a non-negative random variable which is driven by two main factors: market state (default environment) and idiosyncratic component, i.e., the default risk due to its own circumstances. The implied copula model has the following assumptions on these components.

Let M be the discrete random variable of possible market states (default environments) with the set of states $\mathcal{I} = \{1, \ldots, I\}$. If $M = i$, then the time to default T_k of obligor k in a given CDO pool of K assets is assumed to be an exponential random variable with hazard rate λ_i. In other words, in each possible market state $i \in \mathcal{I}$ all obligors have the same probabilities of default by time t equal to

$$P\{T_k \leq t | M = i\} = 1 - e^{-\lambda_i t}, \forall k \in \{1, \ldots, K\}. \tag{1}$$

Normally, the hazard rates λ_i, $i = 1, \ldots, I$ are assumed to be defined and fixed, and chosen in such a way that if $i_1 < i_2 \in \mathcal{I}$, then $\lambda_{i_1} < \lambda_{i_2}$, i.e, the larger the realized value of market state M, the more severe is the credit environment for obligors (larger probability to default earlier). Assume that each market state i has probability p_i to occur:

[1] http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/cs_calibration_copula/.

Table 1 Market quotes for 5, 7, 10-year iTraxx on 20 December, 2006 obtained from Halperin (2009)

Maturity	Low stike (%)	High strike (%)	Bid	Ask
20-Dec-11	0	3	11.75 %	12.00 %
20-Dec-11	3	6	53.75	55.25
20-Dec-11	6	9	14.00	15.50
20-Dec-11	9	12	5.75	6.75
20-Dec-11	12	22	2.13	2.88
20-Dec-11	22	100	0.80	1.30
20-Dec-11	0	100	24.75	25.25
20-Dec-13	0	3	26.88%	27.13%
20-Dec-13	3	6	130.00	132.00
20-Dec-13	6	9	36.75	38.25
20-Dec-13	9	12	16.25	18.00
20-Dec-13	12	22	5.50	6.50
20-Dec-13	22	100	2.40	2.90
20-Dec-13	0	100	33.50	34.50
20-Dec-16	0	3	41.88%	42.13%
20-Dec-16	3	6	348.00	353.00
20-Dec-16	6	9	93.00	95.00
20-Dec-16	9	12	40.00	42.00
20-Dec-16	12	22	13.25	14.25
20-Dec-16	22	100	4.35	4.85
20-Dec-16	0	100	44.50	45.50

Quotes for the 0 to 3 % tranche are the percent of the principal that must be paid up front in addition to 500 basis points per year. Quotes for other tranches are in basis points

$$P\{M = i\} = p_i, \quad \forall i \in \mathcal{I}.$$

Let a_{ij} be the expected net payments (what you expected to pay minus what you expected to get paid) of tranche j in the market state i. Since the time to default of each obligor is completely defined in each market state, then the expected net payments a_{ij} can be calculated using current CDO prices. In the risk-neutral world, under *no-arbitrage* assumptions, the expected net payoff of each tranche should be zero. Therefore,

$$\sum_{i=1}^{I} a_{ij} p_i = 0, \quad j = 1, \ldots, J. \tag{2}$$

Since the tranche prices are usually given in bid and ask quotes, we use $\underline{a}_{ij}$ and $\overline{a}_{ij}$ to denote the expected net payments of tranche j in the market state i for bid and ask quotes respectively. Then, the *no-arbitrage* constraints are

$$\sum_{i=1}^{I} \underline{a}_{ij} p_i \leq 0, \quad j = 1, \ldots, J, \tag{3}$$

$$\sum_{i=1}^{I} \overline{a}_{ij} p_i \geq 0, \quad j = 1, \ldots, J. \tag{4}$$

Indeed, any violation of either (3) or (4) creates an arbitrage opportunity of buying or short-selling the corresponding tranche for the expected net profit. There might be an infinite number of probability distributions $(p_1, p_2, \ldots, p_I)$ satisfying no-arbitrage constraints (2) or (3), (4) or no distributions at all. Many researchers and market participants tried to consider different criteria for choosing the "best" distributions, or trying to find parametric distributions satisfying the aforementioned constraints. Intuitively, the distribution should be quite smooth since it corresponds to movement from a "good" market state (λ is low) to a "bad" market state (λ is high). Therefore, high variations in the probability distribution, as the market state slightly changes, may seem counterintuitive. Moreover, since the only information used by the model is the current CDO prices, then the model should be insensitive to the considered number of market states or other regularization coefficients. Therefore, some of the desirable distribution properties are "smoothness", lack of noise, and robustness to changes to model parameters, such as the number of market states and regularization coefficients.

One of the earlier work (Hull and White 2006) proposes solving the following optimization problem to find the "best" probability distribution:

Problem A

$$\min_p (D(p) + S(p))$$

subject to

probability distribution constraints

$$\sum_{i=1}^{I} p_i = 1, \tag{5}$$

$$p_i \geq 0, \quad i = 1, \ldots, I. \tag{6}$$

where $D(p)$ is a deviation term

$$D(p) = \sum_{j=1}^{J} \left(\sum_{i=1}^{I} p_i a_{ij} \right)^2, \tag{7}$$

and $S(p)$ is a smoothing term

$$S(p) = c \sum_{i=2}^{I-1} \left[\frac{p_{i+1} + p_{i-1} - 2p_i}{0.5(\lambda_{i+1} - \lambda_{i-1})} \right]^2 . \tag{8}$$

The deviation term $D(p)$ penalizes deviations from zero of the net expected payoff of every tranche. The smoothing term $S(p)$ imposes the penalty for every three consecutive points on the distribution not laying on the same line. The coefficient c is a regularization coefficient, which has to be chosen by trial and error. Although this model is rarely used in the literature and usually for comparison reasons with other approaches; moreover, the authors of this method developed other techniques [see, for instance, Hull and White (2010)] to find probability distributions, we use this problem formulation for demonstration purposes only. Specifically, we show that using our methodology we might be able to avoid the typical flaws of Problem A such as the sensitivity of optimal solution to smoothing term coefficient c and the the number of considered market states I.

Remark 1 Although the implied copula model received criticism in the literature, we want to point out that it is quite flexible. Specifically, for any particular market state i, a market participant may assign different hazard rates of obligors depending on his view on the creditworthiness of each company. Moreover, he can use its own idiosyncratic default random variables for each specific obligor, which also may be time-dependent. Thus, as long as the expected net cashflows can be calculated for each obligor in the pool of assets in each market state, the model still applies. Such flexibility may allow to mark-to-market other baskets of similar products, and extract more information from the CDO prices and what the changes of prices may reflect.

3 Maximum entropy principle for recovering probability distributions

As mentioned in the previous section, there might be an infinite number of probability distributions $(p_1, \ldots, p_I)$ satisfying no-arbitrage constraints (2) for exact tranche quotes or (3), (4) for bid and ask quotes. Besides the aforementioned criterion which chooses the distribution with the minimized sum of squared deviations of tranche payoffs from "perfect fit" (7) and smoothing term (8), other possible approaches have been proposed in the literature, see for instance (Bahra 1997; Jackwerth 1999; Monteiro et al. 2008).

This paper considers *entropy*, a measure of uncertainty of random variable. In discrete case, the entropy is formally defined as follows.

Definition 1 The *entropy* $H(p)$ of discrete probability distribution vector $p = (p_1, \ldots, p_I)$ is defined by

$$H(p) = -\sum_{i=1}^{I} p_i \ln p_i \tag{9}$$

To obtain the most unbiased probability distribution, i.e., the most uncertain distribution among those satisfying the available information, one would choose the distribution with maximum entropy. This logic lies behind the *maximum entropy principle*. The *Maximum Entropy Principle* [first introduced by Shannon Shannon (1948)] is popular in information theory. This principle is actively used in financial applications for recovering probability distributions; see for instance (Avellaneda 1998; Avellaneda et al. 2001; Golan 2002; Meyer-Dautrich and Wagner 2007; Miller and Liu 2002; Dempster et al. 2007; Nedeljkovic et al. 2010). The essence of the maximum entropy principle is that, with a given *data* and *non-data* information about the distribution (specified through equations and constraints), we maximize the entropy which selects the most "uncertain" distribution. Therefore, we find the most "unbiased" distribution given the available information about the distribution.

To find the most uncertain distribution of market states in the aforementioned implied copula model we use the maximum entropy principle with no-arbitrage constraints. In other words, the following optimization problem needs to be solved:

Problem B

$$\min_{p} \ -H(p)$$

subject to

no-arbitrage constraints

$$\sum_{i=1}^{I} \underline{a}_{ij} p_i \leq 0, \, j = 1, \ldots, J, \tag{10}$$

$$\sum_{i=1}^{I} \overline{a}_{ij} p_i \geq 0, \, j = 1, \ldots, J, \tag{11}$$

probability distribution constraints

$$\sum_{i=1}^{I} p_i = 1, \tag{12}$$

$$p_i \geq 0, \quad i = 1, \ldots, I. \tag{13}$$

Note that Problem B contains only *data* information about the probability distribution of market states provided by CDO prices and no-arbitrage assumptions. But the observed prices may not fully reflect the information about the distribution of market states. In other words, other *non-data* information may be incorporated into the entropy maximization problem such as the shape of distribution, smoothness, bounds, etc. For example, one may reasonably assume that the probability distribution should be unimodal or two-modal. Although, the explanations of such assumptions might be arguable, the results provided by the corresponding models may still be worth to consider as they reflect the effect of adding various non-data information.

Problem B is an optimization problem with convex objective and linear constraints; thus, it can be solved to optimality using standard optimization solvers. In order to be able to solve entropy maximization problem with extra *non-data* information, it is desirable that this information can be incorporated in terms of linear constraints. Next section introduces a class of functions (CCC functions) which can be embedded as linear constraints and also may help to incorporate an additional information about the nature of the distribution of market states in the corresponding optimization problem.

4 Modeling default probabilities with CCC distributions

In this section, we introduce a class of convex-concave-convex (CCC) distributions and demonstrate its application to the modeling of probability distributions of market states (hazard rates) with single and multiple humps. CCC is a wide class of distributions including normal, log-normal, gamma, and F distributions. Below we give several definitions that specify the class of CCC distributions. For the sake of readability, the definitions and further discussion are made for unimodal CCC distributions, meaning that they contain one hump. Generalization to multiple humps is straightforward: one needs to combine unimodal CCC functions on separate intervals.

Definition 2 (*convex-concave-convex Function*) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We call function $f(x)$ convex-concave-convex (CCC) if only if there exist $w_l, w_r \in \mathbb{R}$ such that $w_l \leq w_r$ and the following inequalities hold:

1. **Convexity on** $(-\infty, w_l) \cap X$: $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in (-\infty, w_l) \cap X$ and all $\lambda \in [0, 1]$;
2. **Concavity on** $[w_l, w_r] \cap X$: $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in [w_l, w_r) \cap X$ and all $\lambda \in [0, 1]$;
3. **Convexity on** $(w_r, +\infty) \cap X$: $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in (w_r, +\infty) \cap X$ and all $\lambda \in [0, 1]$.

Figure 1 shows an example of CCC function, which is convex function from the beginning to the point w_l, then it is concave to the point w_r, and then it is again convex to the end. The class of continuous CCC distributions can be specified in terms of Definition 2.

Definition 3 (*CCC distribution in continuous case*) A continuous random variable with probability density function $f : X \to \mathbb{R}$ belongs to the CCC class of continuous distributions if function $f(x)$ is CCC function.

When the function $f : X \to \mathbb{R}$ is defined on the finite set X, it is convenient to express definition of CCC function only in term of points of the set X (other than λ).

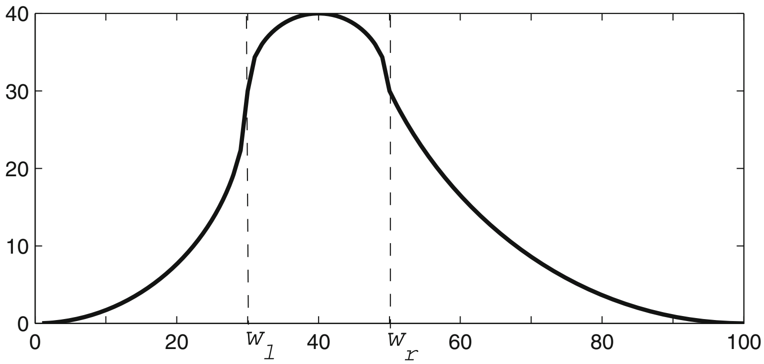

Fig. 1 Example of a CCC function, the first inflection point $w_l = 30$ and the second inflection point $w_r = 50$

We provide an equivalent alternative definition of CCC function with such a property and show its equivalence to Definition 2.

Definition 4 (*convex-concave-convex function*) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We call function $f(x)$ convex-concave-convex (CCC) if only if there exist $w_l, w_r \in \mathbb{R}$ such that $w_l \le w_r$ and the following inequalities hold:

1. **Convexity on** $(-\infty, w_l) \cap X$: $(x_2 - x_1) f(x_3) \le (x_2 - x_3) f(x_1) + (x_3 - x_1) f(x_2)$ for all $x_1, x_2, x_3 \in (w_r, +\infty) \cap X$ such that $x_1 \le x_2 \le x_3$ for all $x_1, x_2, x_3 \in (-\infty, w_l) \cap X$ such that $x_1 \le x_2 \le x_3$;
2. **Concavity on** $[w_l, w_r] \cap X$: $(x_2 - x_1) f(x_3) \ge (x_2 - x_3) f(x_1) + (x_3 - x_1) f(x_2)$ for all $x_1, x_2, x_3 \in [w_l, w_r) \cap X$ such that $x_1 \le x_2 \le x_3$;
3. **Convexity on** $(w_r, +\infty) \cap X$: $(x_2 - x_1) f(x_3) \le (x_2 - x_3) f(x_1) + (x_3 - x_1) f(x_2)$ for all $x_1, x_2, x_3 \in (w_r, +\infty) \cap X$ such that $x_1 \le x_2 \le x_3$.

Definitions 2 and 4 are equivalent. Indeed, if $x_1 = x_2$, then both definitions are the same. If $x_2 \ne x_1$, then there is a one-to-one correspondence between parameters λ (in Definition 2) and x_3 (in Definition 4) given by an equation $x_3 = \lambda x_1 + (1 - \lambda) x_2$ which establishes the equivalence of formulas in both definitions. A discrete class of CCC distributions with a finite number of atoms can specified in terms of Definition 4.

Definition 5 (*CCC distribution in discrete case*) Let $X = \{d_1, \ldots, d_I\} \subset \mathbb{R}$ be a finite set such that $d_1 < d_2 < \ldots < d_I$ and $f : X \to [0, 1]$ be a probability measure function i.e. $\sum_{i=1}^{I} f(d_i) = 1$. Then probability measure f belongs to CCC class of discrete distributions if the function f satisfies Definition 4 of CCC function.

Observe that in case of finite set $X = \{d_1, \ldots, d_I\} \subset \mathbb{R}$, the probability measure function $f : X \to \mathbb{R}$ immediately satisfies Definition 4 if conditions 1–3 of Definition 4 hold only for every three consecutive points d_{i-1}, d_i, d_{i+1} ($i = 2, \ldots, I - 1$). This observation can be summarized in Proposition 1.

Proposition 1 (CCC distribution in discrete case) *Let* $X = \{d_1, \ldots, d_I\} \subset \mathbb{R}$ *be a finite set such that* $d_1 < d_2 < \ldots < d_I$ *and* $f : X \to [0, 1]$ *be a probability measure function i.e.* $\sum_{i=1}^{I} f(d_i) = 1$. *Then* f *belongs to CCC class of discrete distributions if there exist indices* $1 \le w_l, w_r \le I$ *such that* f *satisfies the following inequalities:*

1. ***Convexity on*** $(-\infty, d_{w_l}) \cap X$: $(d_{i+1} - d_i) f(d_{i-1}) + (d_i - d_{i-1}) f(d_{i+1}) \ge (d_{i-1} - d_{i+1}) f(d_i)$, *for all* i: $1 < i < w_l$;
2. ***Concavity on*** $[d_{w_l}, d_{w_r}] \cap X$: $(d_{i+1} - d_i) f(d_{i-1}) + (d_i - d_{i-1}) f(d_{i+1}) \le (d_{i-1} - d_{i+1}) f(d_i)$, *for all* i: $w_l < i < w_r$;
3. ***Convexity on*** $(d_{w_r}, +\infty) \cap X$: $(d_{i+1} - d_i) f(d_{i-1}) + (d_i - d_{i-1}) f(d_{i+1}) \ge (d_{i-1} - d_{i+1}) f(d_i)$, *for all* i: $w_r < i < I$.

The proof of the Proposition 1 is straightforward. Further, we assume that the distance between every two consecutive points d_i, d_{i+1} for $i = 1, \ldots, I - 1$ is the same. In this case, Proposition 1 simplifies to Corollary 1:

Corollary 1 *Let* $X = \{d_1, \ldots, d_I\} \subset \mathbb{R}$ *be a finite set* $d_1 < d_2 < \ldots < d_I$ *such that the distance between every two consecutive points* d_i, d_{i+1} *for* $i = 1, \ldots, I - 1$ *is the* *same and* $f : X \to [0, 1]$ *be a probability measure function i.e.* $\sum_{i=1}^{I} f(d_i) = 1$. *Then probability measure* f *belongs to CCC class of discrete distributions if there exist indices* $1 \leq w_l, w_r \leq I$ *such that function* f *satisfies the following inequality conditions:*

1. ***Convexity on*** $(-\infty, d_{w_l}) \cap X$*:* $f(d_{i-1}) + f(d_{i+1}) \geq 2f(d_i)$, *for all* $i : 1 < i < w_l$;
2. ***Concavity on*** $[d_{w_l}, d_{w_r}] \cap X$*:* $f(d_{i-1}) + f(d_{i+1}) \leq 2f(d_i)$, *for all* $i: w_l < i < w_r$*;*
3. ***Convexity on*** $(d_{w_r}, +\infty) \cap X$*:* $f(d_{i-1}) + f(d_{i+1}) \geq 2f(d_i)$, *for all* $i: w_r < i < I$.

The class of discrete CCC distributions introduced in this section can be applied to the modeling of default probabilities $p_1, \ldots, p_I$ corresponding to default intensities $\lambda_1, \ldots, \lambda_I$. This can be done by assuming that a probability measure function f defined on the finite set $X = \{\lambda_1, \ldots, \lambda_I\}$ $(\lambda_1 < \lambda_2 < \ldots < \lambda_I)$ as $f(\lambda_i) = p_i$ for $i = 1, \ldots, I$, belongs to CCC class of discrete distributions specified by Definition 5. This implies that probabilities $p_1, \ldots, p_I$ should satisfy inequalities 1–3 in Proposition 1. For simplicity, we assume that the distance between every two consecutive default intensities $\lambda_1, \ldots, \lambda_I$ is the same. Then there exist indices $1 \leq w_l, w_r \leq I$ such that default probabilities $p_1, \ldots, p_I$ should satisfy linear inequalities:

Convexity of the left slope:

$$\frac{p_{i-1} + p_{i+1}}{2} \geq p_i,\ i = 2, ..., w_l - 1\ , \tag{14}$$

Concavity of the hump:

$$\frac{p_{i-1} + p_{i+1}}{2} \leq p_i,\ i = w_l + 1, ..., w_r - 1\ , \tag{15}$$

Convexity of the right slope:

$$\frac{p_{i-1} + p_{i+1}}{2} \geq p_i,\ i = w_r + 1, ..., I - 1\ . \tag{16}$$

Inequalities (14)–(16) can be incorporated into an optimization problem as additional linear constraints (further referred to as *CCC constraints*) assuring that distribution of default intensities is found in the class of CCC distributions. By adding the CCC constraints to Problem B with points w_l, w_r as additional variables, we obtain the following optimization problem.

Problem C

$$\min_{w_l, w_r, p} \ -H(p)$$
$$\textit{subject to} \tag{17}$$

no-arbitrage constraints

$$\sum_{i=1}^{I} \underline{a}_{ij} p_i \leq 0 , \tag{18}$$

$$\sum_{i=1}^{I} \overline{a}_{ij} p_i \geq 0 , \tag{19}$$

probability distribution constraints

$$\sum_{i=1}^{I} p_i = 1 , \tag{20}$$

$$p_i \geq 0, \quad i = 1, \ldots, I . \tag{21}$$

CCC constraints:
constraint on inflection points

$$w_l \leq w_r, w_l \in \{1, \ldots, I\}, \;\; w_r \in \{1, \ldots, I\}, \tag{22}$$

convexity of the left slope

$$\frac{p_{i-1} + p_{i+1}}{2} \geq p_i, \; i = 2, ..., w_l - 1 , \tag{23}$$

concavity of the hump

$$\frac{p_{i-1} + p_{i+1}}{2} \leq p_i, \; i = w_l + 1, ..., w_r - 1 , \tag{24}$$

convexity of the right slope

$$\frac{p_{i-1} + p_{i+1}}{2} \geq p_i, \; i = w_r + 1, ..., I - 1 , \tag{25}$$

The formulation of Problem C cannot be implemented using standard solvers since the CCC constraints are dependent on variables w_l, w_r. For this reason, for any pair w_l, w_r we denote Problem $C(w_l, w_r)$ as Problem C with fixed values w_l, w_r. To obtain the solution of Problem C, we can solve Problem $C(w_l, w_r)$ for all possible pairs of integers w_l, w_r such that $1 \leq w_l \leq w_r \leq I$, and then choose the solution with maximum entropy among all these solutions. The total number of subproblems (Problem $C(w_l, w_r)$) which need to be solved is $\Theta(I^2)$. If I is relatively small, then Problem C can be solved using this procedure in a reasonable amount of time. For larger values of I, the solution may not be obtained in a reasonable time as the number of subproblems increases quite fast. For that reason, we provide a heuristic algorithm for solving Problem C. It is based on solving Problem B first and then solving a sequence of Problem C(w_l, w_r) for different pairs of (w_l, w_r).

Here is the formal description of the proposed heuristic algorithm. Explanations are provided after the formal description.

Algorithm:
Step 0. Initial optimal solution.

- Solve Problem B and denote its solution obtained for optimization problem by p^*.
- Initialize $w_l = w_r = argmax\{p_i^* : i = 1, \ldots, I\}$[2], $k = 0$, $H_0 = \infty$.

Step 1. Solve Problem C(w_l, w_r)

- Set $k = k + 1$, $exit\_flag = 0$.
- Solve Problem C(w_l, w_r) and obtain the optimal solution p_k^* and $H_k = H(p_k^*)$.

Step 2. Shifting w_r ***to the right***

- If $w_r < I$ and $H_k \leq H_{k-1}$, then set $w_r = w_r + 1$, $exit\_flag = 1$, and go to Step 1.

Step 3. Initialization of shifting w_l ***to the left***

- If $w_l > 1$, then set $w_l = w_l - 1$.
- If $w_l = 1$, then stop the algorithm, and p_{k-1}^* is an approximation of the optimal solution.

Step 4. Solve Problem C(w_l, w_r) ***(the same as Step 1)***

- Set $k = k + 1$.
- Solve Problem C(w_l, w_r) and obtain the optimal solution p_k^* and $H_k = H(p_k^*)$.

Step 5. Shifting w_l ***to the left***

- If $w_l > 1$ and $H_k \leq H_{k-1}$ then set $w_l = w_l - 1$, $exit\_flag = 1$,and go to Step 4.
- If $exit\_flag = 1$, then go to Step 1.
- If ($w_l = 1$ or $H_k > H_{k-1}$) and $exit\_flag = 0$, then stop the algorithm, and p_{k-1}^* is an approximation of the optimal point.

The idea of this algorithm is that we step-by-step change inflection points w_l, w_r and solve Problem C(w_l, w_r). In Step 0, we solve Problem B and obtain an optimal solution p^*. Then, we set $w_l = w_r = argmax\{p_i^* : i = 1, \ldots, I\}$. In other words, we find the maximum component of optimal vector p^* and make w_l, w_r equal to its index. In Step 1, we solve Problem C(w_l, w_r) with these w_l, w_r and obtain the optimal point and its objective value. Then, we shift w_r to the right, if it is possible, making $w_r = w_r + 1$. After that we go to Step 1 and again solve Problem C(w_l, w_r) to obtain the optimal point and its objective value. Then we compare this objective value with the previous one obtained in Step 1 (H_k and H_{k-1}). The procedure stops when the new objective value is greater then the previous one ($H_k > H_{k-1}$), or $w_r = I$. In

[2] If the maximum is not unique, the algorithm should be performed for eash point in the set $argmax\{p_i^* : i = 1, \ldots, I\}$, and then the solution with the smallest objective value should be chosen.

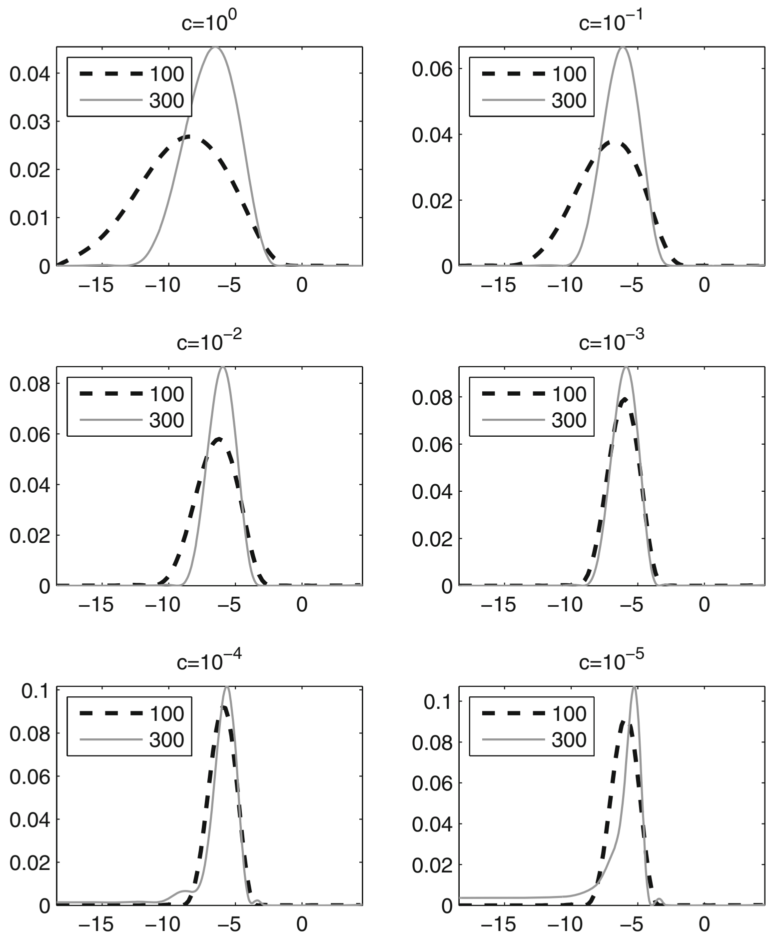

Fig. 2 Distributions of the collateral hazard rate implied by 5-year iTraxx tranche spreads obtained by solving Problem A for 100 and 300 decision variables, and different smoothing term coefficients *c*

Steps 3–5 we run the same procedure, but now we shift w_l to the left. The procedure also stops when the new objective value is larger then the previous one ($H_k > H_{k-1}$), or $w_l = 1$. If during the steps 1 through 4, the smaller objective value is found by shifting w_r or w_l, then these steps should be performed again. In other words, we shift the points w_r and w_l to reach local optimality. Finally, the algorithm returns p^*_{k-1}, which is an approximation of the optimal point. We do not prove that this algorithm provides an optimal solution to Problem C. The case study shows that this algorithm provides reasonable solutions and works quite fast.

Remark 2 Although Fig. 1 illustrates a unimodal CCC function, not all CCC functions have unimodal structure. Intuitively, convexity and concavity constraints can be viewed as the bounds which help to regularize "smoothness" of the calibrated probability distribution, but they cannot guarantee certain increasing or decreasing intervals assumed by unimodality. To ensure unimodality (or multi-

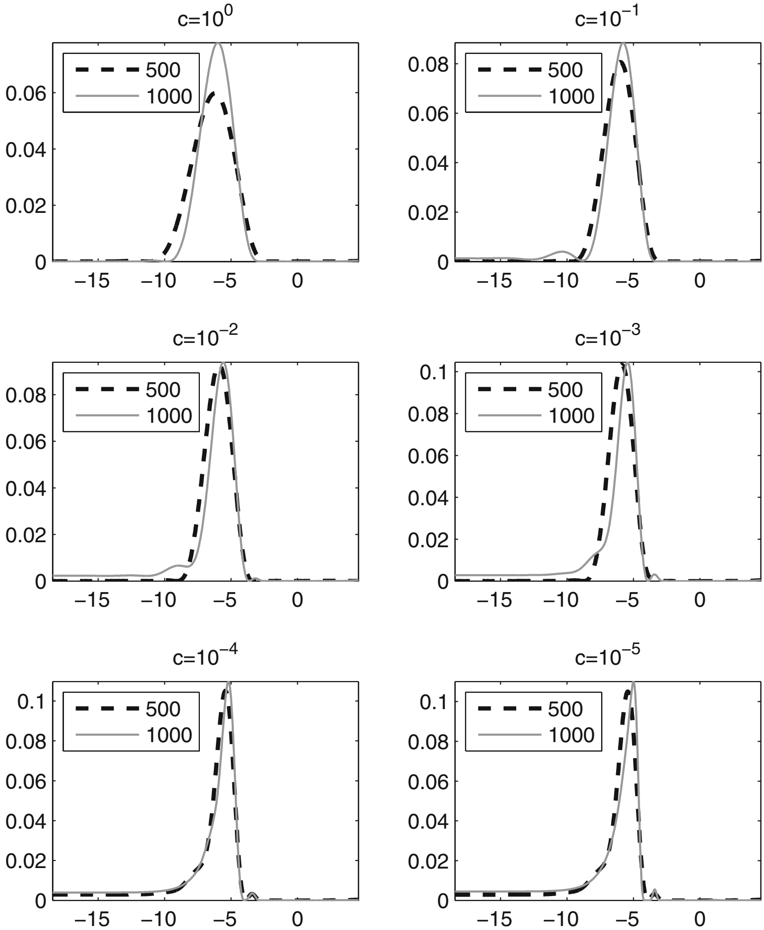

Fig. 3 Distributions of the collateral hazard rate implied by 5-year iTraxx tranche spreads obtained by solving Problem A for 500 and 1,000 decision variables, and different smoothing term coefficients c

modality) one can easily incorporate extra constraints on probability distribution into the proposed optimization problems. For example, if the probability distribution $(p_1, \ldots, p_I)$ needs to be nondecreasing (nonincreasing) from points s to t, the extra constraints would be $p_i \le p_{i+1}, (p_i \ge p_{i+1}), \forall i = s, \ldots,$ t-1. In our computational experiments all CCC distributions have unimodal (two modal) structure, so we do not need to incorporate extra constraints to ensure unimodality.

5 Case study

We implement the proposed methodology using real-life data, solve the corresponding optimization problems (Problem A, Problem B, Problem C), illustrate and discuss

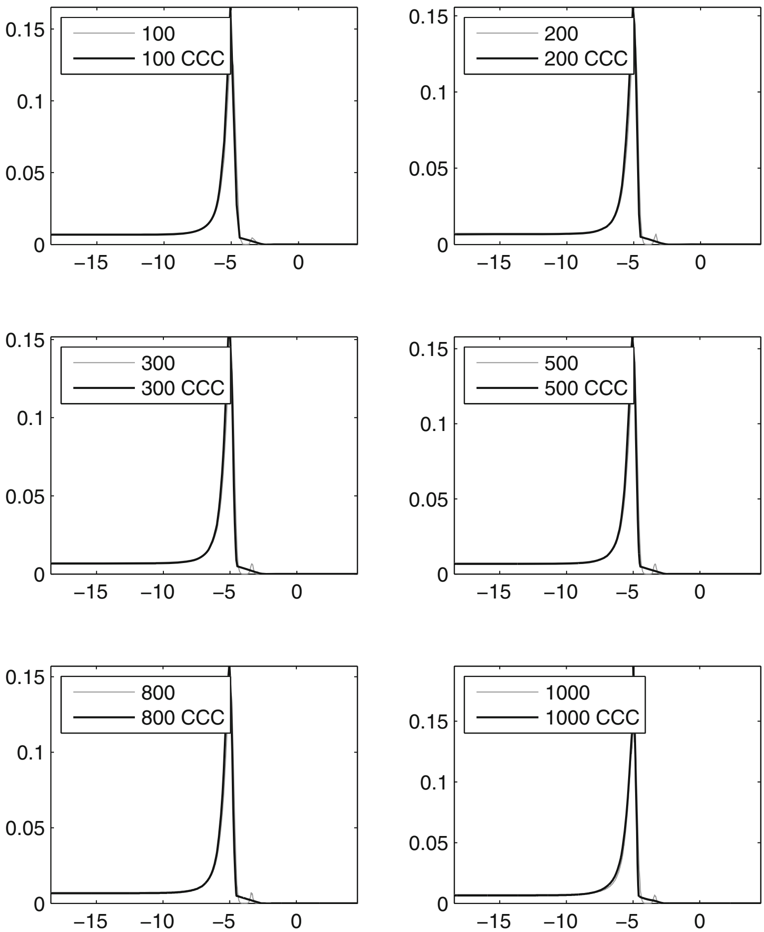

Fig. 4 Distributions of the collateral hazard rate implied by 5-year iTraxx tranche spreads. The plots depict the solutions of Problem B and Problem C obtained with the heuristic algorithm for 6 cases with 100, 200, 300, 500, 800, 1,000 decision variables

the obtained results. We use Portfolio Safeguard (2008) in MATLAB and Run-File Text Environment to solve the optimization problems (MATLAB and PSG Run-File text files are posted at the following link[3]). The provided files can be used for both simulating the expected cash flow matrices based on given tranche quotes and solving the corresponding optimization problems. Appendix 1 contains information on running the case study with PSG. We run the case study on a Windows XP machine with Intel Core 2 CPU @2GHz processor.

We use the data on iTraxx (Europe) index which is composed of the most liquid 125 CDS (Credit Default Swap) referencing European investment grade credits. The number of tranches in the iTraxx index is six ($J = 6$). To get the net expected cash

[3] http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/cs_calibration_copula/.

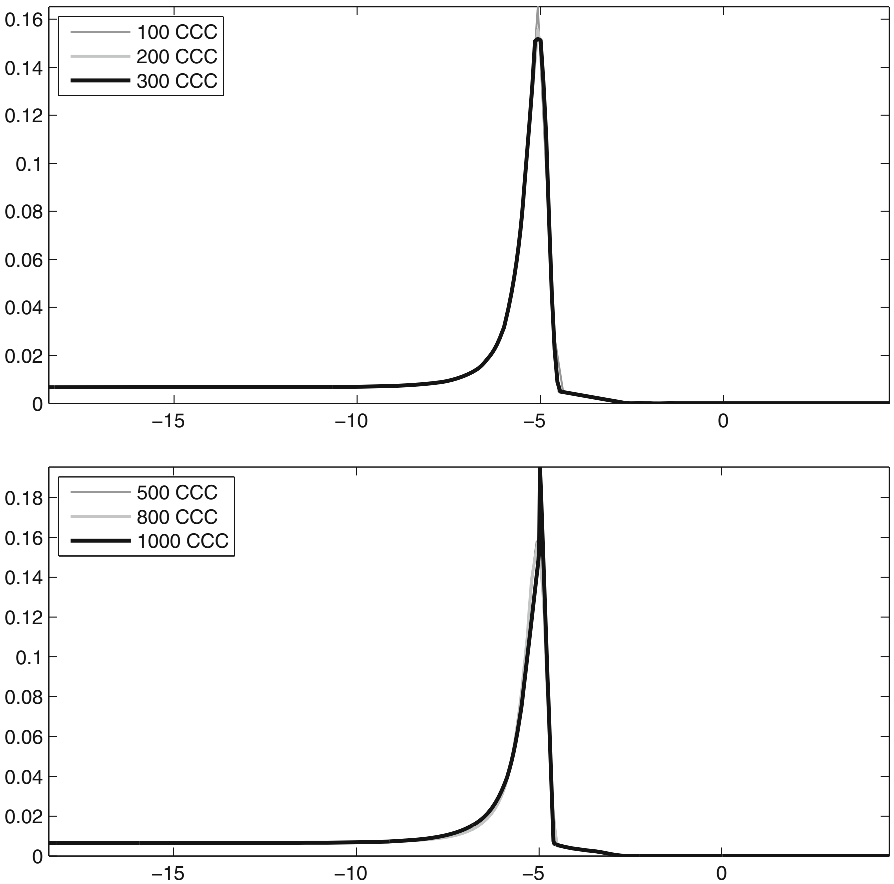

Fig. 5 Distributions of the collateral hazard rate implied by 5-year iTraxx tranche spreads obtained by using proposed heuristic algorithm for solving Problem C for 100, 200, 300, 500, 800, 1,000 decision variables

flow matrices $\left(\overline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$, $\left(\underline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$, for each market state i we simulated the times to default of 125 companies in the iTraxx index and calculated the net cash flow for each tranche. The required matrices are composed by the averages of net cash flows simulated 10,000 times. As assumed by the implied copula model, the time to default of each company in market state i is exponentially distributed with parameter λ_i. We use $\lambda_1 = 10^{-8}$ (almost no companies default before T), $\lambda_I = 100$ (almost all companies default immediately), and choose λ_i in such a way that the distances between two consecutive $ln(\lambda_i)$ are equal [similar to Hull and White (2006)]: $ln(\lambda_i) = ln(10^{-8}) + (i-1)(ln(100) - ln(10^{-8})/(I-1)$. The tranche payments are assumed to be made quarterly, the recovery rate in case of default is 40 % and the annual risk free rate is 4 %. More details on the simulation procedure and the calculation of net cash flows based on time to defaults of obligors can be found in (Hull and White 2006). The figures illustrate the implied probability distribution vectors $(p_1, \ldots, p_I)$ obtained by solving the corresponding optimization problems, where p_i correspond

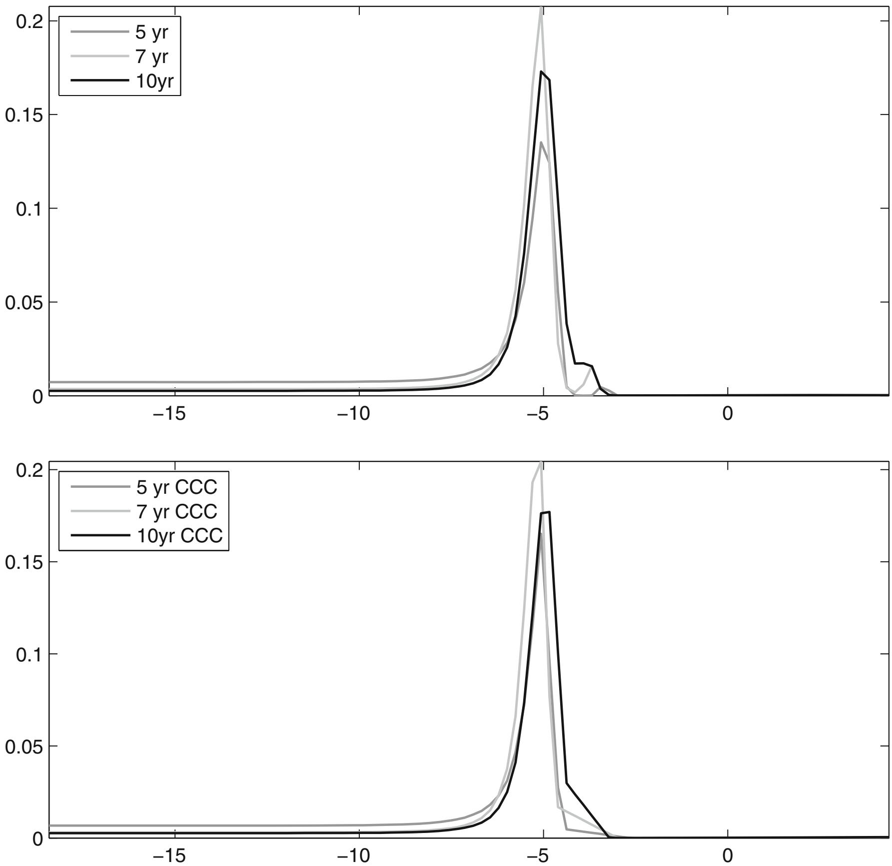

Fig. 6 Distributions of the collateral hazard rate implied by 5, 7 and 10-year iTraxx tranche spreads obtained by solving Problem B (*upper chart*) and Problem C (*lower chart*) using heuristic algorithm for 100 decision variables

to the point $(ln(\lambda_i), p_i), i = 1, \ldots, I$. For comparison reasons, the distributions are drawn as continuous functions and scaled to make areas under the corresponding graphs to be equal.

Figures 2, 3 depict the plots of probability distributions of hazard rates (market states) implied by 5-year iTraxx index tranche quotes given in Table 1. The distributions are obtained by solving Problem A for six different values of smoothing term coefficient $c = 10^0, 10^1, 10^2, 10^3, 10^4, 10^5$ and $I = 100, 300, 500, 1,000$. The corresponding matrices $\left(a_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$ are simulated using mid-prices (the average between bid and ask quotes). Observe that the probability distributions are sensitive to the parameters c and I. Intuitively, in a good model the calibrated probability distribution should not significantly depend on the number of considered hazard rates and the smoothing term coefficient as they contain no additional information about the market states. Figure 4 presents the probability distributions obtained by solving the entropy maximization problem with no-arbitrage constraints (Problem B), and with

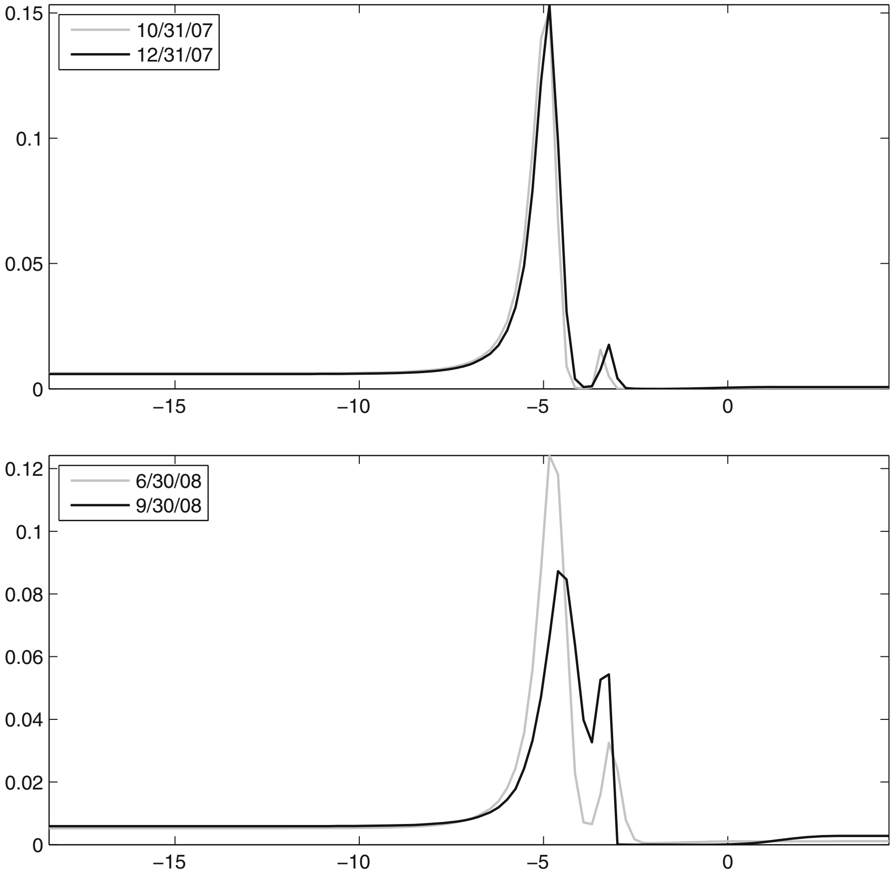

Fig. 7 Distributions of the collateral hazard rate implied in 5-year iTraxx tranche spreads for different dates obtained by solving Problem B with 100 decision variables

CCC constraints for the same dataset and values $I = 100, 200, 300, 500, 800, 1,000$. The only difference from previous settings is that the expected net cash flow matrices $\left(\overline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$, $\left(\underline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$ are simulated for bid and ask quotes, not mid quotes. We use the proposed heuristic algorithm to incorporate CCC constraints described at the end of Section 4; although, for small values of I the Problem C can be solved exactly by solving $O(I^2)$ variants of Problem $C(w_l, w_r)$ and choosing the solution with maximum entropy. The optimization time for Problem B varies from 0.01 sec. for $I = 100$ to 0.06 sec. for $I = 1,000$; for heuristic with CCC constraints time varies from 0.36 sec. for $I = 100$ to 600 sec. for $I = 1,000$. Note that the optimal probability distributions are not visibly sensitive to parameter I (Fig. 5) and imposing CCC constraints does not change significantly the shape of implied density functions. However, irregularities are streamlined.

In the next set of computational experiments the probability distributions are calibrated using the entropy approach (Problem B and the proposed heuristic for

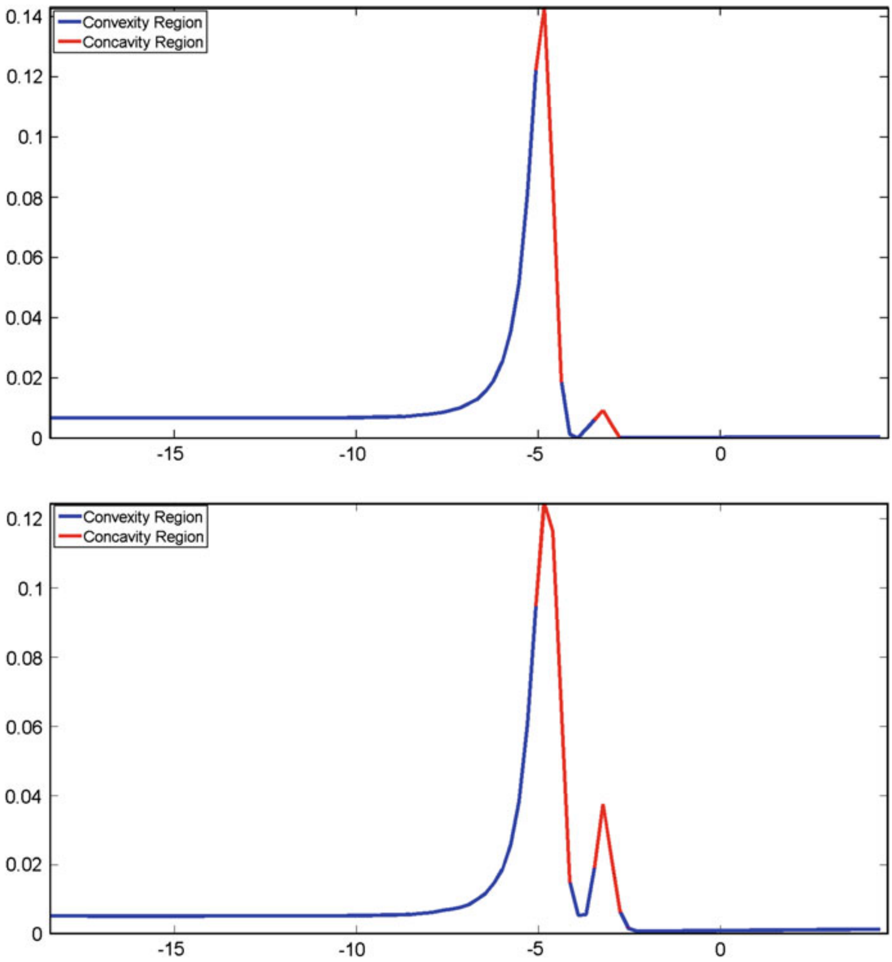

Fig. 8 Distributions of the collateral hazard rate implied in 5-year iTraxx tranche spreads for different dates (10/31/07-*upper chart*, 6/30/08-*lower chart*) obtained using the maximum entropy principle with two-hump CCC model

Problem C) with CCC constraints and the same iTraxx index quotes (Table 1), but with different maturities (5, 7, 10 years). These experiments may show whether the homogeneity assumption proposed in the implied copula model is reasonable. Specifically, the model assumes that in each market state the obligors' hazard rates are the same for all obligors in the CDO pool and they do not depend on the contract period. Therefore, since the set of obligors is the same, the calibrated probability distributions should also be similar (ideally, they should coincide). We simulated the matrices $\left(\overline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$, $\left(\underline{a}_{ij}\right)_{i=1,\ldots,I}^{j=1,\ldots,J}$ of expected net cash flows using the prices from Table 1 for 5, 7 and 10-year iTraxx contracts for $I = 100$. Figure 6 plots the graphs of optimal solutions obtained by solving Problem B and Problem C using the proposed heuristic algorithm. The graphs are visibly similar and show little dependence on the length of the contract period.

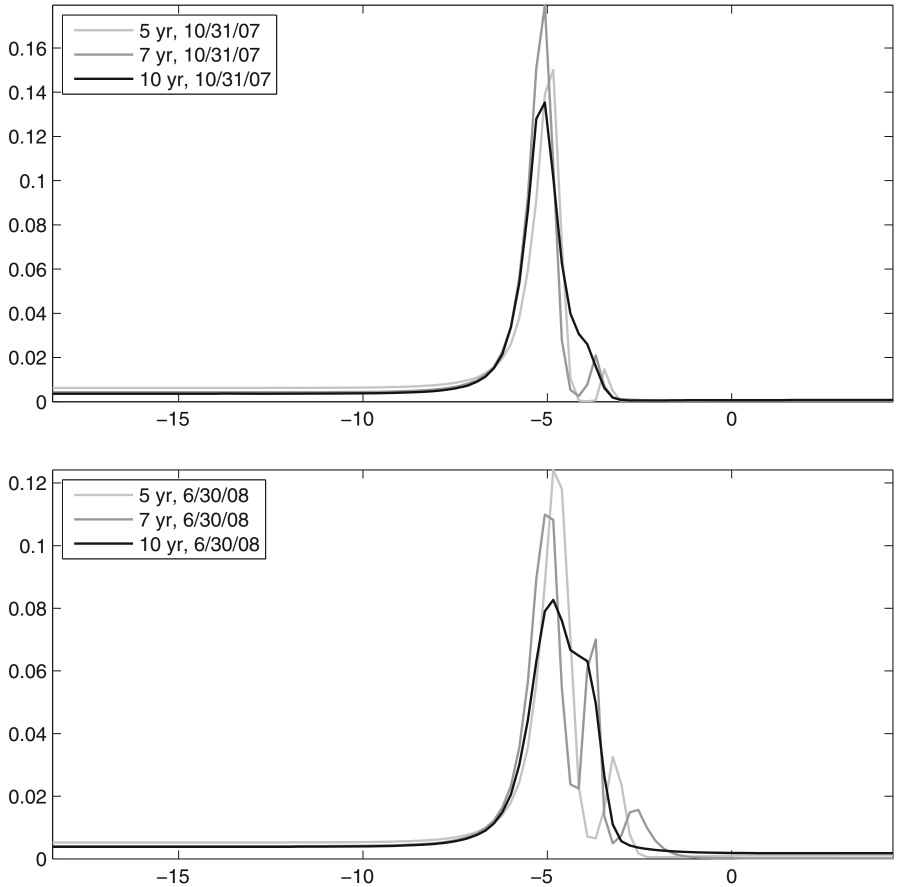

Fig. 9 Distributions of the collateral hazard rate implied by 5, 7 and 10-year iTraxx tranche spreads at two different dates. The distributions were found by solving Problem B with 100 decision variables

The data analyzed in the previous computational experiments are the market quotes for 5, 7, 10-year iTraxx index on 20 December, 2006. At that time, the credit derivatives market was flourishing and expanding very fast. The market became very unstable in the next couple of years, which should also be reflected in the implied probability distribution of market states. Intuitively, the market states corresponding to worse credit environments should have higher probabilities to be realized. Figure 7 shows the graphs of probability distributions of market states calibrated from prices of 5-year iTraxx contract on 4 different dates: 10/31/07, 12/31/07, 6/30/08 and 9/30/08. We also use $I = 100$ and the same simulation procedure to get the net expected cash flow matrices. The optimal distributions are obtained by solving Problem B ($I = 100$). Observe that the plots look intuitively reasonable: as time goes by and the market becomes more unstable, higher chances of bad default environments get reflected in the corresponding implied probability distribution functions in the appearance of a second hump. We could not solve Problem C since it becomes infeasible because of the second hump in the solution of Problem B. It means that the assumption on unimodality of probability distribution of market states may not be reasonable. In this case, we solve two-hump CCC model. Namely, we solve Problem C for two CCC humps for any fixed set of inflection points and choose the solution with maximum entropy. Observe that for two-hump CCC model only 3 inflection points need to be fixed; thus, to solve Problem C with two-hump CCC constraints, we need to solve $O(I^3)$ subproblems similar to Problem $C(w_l, w_r)$. Since we use $I = 100$ the CPU time for subproblem is less than a second, we are able to identify the exact solution in a reasonable time. Figure 8 plots the results. We use different coloring to emphasize the convexity and concavity regions of the corresponding implied distributions.

The last set of computational experiments aim to test how reasonable is the homogeneity assumption proposed in the implied copula model when the market of credit derivatives is unstable. Figure 9 shows the calibrated probability distributions obtained by solving Problem B using the 5, 7, 10-year iTraxx index prices on 10/31/07, 6/30/08. The results suggest that the homogeneity assumption does not work well and may need to be modified to better reflect the nature of underlying assets as, for example, it is discussed in Remark 1.

6 Conclusion

In this paper, we have considered a class of functions, so-called CCC functions, which can be used to calibrate unimodal or multimodal probability distributions. In situations where a discrete probability distribution is being recovered by solving optimization problem, we showed that the CCC class can be incorporated as a set of linear constraints. The application of proposed methodology is demonstrated for the problem of calibrating probabilities of credit environments (market states) in the implied copula CDO pricing model. For the computational experiments we used the historical prices of iTraxx Europe index during stable and unstable times of credit environments, and compared our methodology with the one proposed by Hull and White (Hull and White (2006)). We also demonstrated how to apply two-hump CCC model, discussed its implications and other potential generalizations.

Appendix 1: running case study with portfolio safeguard (PSG)

PSG has several syntax formats for running optimization problems in MATLAB environment:

- Optimization subroutines for optimizing nonlinear functions. Subroutines (e.g., “riskprog”) use as a parameter the name of a nonlinear function (e.g. “entropyr”), which is optimized.
- General PSG format.

With PSG optimization language in general format, the problem solving typically involves three main stages:

1. *Mathematical formulation of a problem with a meta-code using PSG nonlinear functions.* Typically, a problem formulation involves 5–10 operators of a meta-code. See in the end of the Appendix 1 the PSG meta-code for Problem C(w_l, w_r).

2. *Preparation of data for the PSG functions in an appropriate format*. For instance, the *meansquare* error function is defined by the matrix of loss scenarios. One of those matrices should be prepared if we use this function in the problem statement.
3. *Solving the optimization problem with PSG using the predefined problem statement and data for PSG functions.* The problem can be solved in several PSG environments, such as MATLAB environment and Run-File (Text) environment.

Further we present the PSG meta-code for solving Optimization Problem C(w_l,w_r). The meta-code, data and solutions can be downloaded from the link at the bottom of this page[4].

Meta-Code for Optimization Problem C(w_l, w_r)

1 Problem: problem_CCC, type = **minimize**
2 **Objective:** objective_h, linearize = 1
3 **entropyr**_h**(matrix_h)**
4 **Constraint:** constraint_a, lower_bound = vector_bl, upper_bound = vector_b
5 **linearmulti_a (matrix_a)**
6 **Constraint:** constraint_aeq, lower_bound = 1, upper_bound = 1
7 **linearmulti_aeq (matrix_aeq)**
8 **Box_of_Variables:** lowerbounds = 0
9 Solver: VAN, precision = 5

Here is a brief description of the presented meta-code. We boldface the important parts of the code. The keyword **minimize** tells a solver that the Problem C(w_l, w_r) is a minimization problem. The keyword **Objective** is used to define the objective function. The objective function (17), that is a Shannon entropy function, is defined in lines 2,3 with the keyword **entropyr** and the data matrix, located in the file **matrix_h.txt**. Each constraint starts with the keyword **Constraint**. The constraints (18), (19) and (22)–(25) are the system of linear inequalities, defined in lines 4,5 with the keyword **linearmulti**. The coefficients for these linear inequalities are given in the file **matrix_a.txt**. The probability distribution constraint (20) is defined in lines 6,7 with keyword **linearmulti** and the matrix of unit coefficients, located in the file **matrix_aeq.txt**. The **Box_of_Variables** in line 8 sets the non-negativity constraints (21).

References

Portfolio safeguard (2009) version 2.1. http://www.aorda.com/aod/welcome.action

Andersen L, Sidenius J (2004) Extensions to the gaussian copula: random recovery and random factor loadings. J Credit Risk 1(1):29–70

Arnsdorf M, Halperin I (2007) Bslp: Markovian bivariate spread-loss model for portfolio credit derivatives. Quantitative research, JP Morgan

Avellaneda M (1998) Minimum-relative-entropy calibration of asset-pricing models. Intern J Theor Appl Finance 1(4):447

Avellaneda M, Buff R, Friedman C, Grandchamp N, Gr N, Kruk L, Newman J (2001) Weighted monte carlo: a new technique for calibrating asset-pricing models. Intern J Theor Appl Finance 4:1–29

[4] http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/cs_calibration_copula/.

Bahra B (1997) Implied risk-neutral probability density functions from option prices: theory and application. Working paper, Bank of England
Bu R, Hadri K (2007) Estimating option implied risk-neutral densities using spline and hypergeometric functions. Econ J 10:216–244
Burtschell X, Gregory J, Laurent JP (2005) A comparative analysis of cdo pricing models. In: ISFA Actuarial School and BNP Parisbas. ISFA Actuarial School
Campa JM, Chang PK, Reider RL (1998) Implied exchange rate distributions: evidence from otc option markets. J Intern Money Finance 17(1):117–160
Dempster MAH, Medova EA, Yang SW (2007) Empirical copulas for cdo tranche pricing using relative entropy. Intern J Theor Appl Finance (IJTAF) 10(04):679–701
Golan A (2002) Information and entropy econometrics—editor's view. J Econ 107(1–2):1–15
Halperin I (2009) Implied multi-factor model for bespoke cdo tranches and other portfolio credit derivatives. Quantitative research, JP Morgan
Hull J, White A (2010) An improved implied copula model and its application to the valuation of bespoke cdo tranches. J Invest Manag 8(3):11–31
Hull JC, White AD (2006) Valuing credit derivatives using an implied copula approach. J Deriv 14(2):8–28
Jackwerth JC (1999) Option implied risk-neutral distributions and implied binomial trees: a literature review. J Deriv 7:66–82
Jackwerth JC, Rubinstein M (1996) Recovering probability distributions from option prices. J Finance 51(5):1611–1631
Laurent JP, Gregory J (2003) Basket default swaps, cdo's and factor copulas. J Risk 7(4):103–122
Li DX (2000) On default correlation: a copula function approach. J Fixed Income 9(4):43–54
Malz AM (1997) Estimating the probability distribution of the future exchange rate from option prices. J Deriv 5(2):18–36
Meyer-Dautrich S, Wagner C (2007) Minimum entropy calibration of cdo tranches. Working paper, Uni-Credit MIB
Miller D, Liu Wh (2002) On the recovery of joint distributions from limited information. J Econ 107(1): 259–274
Monteiro AM, Tütüncü RH, Vicente LN (2008) Recovering risk-neutral probability density functions from options prices using cubic splines and ensuring nonnegativity. Eur J Oper Res 187(2):525–542
Nedeljkovic J, Rosen D, Saunders D (2010) Pricing and hedging collateralized loan obligations with implied factor models. J Credit Risk 6(3):53–97
Rosen D, Saunders D (2009) Valuing cdos of bespoke portfolios with implied multi-factor models. J Credit Risk 5(3):3–36
Shannon CE (1948) A mathematical theory of communication. Bell Syst Tech J 27:379–423