European Journal of Operational Research 234 (2014) 508–517

Contents lists available at SciVerse ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Capital Asset Pricing Model (CAPM) with drawdown measure

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ARTICLE INFO

Article history:
Available online 27 March 2013

Keywords:
Capital asset pricing model (CAPM)
Drawdown
Conditional drawdown-at-risk (CDaR)
Asset beta
Portfolio theory

ABSTRACT

The notion of drawdown is central to active portfolio management. Conditional Drawdown-at-Risk (CDaR) is defined as the average of a specified percentage of the largest drawdowns over an investment horizon and includes maximum and average drawdowns as particular cases. The necessary optimality conditions for a portfolio optimization problem with CDaR yield the capital asset pricing model (CAPM) stated in both single and multiple sample-path settings. The drawdown beta in the CAPM has a simple interpretation and is evaluated for hedge fund indices from the HFRX database in the single sample-path setting. Drawdown alpha is introduced similarly to the alpha in the classical CAPM and is evaluated for the same hedge fund indices. Both drawdown beta and drawdown alpha are used to prioritize hedge fund strategies and to identify instruments for hedging against market drawdowns.

1. Introduction

The capital asset pricing model (CAPM) is one of the fundamental and most influential concepts in modern finance. It is closely related to portfolio theory and finds its application in portfolio risk management, fund performance measurement, security valuation, etc. The CAPM was developed by Sharpe [35], Lintner [16], and Mossin [22] and can be viewed from two perspectives:

(i) The CAPM is a reformulation of the necessary optimality conditions for the Markowitz's mean–variance portfolio problem and, thus, inherently depends on the definition of risk as variance.
(ii) The CAPM is a single-factor linear model (security market line) that relates the expected returns of an asset and a market portfolio, in which the slope, called asset *beta*, serves as a measure of asset non-diversifiable (systematic) risk.

Owing to its idealized assumptions, e.g. the use of variance as a measure of risk and homogeneity of risk preferences, the CAPM fails to "predict" stock returns accurately enough. Moreover, extensive empirical evidence shows that factors other than market portfolio contribute to stock return variations. Nevertheless, the CAPM, being mathematically simple, still offers a quick quantitative insight into risk-reward interplay and remains a major benchmark for asset pricing. Since its conception, the CAPM has been extended largely in three main directions: (i) relaxing or changing the assumptions under which the CAPM was derived, (ii) identifying new factors as explanatory variables for stock returns, and (iii) using different risk measures for portfolio valuation; see [23,12] for detailed CAPM reviews. Prominent CAPM extensions in the first direction include an intertemporal CAPM [20] and Black-Litterman model [3] that incorporates investor views on asset returns. Also, to lessen beta intertemporal instability, Levy [15] and Fabozzi and Francis [10] introduced several definitions of bear/bull market and applied the CAPM to different time periods separately. The second direction is aligned with the arbitrage pricing theory (APT), which explains asset returns through linear models with several factors and is often viewed as an empirical counterpart of the CAPM. Exemplary contributions to the APT are the works of Ross [32], Banz [2], Rosenberg [31], Burmeister and Wall [4], etc. to name just a few, whereas the Fama and French three-factor model [11] is now widely accepted as a CAPM empirical successor. Recently, Ranaldo [24] suggested higher distribution moments as factors for hedge fund pricing. As for the use of different risk measures in place of variance, Markowitz [18] himself acknowledged the shortcomings of variance and proposed the mean-semivariance approach to portfolio selection. In fact, standard deviation and lower semideviation are particular examples of general deviation measures [27] that are not necessarily symmetric with respect to ups and downs of a random variable and can be customized to tailor investor's risk preferences. Rockafellar et al. [28–30] developed mean-deviation approach to portfolio selection as an extension of the Markowitz's mean–variance ap-

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http://dx.doi.org/10.1016/j.ejor.2013.03.024

proach and generalized a number of classical results, including the one-fund theorem [28], the CAPM [29] and the existence of market equilibrium with investors using different deviation measures [30].

After 60 years of intensive CAPM research, the search for an informative and yet simple one-factor model explaining stock returns continues. So far, the theoretical approach through the portfolio theory to finding risk or deviation measures for a portfolio problem that would improve the CAPM has had limited success. This could be partially explained by the fact that one-period models ignore sequential structure of returns. For example, if pairs of returns for an asset and a market portfolio corresponding to different time moments are reshuffled, the covariance of the returns will not be affected. However, from the investment perspective, the original return sequence and the reshuffled one are unlikely to be viewed equally. In fact, investors are concerned not only about magnitude of asset losses but also about their sequence. One of the measures that capture this phenomenon is based on asset *drawdown*, which at each time moment is the drop of the asset cumulative return from its peak value since the time of investment or monitoring. Usually, investors quit an investment fund after either a single large drawdown (greater than 20%) or a small but prolonged drawdown (over a year). As a result, active portfolio management imposes several constraints on fund drawdowns [5]. Funds can be compared by Managed Account Reports (MAR) ratio, which is similar to Sharpe ratio [36] and is defined as the ratio of fund compound annual return from inception to the fund maximum drawdown from inception. However, the MAR ratio and similar Calmar (Sterling) ratio (also based on the notion of drawdown) do not involve market portfolio, and thus, are unable to identify funds producing positive returns when the market portfolio is in drawdown. In other words, these ratios are not substitutes for *drawdown beta*, which can identify instruments for hedging against market drawdowns.

The goal of this work is twofold: (i) to define drawdown beta β_{DD} and to demonstrate empirically that β_{DD} is an informative measure of asset performance for hedging against market drawdowns and (ii) to lay down a theoretical foundation for β_{DD} through portfolio theory. Grossman and Zhou [13] were arguably the first to solve a portfolio problem with a constraint on portfolio *relative* drawdown with a single risky asset in a continuous-time setting, whereas Cvitanic and Karatzas [8] extended Grossman and Zhou's approach to the case of several risky assets, and recently, Yang and Zhong [37] adopted the Cvitanic and Karatzas' continuous-time solution to a discrete-time setting with a rolling time window. The line of research on portfolio optimization under drawdown constraints was further advanced in [34,7], where in particular, Cherny and Obloj [7] showed that in the setup of Grossman and Zhou, the problem of maximizing the long-term growth rate of the expected utility of wealth under a drawdown constraint could be reduced to an unconstrained problem with a modified utility function. Also, Elie and Touzi [9] studied an optimal consumption-investment problem with an infinite horizon and a drawdown constraint. Other relevant works on drawdown include capital allocation strategies with a drawdown constraint [21], mean–variance portfolio problem with drawdown constraints [1], drawdown modeling and estimation [19,17], and empirical studies [14]. At this point, it should be noted that for a given investment horizon, portfolio drawdowns are either a function in a continuous-time setting or a time series in a discrete-time setting, and while portfolio drawdowns can be constrained at each time period, they cannot be easily compared to drawdowns of other portfolios even for the same investment horizon. In other words, to characterize portfolio risk for the whole investment period, drawdowns (function or time series) should be translated into a single number, or, equivalently, into *drawdown measure*. For a single sample-path of portfolio return, Chekhlov et al. [5] proposed a drawdown measure, called *Conditional Drawdown-at-Risk (CDaR)*, as the average of a specified percentage of the largest drawdowns over the investment horizon. In the followup work, Chekhlov et al. [6] extended CDaR definition for the multiple sample-path case and formulated a portfolio optimization problem with CDaR, which was reduced to linear programming with the Rockafellar–Uryasev formula for *Conditional Value-at-Risk (CVaR)* [25,26]. In fact, CDaR possesses all the properties of a deviation measure: it is convex, positive homogeneous, nonnegative, and invariant to constant translation. This suggests that the approach used to derive the CAPM with general deviation measures in [28,29] can be applied to obtain the CAPM with CDaR. Several case studies on portfolio optimization with CDaR and real data from hedge funds are available at the University of Florida Financial Engineering Test Problems webpage.[1] They provide codes, data, and optimization results.

This work derives necessary optimality conditions for a portfolio optimization problem with CDaR, similar to the one addressed in [6], and reformulates the conditions in the form of CAPM yielding definition for the drawdown beta β_{DD}. The drawdown beta has a simple interpretation and depends on the confidence level $\alpha \in [0,1]$ that determines the percentage of the largest drawdowns in CDaR with $\alpha = 0$ and $\alpha = 1$ corresponding to the average and maximum drawdowns, respectively. The drawdown beta and classical beta have a clear distinction: drawdown beta accounts only for periods when the market portfolio is in drawdown and ignores asset performance when the market portfolio goes up, whereas the classical beta evaluates correlation between returns of the asset and market portfolio over the whole period. To some extent, β_{DD} is similar to the bear market beta introduced in [10], which also concentrates on poor market performance, but in contrast to β_{DD}, has no portfolio optimization rationale.

The paper is organized into four sections. Section 2 formulates the portfolio optimization problem with CDaR and derives the necessary optimality conditions, which are then restated as CAPM in the multiple sample-path setting. Section 3 presents the CAPM with CDaR in the case of a single sample-path and evaluates β_{DD} for hedge fund indices from the HFRX database. It also introduces the *drawdown alpha* (similar to the classical alpha) as asset excess return compared to the CAPM prediction and evaluates drawdown alphas for the same hedge fund indices. Section 4 concludes the paper.

2. Drawdown CAPM with CDaR: theoretical background

The classical CAPM establishes a linear relationship between the expected rate of return of an asset and the expected rate of return of a market portfolio, where the slope of the linear relationship is called asset beta and is defined as the ratio of the covariance of the asset and market portfolio rates to the variance of the market rate. If the asset beta and the expected rate of the market portfolio are known or given, the CAPM "predicts" the asset expected rate of return. However, asset beta can be defined differently thereby leading to a non-classical CAPM. In fact, it is closely related to a risk or deviation measure used in a portfolio selection problem yielding an optimal portfolio. This section formulates portfolio optimization problems with CDaR and presents their necessary optimality conditions in the form of the CAPM in the multiple sample-path setting.

[1] Case studies website: http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/, case study Portfolio Optimization with Drawdown Constraints on a Single Path, case study Portfolio Optimization with Drawdown Constraints on Multiple Paths, case study Portfolio Optimization with Drawdown Constraints, Single Path vs Multiple Paths.

2.1. Definition of CDaR

Suppose $r_1, \ldots, r_T$ are the rates of return of a risky instrument for T consecutive time moments with r_k being the rate of return for period k, $k = 1, \ldots, T$. Let w_t be the cumulative rate of return of the instrument for time t, which can be either uncompounded and defined by $w_t = \sum_{k=1}^t r_k$, or compounded and defined by $w_t = \prod_{k=1}^t (1 + r_k) - 1, k = 1, \ldots, T$. Further analysis in this section holds for either definition of the cumulative return, however, for the sake of simplicity, it is conducted for w_t defined as *uncompounded cumulative rate of return*.

The drawdown of the instrument at time t with τ-window is the instrument's loss from a peak of the cumulative rate of return that occurs within $[t_\tau, t]$, where $t_\tau = 1$ for $t \leqslant \tau$ and $t_\tau = t - \tau$ for $t > \tau$. Formally, it is defined by

$$d_t = \max_{t_\tau \leqslant k \leqslant t} w_k - w_t, \qquad t_\tau = \max\{t - \tau, 1\}, \qquad t = 1, \ldots, T, \quad \tau \in \{1, \ldots, T\}. \tag{1}$$

The drawdown is always nonnegative and is often referred to as *underwater curve*. If at time t, the cumulative rate of return is highest on $[t_\tau, t]$, then $d_t = 0$. Moreover, $d_t = 0$ for all time moments only if the instrument rate of return is nonnegative for each period; see Fig. 1 for the illustration of the drawdown definition.

The performance of the instrument over T periods with the vector $w = (w_1, \ldots, w_T)$ of cumulative rates of return can be characterized by three drawdown measures: maximum drawdown (MaxDD), defined by $\text{MaxDD}(w) = \max_{1 \leqslant t \leqslant T} d_t$, average drawdown (AvDD), defined by $\text{AvDD}(w) = \frac{1}{T}\sum_{t=1}^T d_t$, and Conditional Drawdown-at-Risk (CDaR), introduced in [5] and defined as follows.

Definition 1 (*CDaR on a single sample-path when $\alpha \cdot T$ is an integer*). For a given $\alpha \in [0, 1)$ such that $\alpha \cdot T$ is an integer, single sample-path Conditional Drawdown-at-Risk (CDaR), denoted by $D_\alpha(w)$, is the average of $(1 - \alpha) \cdot 100\%$ largest drawdowns, selected from $d_1, \ldots, d_T$, and is formally defined by

$$D_\alpha(w) = \sum_{t=1}^T q_t^* d_t, \tag{2}$$

where $q_t^* = 1/((1 - \alpha)T)$ if d_t is one of the $(1 - \alpha) \cdot T$ largest portfolio drawdowns and $q_t^* = 0$ otherwise.

The maximum and average drawdowns are particular cases of CDaR that correspond to $\alpha = 1$ and $\alpha = 0$, respectively.

For arbitrary $\alpha \in [0, 1]$ and for multiple sample-paths, CDaR is defined as in [6]. Suppose there are S sample-paths (possible realizations) for the sequence $r_1, \ldots, r_T$ with probability $p_s > 0$ and $r_{s1}, \ldots, r_{sT}$ corresponding to sample path s, $s = 1, \ldots, S$. Then $w_{st} = \sum_{k=1}^t r_{sk}$ (or $w_{st} = \prod_{k=1}^t (1 + r_{sk}) - 1$) and $d_{st} = \max_{t_\tau \leqslant k \leqslant t} w_{sk} - w_{st}$ are the uncompounded (or compounded) cumulate rate of return and drawdown of the instrument, respectively, at time t for sample path s, where $t = 1, \ldots, T$, $s = 1, \ldots, S$, and t_τ is defined as in (1).

Definition 2 (*CDaR for multiple sample-paths when $\alpha \cdot T$ is arbitrary*). For a given $\alpha \in [0, 1)$ in the multiple sample-path setting, CDaR, denoted by $D_\alpha(w)$, is the average of $(1 - \alpha) \cdot$ 100% drawdowns of the set $\{d_{st} | t = 1, \ldots, T, s = 1, \ldots, S\}$, and is defined by

$$D_\alpha(w) = \max_{\{q_{st}\} \in Q} \sum_{s=1}^S \sum_{t=1}^T p_s q_{st} d_{st}, \tag{3}$$

where

$$Q = \left\{ \{q_{st}\}_{s,t=1}^{S,T} \,\middle|\, \sum_{s=1}^S \sum_{t=1}^T p_s q_{st} = 1, \ 0 \leqslant q_{st} \leqslant \frac{1}{(1 - \alpha)T}, \quad s = 1, \ldots, S, \ t = 1, \ldots, T \right\}.$$

For $\alpha = 1$, $D_\alpha(w)$ is defined by (3) with the constraint $0 \leqslant q_{st} \leqslant 1/((1 - \alpha)T)$ in Q replaced by $q_{st} \geqslant 0$.

As in the case of a single sample-path, the CDaR definition (3) includes two special cases: (i) for $\alpha = 1$, $D_1(w)$ is the *maximum drawdown*, also called drawdown from peak-to-valley, and (ii) for $\alpha = 0$, $D_0(w)$ is the *average drawdown*.

2.2. Portfolio optimization

Suppose there are n risky instruments available on the market, and suppose there is a *zero-rate* risk-free instrument ($r_0 = 0$). An investment problem is planned for T consecutive time periods $t = 1, \ldots, T$. Suppose there are S future sample-paths $r_{s1}, \ldots, r_{sT}$, $s = 1, \ldots, S$, where $r_{st} = \left(r_{st}^1, \ldots, r_{st}^n\right)$ and r_{st}^i is the rate of return of instrument i for $i = 1, \ldots, n$, $s = 1, \ldots, S$, $t = 1, \ldots, T$. Sample-path s has the probability of occurrence $p_s > 0$, where $\sum_{s=1}^S p_s = 1$. Let w_{st}^i denote the cumulative rate of return of instrument i in sample-path s at time t, which can be either uncompounded rate of return, defined by $w_{st}^i = \sum_{j=1}^t r_{sj}^i$, or compounded cumulative rate of return, defined by $w_{st}^i = \prod_{j=1}^t \left(1 + r_{sj}^i\right) - 1$. Further, w_{st}^i will be referred simply to as cumulative rate of return, and all results will hold for both definitions. For any sample-path and time moment, the

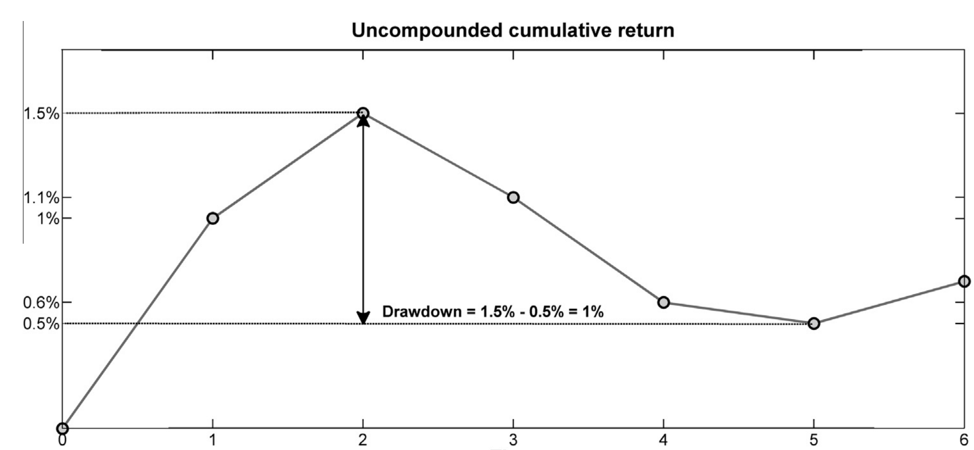

Fig. 1. Drawdown example: the solid line is the uncompounded cumulative rate of return, which at time t is the sum of rates of return over periods $1, \ldots, t$. Here, $\tau = 6$. For $t = 5$, $w_5 = 0.5\%$, whereas the maximum of w_t over time moments preceding $t = 5$ occurs at $t = 2$ with $w_2 = 1.5\%$. Consequently, $d_5 = 1.5 - 0.5\% = 1\%$. The instrument maximum drawdown over time period $[0, 6]$ occurs at $t = 5$.

zero-rate risk-free instrument has zero compounded and uncompounded cumulative rates of return: $w^0_{st}=0,\ s=1,\dots,S,\ t=1,\dots,T$. Next two examples illustrate portfolios with compounded and uncompounded cumulative rates of return.

Example 1 (*portfolio with compounded cumulative rate of return*). In this example, a portfolio follows a traditional setting, where capital C is invested in the risk-free instrument and in the n risky instruments (e.g. stocks and bonds) with capital weights x_0 and $x_1,\dots,x_n$, respectively. Note that x_i can be negative, which corresponds to shorting, and weights sum up to one: $x_0+\sum_{i=1}^n x_i=1$ (budget constraint). At the end of period t in sample-path s, instrument i accumulates $(1+w^i_{st})x_iC$, and the portfolio has the value $C\left[(1+w^0_{st})x_0+\sum_{i=1}^n(1+w^i_{st})x_i\right]\equiv C[1+\sum_{i=1}^n w^i_{st}x_i]$. Consequently, for sample-path s at time t, the portfolio cumulative rate of return is given by

$$w^p_{st}(x)=\sum_{i=1}^n x_i w^i_{st}, \tag{4}$$

and the corresponding portfolio drawdown is defined by

$$d^p_{st}(x)=\max_{t_\tau\leqslant k\leqslant t} w^p_{sk}(x)-w^p_{st}(x),\qquad t=1,\dots,T. \tag{5}$$

Observe that since the risk-free instrument has zero rate of return, the portfolio rate of return (4) does not involve the risk-free instrument. Moreover, x_0 can be expressed from the budget constraint as $x_0=1-\sum_{i=1}^n x_i$, and consequently, a portfolio problem can be formulated only in terms of weights $x=(x_1,\dots,x_n)$ with no budget constraint on x. Also, it is worth mentioning that $x_1,\dots,x_n$ have the meaning of the portfolio weights only at the initial time moment, because the proportions of the portfolio capital in each instrument may change in time, and $x_1,\dots,x_n$ may not coincide with the true portfolio weights at times $t=1,\dots,T$.

Example 2 (*portfolio with uncompounded cumulative rate of return*). This setting is typical for a portfolio with positions in futures and other derivatives. For instance, a portfolio has positions in n risky instruments, which are futures contracts. In this case, initial capital C is not invested, but is used rather as a deposit, and x_i is a position (positive or negative) opened in instrument i. Also, there is no position in the zero-rate risk-free instrument, and the sum of the positions $x_1,\dots,x_n$ is not constrained. The profit (or loss) that the portfolio $x=(x_1,\dots,x_n)$ makes over period t in sample-path s, i.e. $\sum_{i=1}^n r^i_{st}\,x_i$, is transferred from the portfolio to a separate account, so that in the beginning of period $t+1$, the positions $x_1,\dots,x_n$ remain the same. Then, over t periods since investment, the account accumulates *uncompounded cumulative profit* $\sum_{k=1}^t\left(\sum_{i=1}^n r^i_{sk}\,x_i\right)\equiv\sum_{i=1}^n\left(\sum_{k=1}^t r^i_{sk}\right)x_i\equiv\sum_{i=1}^n w^i_{st}\,x_i\equiv w^p_{st}(x)$. In other words, the profit of the futures contract portfolio is mathematically identical to the cumulative rate of return (4), although $x_1,\dots,x_n$ and w^i_{st} here and in (4) have different meaning. Consequently, the drawdown for the futures contract portfolio is defined by (5) with $w^p_{st}(x)$ being the portfolio uncompounded cumulative profit.

Further analysis does not distinguish portfolios with compounded and uncompounded cumulative rates of return.

A portfolio optimization problem with CDaR can be formulated either as minimizing the portfolio CDaR over T periods subject to a constraint on the portfolio expected rate of return at time T:

$$\min_x D_\alpha(w^p(x))\quad \text{s.t.}\quad \sum_{s=1}^S p_s w^p_{sT}(x)\geqslant \Delta, \tag{6}$$

or as maximizing the portfolio expected rate of return at time T subject to a constraint on the portfolio CDaR over T periods:

$$\max_x \sum_{s=1}^S p_s w^p_{sT}(x)\quad \text{s.t.}\quad D_\alpha(w^p(x))\leqslant \nu, \tag{7}$$

where $x=(x_1,\dots,x_n)$ is a vector of the portfolio weights, $\Delta>0$, $\nu>0$, and $D_\alpha(w^p(x))$ is a convex and positively homogeneous function of x (see [6]). The problem (6) is similar to Markowitz's portfolio selection, in which portfolio variance is replaced by portfolio CDaR, whereas the problem (7) is used in active portfolio management, where $w^p(x)$ is the portfolio profit; see [5]. The problems (6) and (7) are both convex and are equivalent in the sense that by varying Δ in (6) and ν in (7), the same set of optimal solutions (called efficient frontier) will be generated.

The portfolio problems (6) and (7) lead to the same necessary optimality conditions for portfolio weights, which can be stated in the form of CAPM. It is assumed that the constraint $\sum_{s=1}^S p_s w^p_{sT}(x)\geqslant\Delta$ is feasible and that $D_\alpha(w^p(x))>0$ for any $x\neq 0$.

Theorem 1 (*CAPM with CDaR for multiple sample-paths and arbitrary* α). *Let $x^*=(x^*_1,\dots,x^*_n)$ be an optimal solution to either (6) or (7) and let $w^M=w^p(x^*)\equiv\{w^p_{st}\}_{s,t=1}^{S,T}$ be the cumulative rates of return of the optimal portfolio, then the necessary optimality conditions for x^* can be restated in the form of CAPM*

$$\sum_{s=1}^S p_s w^i_{sT}=\beta^i_{\rm DD}\sum_{s=1}^S p_s\,w^M_{sT},\quad \beta^i_{\rm DD}=\frac{\sum_{s=1}^S\sum_{t=1}^T p_s\,q^*_{st}\left(w^i_{s,k^*(s,t)}-w^i_{st}\right)}{D_\alpha(w^M)},\quad i=1,\dots,n, \tag{8}$$

where $k^*(s,t)\in\operatorname{argmax}_{t_\tau\leqslant k\leqslant t}\,w^p_{sk}(x^*)$; $\{q^*_{st}\}_{s,t=1}^{S,T}$ is an element from Q at which maximum is attained in (3) for $w^p(x^*)$; and $D_\alpha(w^M)\neq 0$ by the assumption that $D_\alpha(w^p(x))>0$ for any $x\neq 0$.

Proof. CAPM derivation is conducted for the problem (6) and is completely analogous to that for the problem (7). The necessary optimality conditions for (6) are given by Theorem 3.34 in [33]:

$$0\in\partial_x D_\alpha(w^p(x^*))-\lambda^*\sum_{s=1}^S p_s\nabla w^p_{sT}(x^*), \tag{9}$$

where $x^*=(x^*_1,\dots,x^*_n)$ is an optimal solution to (6); $\nabla w^p_{sT}(x)=(w^1_{sT},\dots,w^n_{sT}),\ s=1,\dots,S$; λ^* is a Lagrange multiplier such that $\lambda^*\geqslant 0$ and $\lambda^*\left(\Delta-\sum_{s=1}^S\sum_{i=1}^n p_s w^i_{sT}x_i\right)=0$; and $\partial_x D_\alpha(w^p(x^*))$ is the subdifferential (subgradient set) of $D_\alpha(w^p(x))$ with respect to x at $x=x^*$. The set $\partial_x D_\alpha(w^p(x^*))$ is evaluated as follows. With the drawdown $d^p_{st}(x)$ represented by

$$d^p_{st}(x)=\max_{\{z_{skt}\}\in Z_{st}}\sum_{k=t_\tau}^t z_{skt}w^p_{sk}(x)-w^p_{st}(x),$$

$$Z_{st}=\left\{\{z_{skt}\}_{k=t_\tau}^t\,\middle|\,\sum_{k=t_\tau}^t z_{skt}=1,\ z_{skt}\geqslant 0,\ k=t_\tau,\dots,t\right\},$$

the CDaR (3) takes the form

$$D_\alpha(w^p(x))=\max_{\substack{\{q_{st}\}\in Q\\ \{z_{skt}\}\in Z_{st}}}\ \sum_{s=1}^S\sum_{t=1}^T p_s q_{st}\left(\sum_{k=t_\tau}^t z_{skt}w^p_{sk}(x)-w^p_{st}(x)\right). \tag{10}$$

The representation (10) and Theorem 2.87 in [33] imply that $\partial_x D_\alpha(w^p(x^*))$ is the set of vectors $g=(g_1,\dots,g_n)$ with components

$$g_i=\sum_{s=1}^S\sum_{t=1}^T p_s q_{st}\left(\sum_{k=t_\tau}^t z_{skt}w^i_{sk}-w^i_{st}\right),\quad i=1,\dots,n,$$

such that $g^\top x^* = D_\alpha(w^p(x^*))$ and $\{q_{st}\}_{s,t=1}^{S,T} \in Q$ and $\{z_{skt}\}_{k=t_\tau}^{t} \in Z_{st}$ for $s = 1, \dots, S$, $t = 1, \dots, T$.

Thus, the necessary optimality conditions (9) simplify to

$$g_i^* = \lambda^* \sum_{s=1}^{S} p_s w_{sT}^i, \qquad i = 1, \dots, n, \tag{11}$$

where g_i^* are the components of $g^* \in \partial_x D_\alpha(w^p(x^*))$, or equivalently, there exist $\{q_{st}^*\}_{s,t=1}^{S,T} \in Q$ and $\{z_{skt}^*\}_{k=t_\tau}^{t} \in Z_{st}$ such that $g_i^* = \sum_{s=1}^{S} \sum_{t=1}^{T} p_s q_{st}^* \left(\sum_{k=t_\tau}^{t} z_{skt}^* w_{sk}^i - w_{st}^i\right)$, $i = 1, \dots, n$. Observe that $\sum_{k=t_\tau}^{t} z_{skt}^* w_{sk}^i = w_{s,k^*(s,t)}^i$, where $k^*(s,t) \in \operatorname{argmax}_{t_\tau \leqslant k \leqslant t} w_{sk}^p(x^*)$.

Multiplying (11) by x_i^* and summing over $i = 1, \dots, n$, we obtain

$$\lambda^* = \frac{D_\alpha(w^p(x^*))}{\sum_{s=1}^{S} p_s w_{sT}^p(x^*)},$$

and consequently, (11) can be recast in the form (8). □

Remark 1. Observe that $\sum_{i=1}^{n} x_i^*$ in (8) can be arbitrary: either positive or negative, or even zero. The CAPM (8) holds if x^* is rescaled by any positive multiplier. It is customary to normalize x^* so that the sum of normalized components is either 1 if $\sum_{i=1}^{n} x_i^* > 0$ or -1 if $\sum_{i=1}^{n} x_i^* < 0$. If $\sum_{i=1}^{n} x_i^* > 0$, then the portfolio with normalized weights $\hat{x}_i = x_i^* \Big/ \sum_{j=1}^{n} x_j^*$, $i = 1, \dots, n$, is called a *master fund of positive type* (see [28]), and in this case, the CAPM (8) can be restated in the same form with $w^M = w^p(\hat{x})$ and $w_{sT}^M = w_{sT}^p(\hat{x})$, $s = 1, \dots, S$, where q_{st}^* and $k^*(s,t)$ are not affected by the rescaling.

3. Drawdown CAPM for a single sample-path: examples

This section presents empirical evidence demonstrating the informativeness of *drawdown beta* and *drawdown alpha* compared to the classical (standard) beta and alpha.

In the case of a single sample-path (i.e. when $S = 1$), let w^M, d^M, and $w_1^i, \dots, w_T^i$ be defined as in Theorem 1. If $\alpha \cdot T$ is an integer, then $q_t^* = 1/((1-\alpha)T)$ if d_t^M is one of the $(1-\alpha) \cdot T$ largest portfolio drawdowns and $q_t^* = 0$ otherwise, and if, in addition, $k^*(t)$ and $\{q_t^*\}_{t=1}^{T}$ are *uniquely* determined, the CAPM (8) in Theorem 1 can be restated in a simplified form.

Corollary 1 (*CAPM with CDaR for a single sample-path when $\alpha \cdot T$ is integer*). *Let $(1-\alpha) \cdot T$ be integer. Then for a single sample-path, the necessary optimality conditions for $w^M = (w_1^M, \dots, w_T^M)$ to be the cumulative rates of return of an optimal portfolio either in the problem (6) or in the problem (7) can be formulated in the form*

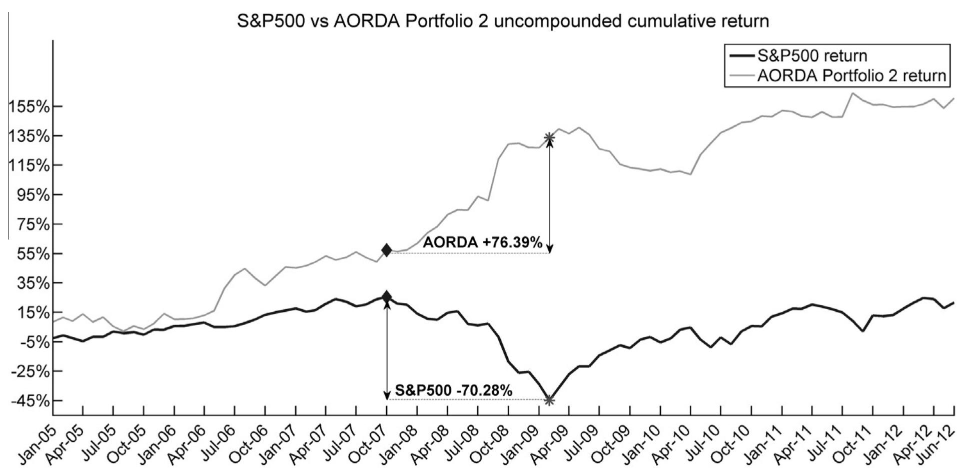

Fig. 2. Uncompounded monthly cumulative rates of return of the S&P500 index and AORDA Portfolio 2 ($T = \tau = 90$). The S&P500 index had its largest drawdown in February 2009, marked by (∗): it peaked in October 2007 (♦) and lost 70.28% from October 2007 to February 2009. During the same period, the AORDA Portfolio 2 earned 76.39%; therefore, the MaxDD beta of the AORDA Portfolio 2 is given by $\beta_{DD} = -76.39\%/70.28\% = -1.09$.

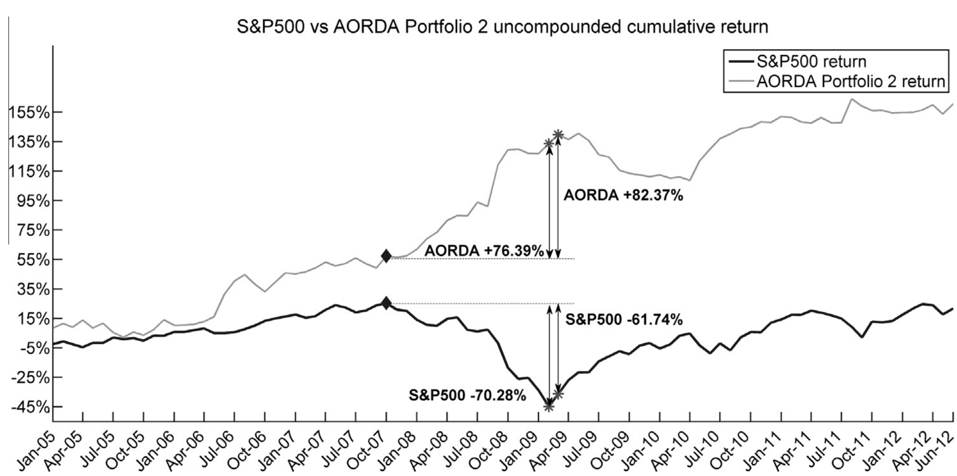

Fig. 3. Uncompounded monthly cumulative rates of return of the S&P500 index and AORDA Portfolio 2 ($T = \tau = 90$). The S&P500 index had two largest drawdowns in February 2009 (70.28%) and in March 2009 (61.74%) marked by (∗): it peaked in October 2007 (♦) and lost 70.28% from October 2007 to February 2009 and 61.74% from October 2007 to March 2009. During the same periods, the AORDA Portfolio 2 earned 76.39% and 82.37%, respectively. The CDaR beta of the AORDA Portfolio 2 is given by $\beta_{DD} = \frac{1}{2}(-76.39\% - 82.37\%)/\left(\frac{1}{2}(70.28\% + 61.74\%)\right) = -1.20$.

Table 1
Betas (AvDD, 90%-CDaR, MaxDD and standard) for 80 HFRX Hedge Fund Indices and for the AORDA Portfolio 2. The rows are ordered by the AvDD beta. Numbers in brackets are ranks in the ordering by the corresponding column; e.g. EH: Short Bias Index has 33rd lowest AvDD beta and the 1st lowest standard beta.

#	Hedge fund index	AvDD β	90%-CDaR β	MaxDD β	Standard β
1	AORDA Portfolio 2	−2.997	−1.412 (1)	−1.087 (1)	−0.414 (2)
2	Macro: Commodity-Metals Index	−1.146	−0.062 (13)	−0.037 (12)	0.354 (55)
3	RV: FI-Asset Backed Index	−1.094	−0.096 (9)	−0.059 (9)	0.098 (14)
4	Macro: Systematic Diversified CTA Index	−0.943	−0.503 (3)	−0.441 (4)	−0.166 (3)
5	Macro/CTA Index	−0.789	−0.35 (4)	−0.114 (8)	0.057 (9)
6	RV: Multi-Strategy Index	−0.615	0.217 (38)	0.197 (31)	0.259 (36)
7	ED: Credit Arbitrage Index	−0.612	0.075 (22)	0.11 (24)	0.174 (21)
8	Fixed Income – Credit Index	−0.561	0.175 (33)	0.163 (27)	0.243 (32)
9	Opportunity EUR Index	−0.537	−0.073 (12)	−0.042 (11)	0.17 (20)
10	Macro: Active Trading Index	−0.525	−0.228 (6)	−0.188 (6)	0.02 (5)
11	ED: Merger Arbitrage Index	−0.427	−0.087 (10)	−0.026 (13)	0.098 (13)
12	Diversity Index	−0.376	0.026 (19)	0.068 (19)	0.241 (30)
13	Macro: Multi-Strategy Index	−0.375	0.007 (18)	0.049 (17)	0.164 (19)
14	Macro: Commodity Index	−0.363	−0.269 (5)	−0.21 (5)	0.041 (6)
15	Market Directional Index	−0.336	0.196 (35)	0.497 (65)	0.383 (60)
16	Global Hedge Fund Index	−0.333	0.053 (21)	0.39 (51)	0.223 (28)
17	ED: Distressed Restructuring Index	−0.327	0.087 (25)	0.547 (70)	0.212 (26)
18	Latin America Index	−0.323	0.417 (48)	0.35 (40)	0.497 (69)
19	Macro: Discretionary Thematic Index	−0.312	−0.016 (16)	−0.024 (14)	0.221 (27)
20	Equal Weighted Strategies Index	−0.301	0.048 (20)	0.352 (41)	0.184 (22)
21	Absolute Return Index	−0.285	−0.045 (15)	0.198 (32)	0.085 (12)
22	MLP Index	−0.278	−0.18 (7)	−0.479 (3)	0.235 (29)
23	Relative Value Arbitrage Index	−0.254	0.198 (36)	0.627 (72)	0.208 (25)
24	Event Driven Index	−0.233	0.16 (31)	0.376 (46)	0.271 (40)
25	Northern Europe Index	−0.213	0.092 (26)	0.083 (20)	0.124 (15)
26	Macro: Commodity-Agriculture Index	−0.202	−0.012 (17)	0.016 (15)	0.078 (11)
27	EH: Equity Market Neutral Index	−0.169	−0.104 (8)	0.017 (16)	0.001 (4)
28	Emerging Markets Composite Index	−0.166	0.381 (45)	0.36 (44)	0.374 (57)
29	Asia with Japan Index	−0.135	0.171 (32)	0.17 (29)	0.249 (34)
30	Equity Hedge Index	−0.124	0.155 (30)	0.479 (62)	0.35 (53)
31	ED: Private Issue/Regulation D Index	−0.111	0.147 (28)	0.117 (25)	0.159 (17)
32	Macro: Commodity-Energy Index	−0.109	0.083 (23)	0.092 (22)	0.162 (18)
33	EH: Short Bias Index	−0.076	−0.545 (2)	−0.56 (2)	−0.679 (1)
34	RV: Volatility Index	−0.073	−0.054 (14)	−0.054 (10)	0.043 (7)
35	RV: Energy Infrastructure Index	−0.045	0.575 (68)	0.534 (68)	0.429 (66)
36	China Index	−0.035	0.422 (49)	0.4 (54)	0.401 (63)
37	MENA Index	0.006	0.443 (51)	0.487 (64)	0.626 (73)
38	ED: Activist Index	0.05	0.59 (71)	0.614 (71)	0.802 (79)
39	EH: Technology/Healthcare Index	0.051	0.211 (37)	0.188 (30)	0.296 (46)
40	Brazil Index	0.067	0.583 (70)	0.473 (61)	0.6 (72)
41	RV: FI-Sovereign Index	0.081	0.228 (39)	0.169 (28)	0.28 (42)
42	Macro/CTA EUR Index	0.083	−0.082 (11)	−0.117 (7)	0.046 (8)
43	RV: FI-Corporate Index	0.092	0.444 (52)	0.357 (43)	0.282 (43)
44	RV: FI-Convertible Arbitrage Index	0.098	0.409 (47)	1.054 (81)	0.289 (44)
45	North America Index	0.104	0.135 (27)	0.092 (21)	0.194 (24)
46	Total Emerging Market Index	0.109	0.456 (55)	0.4 (53)	0.458 (67)
47	Russia Index	0.15	0.937 (81)	0.967 (80)	0.91 (80)
48	Macro: Currency Index	0.174	0.085 (24)	0.054 (18)	0.066 (10)
49	Aggregate Index	0.235	0.328 (41)	0.277 (34)	0.279 (41)
50	EH: Multi-Strategy Index	0.271	0.454 (53)	0.449 (59)	0.643 (75)
51	ED: Multi-Strategy Index	0.272	0.457 (56)	0.437 (57)	0.393 (62)
52	RV: Real Estate Index	0.299	0.346 (42)	0.304 (35)	0.377 (58)
53	RV: Yield Alternative Index	0.3	0.516 (64)	0.387 (50)	0.378 (59)
54	Asia Equally Weighted Index	0.308	0.362 (43)	0.32 (37)	0.319 (50)
55	Asia Composite Hedge Fund Index	0.313	0.378 (44)	0.335 (38)	0.327 (52)
56	BRIC Index	0.326	0.761 (76)	0.722 (77)	0.737 (77)
57	Multi-Region Index	0.334	0.152 (29)	0.099 (23)	0.148 (16)
58	Japan Index	0.386	0.294 (40)	0.243 (33)	0.242 (31)
59	EH: Quantitative Directional Index	0.392	0.187 (34)	0.147 (26)	0.187 (23)
60	EH: Energy/Basic Materials Index	0.465	0.531 (65)	0.523 (67)	0.531 (70)
61	Western/Pan Europe Index	0.486	0.385 (46)	0.305 (36)	0.25 (35)
62	Multi-Emerging Markets Index	0.525	0.749 (75)	0.676 (74)	0.557 (71)
63	Equal Weighted Strategies GBP Index	0.529	0.425 (50)	0.341 (39)	0.262 (38)
64	Event Driven EUR Index	0.558	0.459 (57)	0.375 (45)	0.325 (51)
65	Equal Weighted Strategies EUR Index	0.571	0.456 (54)	0.353 (42)	0.262 (39)
66	Global Hedge Fund GBP Index	0.597	0.46 (58)	0.376 (47)	0.3 (48)
67	Relative Value Arbitrage EUR Index	0.636	0.761 (77)	0.63 (73)	0.355 (56)
68	ED: Special Situations Index	0.641	0.542 (66)	0.443 (58)	0.352 (54)
69	Equal Weighted Strategies CHF Index	0.669	0.472 (59)	0.377 (48)	0.259 (37)
70	Global Hedge Fund EUR Index	0.671	0.499 (61)	0.395 (52)	0.308 (49)
71	Equal Weighted Strategies JPY Index	0.679	0.482 (60)	0.384 (49)	0.248 (33)
72	Asia ex-Japan Index	0.681	0.612 (72)	0.539 (69)	0.467 (68)
73	India Index	0.688	0.878 (80)	0.881 (79)	0.919 (81)

(continued on next page)

Table 1 (*continued*)

#	Hedge fund index	AvDD β	90%-CDaR β	MaxDD β	Standard β
74	Russia/Eastern Europe Index	0.689	0.789 (78)	0.706 (76)	0.643 (74)
75	Alternative Energy Index	0.709	0.864 (79)	0.816 (78)	0.757 (78)
76	Global Hedge Fund CHF Index	0.74	0.506 (62)	0.412 (55)	0.3 (47)
77	Global Hedge Fund JPY Index	0.742	0.513 (63)	0.417 (56)	0.291 (45)
78	Korea Index	0.827	0.71 (74)	0.703 (75)	0.708 (76)
79	Equity Hedge EUR Index	0.899	0.579 (69)	0.483 (63)	0.425 (65)
80	EH: Fundamental Growth Index	0.914	0.643 (73)	0.516 (66)	0.406 (64)
81	EH: Fundamental Value Index	0.916	0.543 (67)	0.466 (60)	0.389 (61)

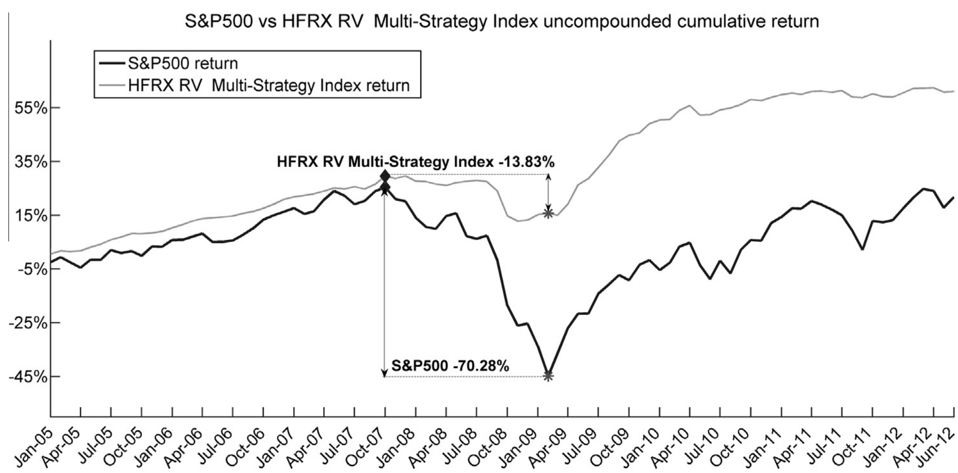

Fig. 4. Uncompounded monthly cumulative rates of return of the S&P500 index and RV Multi-Strategy Index from the HFRX Hedge Fund database ($T = \tau = 90$). Asterisk ($*$) marks the time moment corresponding to the largest (maximum) drawdown of the S&P500 index, i.e. February 2009 (70.28%): the market peaked in October 2007 (marked by (♦)) and lost 70.28% from October 2007 to February 2009. During the same period, RV Multi-Strategy Index lost 13.83%; hence, its MaxDD beta is given by β_{DD} = 13.83%/70.28% = 0.197. The positive MaxDD beta indicates that RV Multi-Strategy Index was in drawdown when the market had the largest drawdown. The figure shows that 10% of the largest market drawdowns happened from October 2008 to July 2009. During this period, RV Multi-Strategy Index was also in a light drawdown. This explains that its 90%-CDaR beta is also positive and is equal to 0.217. However, during January 2005–October 2008 and July 2009–June 2012, RV Multi-Strategy Index successfully overcame light market drawdowns and resulted in the negative AvDD beta.

$$w_T^i = \beta_{\mathrm{DD}}^i \cdot w_T^M \qquad \text{with} \qquad \beta_{\mathrm{DD}}^i = \frac{\sum_{t=1}^T q_t^* \left(w_{k^*(t)}^i - w_t^i \right)}{D_\alpha(w^M)}, \tag{12}$$

where β_{DD}^i *is the drawdown beta of instrument i,* $k^*(t) \in \operatorname{argmax}_{t_\tau \leqslant k \leqslant t} w_k^M$, *and* $q_t^* = 1/((1-\alpha)T)$ *if* d_t^M *is one of the* $(1-\alpha)\cdot T$ *largest drawdowns* $d_1^M, \ldots, d_T^M$ *of the optimal portfolio and* $q_t^* = 0$ *otherwise. It is assumed in (12) that* $D_\alpha(w^M) \neq 0$ *and that* q_t^* *and* $k^*(t)$ *are uniquely determined for all t = 1, ..., T.*

The numerator in β_{DD}^i is the average rate of return of instrument i over time periods corresponding to the $(1-\alpha)\cdot T$ largest drawdowns of the optimal portfolio, where $w_t^i - w_{k^*(t)}^i$ is the cumulative rate of return of instrument i from the optimal portfolio peak time $k^*(t)$ to time t.

The CAPM (12) is further considered as a pricing model in a wider sense, where w^i and w^M are replaced by the rates of return $\widehat{w}$ and $\widehat{w}^M$ of some instrument and some portfolio which is assumed to be optimal. For example, $\widehat{w}$ and $\widehat{w}^M$ can be based on historical or simulated rates of return (if $\widehat{w_k}$ is defined as compounded cumulative rate of return then $\widehat{w_k}^M$ should also be compounded cumulative rate of return). Of course, in this case, the relation (12) may not hold. Then the difference between the actual rate of return and the rate of return of the instrument estimated by $\beta_{\mathrm{DD}} \cdot \widehat{w}_T^M$ is called *CDaR alpha*:

$$\alpha_{\mathrm{DD}} = \widehat{w}_T - \beta_{\mathrm{DD}} \cdot \widehat{w}_T^M, \tag{13}$$

where β_{DD} is evaluated as in (12) with w^i and w^M replaced by $\widehat{w}$ and $\widehat{w}^M$, respectively. Positive α_{DD} implies that the instrument did better than it was predicted, and consequently, α_{DD} can be used as a performance measure to rank instruments and to identify those that outperformed their CAPM "predictions".

The drawdown beta β_{DD} is illustrated with the AORDA[2] Portfolio 2 and S&P500 index being the instrument and the "optimal" portfolio, respectively, for the period from January 2005 to June 2012 with 90 monthly periods (T = 90) and τ = 90. Figs. 2 and 3 both show the uncompounded cumulative rates of return for the AORDA Portfolio 2 and for the S&P500 index from January 2005 to June 2012. The S&P500 index peaked in October 2007, and its uncompounded cumulative rate of return from January 2005 to October 2007 was about 25.4%. From October 2007 to February 2009, the S&P500 index lost 70.28%, which was its maximum drawdown. During the same period, the AORDA Portfolio 2 earned 76.39%. Consequently, the MaxDD beta (α = 1) for the AORDA Portfolio 2 is given by β_{DD} = −76.39%/70.28% = − 1.09; see Fig. 2. If α = 0.978, then $(1-\alpha)\cdot T$ = 2, and we should pick the two largest drawdowns in the history of the S&P500 index. From October 2007 to February 2009 and from October 2007 to March 2009, the S&P500 index lost 70.28% and 61.74%, respectively, which were the two largest drawdowns. During the same periods, the AORDA Portfolio 2 earned 76.39% and 82.37%, respectively. Therefore, the CDaR beta for the AORDA Portfolio 2 with α = 0.978 is given by $\beta_{\mathrm{DD}} = \frac{1}{2}(-76.39\% - 82.37\%)/\left(\frac{1}{2}(70.28 + 61.74\%)\right) = -1.20$; see Fig. 3.

Further, drawdown betas are used to analyze the performance of hedge fund indices from the HFRX database, which are bench-

[2] AORDA Portfolio 2 is described at http://www.aorda.com.

Table 2
Annualized alphas (AvDD, 90%-CDaR, MaxDD, standard), average annual return and AvDD beta for 80 HFRX Hedge Fund Indices and for the AORDA Portfolio 2. Alphas are evaluated by (13) with $\hat{w}_T^M = 12\%$ (the average long-term annual rate of return of the S&P500 index) and $\hat{w}_T$ being the historical rate of return of a hedge fund index. The rows are ordered by the AvDD alpha. Numbers in brackets are ranks in the ordering by the corresponding column.

#	Hedge fund index	AvDD α (%)	90%-CDaR α	MaxDD α	Standard α	Average return	AvDD β
1	AORDA Portfolio 2	57.4	38.4% (1)	34.5% (1)	26.4% (1)	21.4% (1)	−2.997 (1)
2	Macro: Commodity-Metals Index	29.3	16.3% (3)	16% (3)	11.3% (3)	15.5% (3)	−1.146 (2)
3	RV: FI-Asset Backed Index	23.9	11.9% (5)	11.5% (5)	9.6% (4)	10.7% (8)	−1.094 (3)
4	MLP Index	20.8	19.6% (2)	23.2% (2)	14.7% (2)	17.5% (2)	−0.278 (22)
5	Macro: Systematic Diversified CTA Index	17.9	12.6% (4)	11.9% (4)	8.6% (5)	6.6% (27)	−0.943 (4)
6	Macro/CTA Index	15.8	10.5% (6)	7.7% (9)	5.6% (10)	6.3% (32)	−0.789 (5)
7	RV: Multi-Strategy Index	15.5	5.5% (19)	5.8% (19)	5% (14)	8.1% (13)	−0.615 (6)
8	Fixed Income–Credit Index	14.8	5.9% (18)	6.1% (16)	5.1% (13)	8% (14)	−0.561 (8)
9	ED: Credit Arbitrage Index	14.8	6.5% (14)	6.1% (15)	5.3% (11)	7.4% (20)	−0.612 (7)
10	Emerging Markets Composite Index	13.1	6.5% (15)	6.8% (12)	6.6% (7)	11.1% (7)	−0.166 (28)
11	China Index	12.9	7.4% (10)	7.7% (10)	7.7% (6)	12.5% (5)	−0.035 (36)
12	Opportunity EUR Index	12.5	6.9% (12)	6.5% (13)	4% (23)	6% (38)	−0.537 (9)
13	Macro: Active Trading Index	12.2	8.7% (8)	8.2% (7)	5.7% (9)	5.9% (39)	−0.525 (10)
14	Diversity Index	12.2	7.3% (11)	6.8% (11)	4.8% (16)	7.6% (16)	−0.376 (12)
15	Latin America Index	11.7	2.8% (33)	3.6% (26)	1.9% (40)	7.8% (15)	−0.323 (18)
16	Russia Index	11.7	2.2% (38)	1.9% (38)	2.6% (31)	13.5% (4)	0.15 (47)
17	Macro: Discretionary Thematic Index	11.2	7.6% (9)	7.7% (8)	4.8% (17)	7.4% (21)	−0.312 (19)
18	Macro: Multi-Strategy Index	10.9	6.3% (16)	5.8% (18)	4.4% (21)	6.4% (31)	−0.375 (13)
19	ED: Merger Arbitrage Index	10.8	6.7% (13)	6% (17)	4.5% (19)	5.7% (43)	−0.427 (11)
20	Macro: Commodity Index	10.6	9.5% (7)	8.8% (6)	5.8% (8)	6.3% (33)	−0.363 (14)
21	RV: Energy Infrastructure Index	10.6	3.2% (30)	3.7% (25)	4.9% (15)	10.1% (9)	−0.045 (35)
22	Market Directional Index	9.9	3.5% (29)	−0.1% (55)	1.3% (44)	5.9% (41)	−0.336 (15)
23	Asia with Japan Index	9.9	6.2% (17)	6.2% (14)	5.3% (12)	8.3% (12)	−0.135 (29)
24	Global Hedge Fund Index	9.5	4.8% (20)	0.8% (47)	2.8% (29)	5.5% (45)	−0.333 (16)
25	Northern Europe Index	8.4	4.7% (21)	4.8% (21)	4.3% (22)	5.8% (42)	−0.213 (25)
26	Equal Weighted Strategies Index	8	3.8% (27)	0.2% (52)	2.2% (34)	4.4% (51)	−0.301 (20)
27	BRIC Index	7.9	2.7% (34)	3.2% (27)	3% (26)	11.9% (6)	0.326 (56)
28	Event Driven Index	7.8	3.1% (31)	0.5% (50)	1.8% (42)	5.1% (47)	−0.233 (24)
29	ED: Private Issue/Regulation D Index	7.8	4.7% (22)	5% (20)	4.5% (20)	6.4% (30)	−0.111 (31)
30	ED: Distressed Restructuring Index	7.6	2.6% (35)	−2.9% (65)	1.1% (45)	3.7% (55)	−0.327 (17)
31	Relative Value Arbitrage Index	7.5	2.1% (40)	−3.1% (66)	2% (39)	4.5% (48)	−0.254 (23)
32	Equity Hedge Index	7.4	4% (24)	0.1% (53)	1.7% (43)	5.9% (40)	−0.124 (30)
33	MENA Index	7.3	2% (42)	1.5% (39)	−0.2% (57)	7.3% (22)	0.006 (37)
34	Absolute Return Index	6.5	3.6% (28)	0.7% (49)	2% (37)	3.1% (58)	−0.285 (21)
35	ED: Activist Index	6.4	−0.1% (55)	−0.3% (56)	−2.6% (68)	7% (24)	0.05 (38)
36	Macro: Commodity-Agriculture Index	6.3	4% (25)	3.7% (24)	2.9% (28)	3.9% (53)	−0.202 (26)
37	Total Emerging Market Index	6.2	2.1% (41)	2.7% (30)	2% (36)	7.5% (19)	0.109 (46)
38	EH: Technology/Healthcare Index	5.9	4% (26)	4.2% (23)	3% (27)	6.5% (29)	0.051 (39)
39	Brazil Index	5.8	−0.4% (56)	0.9% (45)	−0.6% (59)	6.6% (25)	0.067 (40)
40	RV: Volatility Index	4.7	4.4% (23)	4.4% (22)	3.3% (24)	3.8% (54)	−0.073 (34)
41	Macro: Commodity-Energy Index	4.3	2% (43)	1.9% (37)	1% (47)	3% (60)	−0.109 (32)
42	RV: Yield Alternative Index	3.7	1.1% (50)	2.6% (32)	2.7% (30)	7.3% (23)	0.3 (53)
43	ED: Multi-Strategy Index	3.3	1.1% (48)	1.4% (41)	1.9% (41)	6.6% (26)	0.272 (51)
44	EH: Multi-Strategy Index	3.3	1.1% (49)	1.2% (42)	−1.2% (62)	6.5% (28)	0.271 (50)
45	RV: FI-Corporate Index	3.2	−1% (59)	0% (54)	0.9% (50)	4.3% (52)	0.092 (43)
46	EH: Equity Market Neutral Index	3	2.2% (37)	0.8% (48)	1% (49)	1% (68)	−0.169 (27)
47	Asia Equally Weighted Index	2.6	1.9% (44)	2.4% (34)	2.4% (32)	6.3% (34)	0.308 (54)
48	RV: FI-Sovereign Index	2.5	0.7% (51)	1.4% (40)	0.1% (56)	3.5% (56)	0.081 (41)
49	Asia Composite Hedge Fund Index	2.3	1.5% (47)	2.1% (36)	2.2% (35)	6.1% (36)	0.313 (55)
50	North America Index	2	1.6% (46)	2.1% (35)	0.9% (53)	3.2% (57)	0.104 (45)
51	Aggregate Index	1.6	0.5% (52)	1.1% (44)	1.1% (46)	4.5% (49)	0.235 (49)
52	Multi-Emerging Markets Index	1.3	−1.4% (63)	−0.5% (58)	0.9% (51)	7.6% (17)	0.525 (62)
53	India Index	1	−1.3% (62)	−1.3% (62)	−1.8% (65)	9.2% (10)	0.688 (73)
54	RV: FI-Convertible Arbitrage Index	0.7	−3% (67)	−10.7% (81)	−1.6% (64)	1.9% (63)	0.098 (44)
55	Russia/Eastern Europe Index	0.5	−0.7% (57)	0.3% (51)	1% (48)	8.7% (11)	0.689 (74)
56	Macro/CTA EUR Index	0.4	2.4% (36)	2.8% (29)	0.9% (52)	1.4% (66)	0.083 (42)
57	Western/Pan Europe Index	0.4	1.6% (45)	2.6% (33)	3.2% (25)	6.2% (35)	0.486 (61)
58	EH: Quantitative Directional Index	−0.3	2.2% (39)	2.7% (31)	2.2% (33)	4.4% (50)	0.392 (59)
59	EH: Energy/Basic Materials Index	−0.4	−1.2% (61)	−1.1% (61)	−1.2% (63)	5.2% (46)	0.465 (60)
60	Asia ex-Japan Index	−0.6	0.3% (53)	1.1% (43)	2% (38)	7.6% (18)	0.681 (72)
61	RV: Real Estate Index	−1.9	−2.5% (65)	−2% (64)	−2.8% (71)	1.7% (64)	0.299 (52)
62	Multi-Region Index	−2	0.2% (54)	0.8% (46)	0.3% (55)	2% (62)	0.334 (57)
63	Japan Index	−2.1	−1% (58)	−0.4% (57)	−0.4% (58)	2.6% (61)	0.386 (58)
64	Macro: Currency Index	−2.2	−1.1% (60)	−0.7% (60)	−0.9% (60)	−0.1% (74)	0.174 (48)
65	Alternative Energy Index	−2.4	−4.3% (68)	−3.7% (69)	−3% (74)	6.1% (37)	0.709 (75)
66	EH: Short Bias Index	−2.7	2.9% (32)	3.1% (28)	4.5% (18)	−3.6% (81)	−0.076 (33)
67	Event Driven EUR Index	−3.7	−2.5% (66)	−1.5% (63)	−0.9% (61)	3% (59)	0.558 (64)
68	EH: Fundamental Growth Index	−5.4	−2.2% (64)	−0.7% (59)	0.7% (54)	5.5% (44)	0.914 (80)
69	Equal Weighted Strategies GBP Index	−5.7	−4.5% (69)	−3.5% (68)	−2.5% (67)	0.6% (72)	0.529 (63)
70	Equal Weighted Strategies EUR Index	−6.1	−4.7% (70)	−3.4% (67)	−2.3% (66)	0.8% (70)	0.571 (65)
71	Relative Value Arbitrage EUR Index	−6.2	−7.7% (78)	−6.1% (76)	−2.8% (69)	1.4% (65)	0.636 (67)
72	ED: Special Situations Index	−6.3	−5.1% (72)	−3.9% (72)	−2.8% (72)	1.4% (67)	0.641 (68)

(continued on next page)

Table 2 (*continued*)

#	Hedge fund index	AvDD α (%)	90%-CDaR α	MaxDD α	Standard α	Average return	AvDD β
73	Global Hedge Fund GBP Index	−6.5	−4.9% (71)	−3.9% (70)	−3% (73)	0.6% (71)	0.597 (66)
74	Global Hedge Fund EUR Index	−7.2	−5.1% (73)	−3.9% (71)	−2.8% (70)	0.9% (69)	0.671 (70)
75	Equal Weighted Strategies CHF Index	−9.3	−6.9% (74)	−5.8% (73)	−4.4% (75)	−1.3% (76)	0.669 (69)
76	Equal Weighted Strategies JPY Index	−10.1	−7.7% (79)	−6.6% (78)	−4.9% (77)	−1.9% (79)	0.679 (71)
77	Global Hedge Fund CHF Index	−10.2	−7.4% (77)	−6.3% (77)	−4.9% (76)	−1.3% (77)	0.74 (76)
78	Equity Hedge EUR Index	−10.8	−7% (76)	−5.8% (74)	−5.1% (79)	0% (73)	0.899 (79)
79	Global Hedge Fund JPY Index	−10.9	−8.1% (80)	−7% (79)	−5.5% (80)	−2% (80)	0.742 (77)
80	EH: Fundamental Value Index	−11.5	−7% (75)	−6.1% (75)	−5.1% (78)	−0.5% (75)	0.916 (81)
81	Korea Index	−11.6	−10.2% (81)	−10.2% (80)	−10.2% (81)	−1.7% (78)	0.827 (78)

marks characterizing performance of a large universe of hedge fund strategies. Funds, included into an HFRX index, have either (a) at least $50 million under management or (b) a track record of greater than twelve months. For each hedge fund index, four betas are evaluated: average drawdown (AvDD) beta ($\alpha = 0$), CDaR beta with $\alpha = 0.9$, maximum drawdown (MaxDD) beta ($\alpha = 1$), and the standard beta from the classical CAPM. The S&P500 index is used as a proxy for the market portfolio in place of the optimal portfolio. The results are shown in Table 1. Of special interest are hedge fund indices with negative betas, which, in general, indicate the ability of an index to generate positive returns when the market goes down. The question is whether CDaR betas provide additional information on hedging capabilities, compared to the standard beta. Table 1 is ordered by the AvDD beta (third column). Numbers in brackets next to the 90%-CDaR beta, MaxDD beta, and the standard beta are ranks in the ordering by the corresponding column; e.g. EH: Short Bias Index is at the 33rd place according to the AvDD beta, however, it has the lowest standard beta that puts it on the first place in the ordering by standard beta. Table 1 shows that in the majority of cases, CDaR betas with different confidence levels demonstrate consistent performance, i.e. all three CDaR betas are either positive or negative. Nevertheless, this is not always the case. For instance, RV: Multi-Strategy Index shows the positive MaxDD beta, positive 90%-CDaR beta, and negative AvDD beta (see Fig. 4). This fund has good hedging characteristics and generates positive returns in the majority of cases when the market goes down (AvDD beta is negative). However, the positive MaxDD beta indicates that when the market was in the largest (maximum) drawdown, the fund lost money.

The absolute majority of funds in Table 1 have positive standard betas, which is not encouraging for hedging purposes; still, these funds have the negative AvDD betas, which indicates that, on average, they hedge the market. The conceptual difference between CDaR betas and the standard betas is that the standard beta accounts for the fund returns over the whole return history, including the periods when the market goes up, whereas CDaR betas focus only on market drawdowns and, thus, are not affected when the market performs well.

Obviously, the best hedging fund provides positive payoffs both for light and severe market drawdowns and, therefore, has negative CDaR betas for all confidence levels. The AORDA Portfolio 2 shows superior hedging characteristics and has the most negative CDaR betas for all confidence levels α. Nevertheless, it does not have the lowest standard beta, which was demonstrated by EH: Short Bias Index.

Similar to CDaR betas, CDaR alphas are used to prioritize the performance of 80 hedge fund indices from the HFRX database and are evaluated by (13), where the corresponding CDaR betas are given in Table 1, $\hat{w}_T$ is the historical rate of return of an HFRX index, and $\hat{w}_T^M = 12\%$ (the average long-term annual rate of return of the S&P500 index). Table 2 presents annualized average drawdown (AvDD) alpha ($\alpha = 0$), 90%-CDaR alpha, maximum drawdown (MaxDD) alpha ($\alpha = 1$) and standard alpha for 80 HFRX indices and for the AORDA Portfolio 2. The indices are ordered with respect to the AvDD alpha, but they could also be ordered by other columns. The rank of each index with respect to other alphas is indicated in brackets next to the corresponding value of alpha. The orderings by AvDD alpha and AvDD beta are close with few exceptions. For instance, MLP Index, located in the 4th line of Table 2, has the 4th largest AvDD alpha (20.8%) and only 22nd lowest AvDD beta (−0.278).

4. Conclusions

The necessary optimality conditions for a portfolio problem that either minimizes or constrains the CDaR measure of portfolio rates of return over a given time period have been formulated in the form of CAPM in the single and multiple sample-path settings. The percentage of the worst drawdowns to be accounted for in the CDaR is controlled by the confidence level $\alpha \in [0,1]$ with $\alpha = 1$ and $\alpha = 0$ corresponding to the maximum and average drawdowns, respectively. The CDaR beta and CDaR alpha can be used to prioritize financial instruments, but in contrast to the standard beta based on standard deviation, they can identify instruments that hedge against market drawdowns, which is a critical advantage when hedging is especially needed. For instance, the maximum drawdown beta of −1 indicates that an instrument generated positive return equal to the largest market drawdown over the market peak-to-valley period. The case study, based on the HFRX database of hedge fund indices, has shown that the CDaR beta can be negative, while the standard beta is positive, which provides an alternative view on hedging capabilities of instruments.

Acknowledgment

The authors are grateful to the anonymous referees for their valuable comments and suggestions, which helped to improve the quality of the paper.

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