Risk Management with POE, VaR, CVaR, and bPOE

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Lower Bound vs. Average of Tail

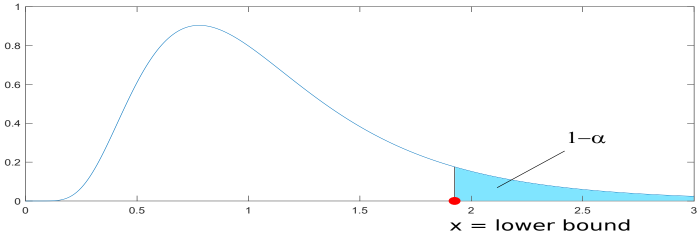
Presentation compares two concepts for measuring and optimization of tails of probabilistic distributions

- **Optimistic concept** based on **lower bound** of outcomes in the tail
- **Conservative concept** based on **average value** of the tail

Concept 1: Lower Bound of the Tail

Two equivalent variants:

- Fix threshold x, which is the lower bound of outcomes in the tail, and constrain Probability of Exceedance (POE): $p(x) \leq 1 - \alpha$
- Fix probability of the tail, $1 - \alpha$, and constrain quantile, called Value-at-Risk (VaR) in finance: $q(\alpha) \leq x$

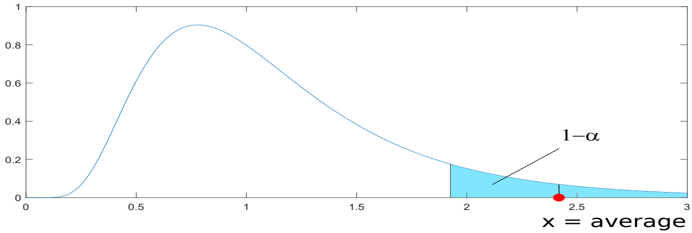


$$q(\alpha) \leq x \quad \Longleftrightarrow \quad p(x) \leq 1 - \alpha$$

Concept 2: Average Value of the Tail

Two equivalent variants:

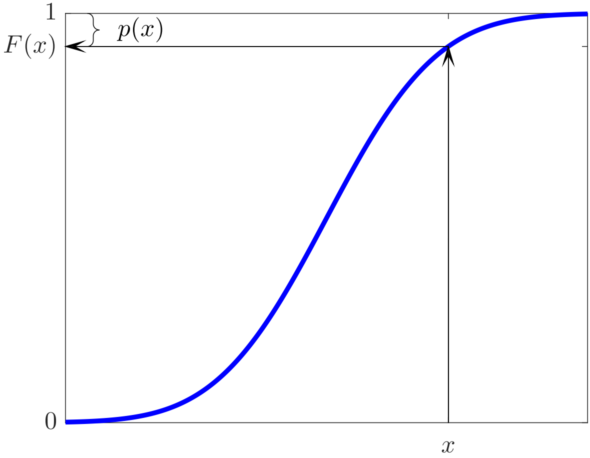
- Fix mean value of the tail, x, and constrain tail probability, called Buffered Probability of Exceedance (bPOE): $\bar{p}(x) \leq 1 - \alpha$
- Fix probability of the tail, $1 - \alpha$, and constrain mean value of the tail, called Superquantile, Average VaR, Tail VaR, CVaR, and Expected Shortfall in finance: $\bar{q}(\alpha) \leq x$



$$\bar{q}(\alpha) \leq x \quad \Longleftrightarrow \quad \bar{p}(x) \leq 1 - \alpha$$

CDF and POE

- X = random "loss"
- Cumulative Distribution Function (CDF) $= F(x) = \mathbb{P}\{X \leq x\}$
- Probability of Exceedance (POE) $= p(x) = \mathbb{P}\{X > x\} = 1 - F(x)$, also known as Survival, Survivor, or Reliability function.



Risk Management with POE and CDF

Requirement: probability that loss exceeds threshold x is small

$$p(x) \leq 1 - \alpha \quad \text{e.g., } 1 - \alpha = 1 - 0.95 = 0.05$$

- Nuclear: probability that release of radiation exceeds some level
- Finance: default probability of a company (Assets-Liability < 0)

Equivalently: probability that loss is below threshold x is large

$$p(x) = 1 - F(x) \leq 1 - \alpha \quad \Longrightarrow$$

$$F(x) \geq \alpha \quad \text{e.g., } \alpha = 0.95$$

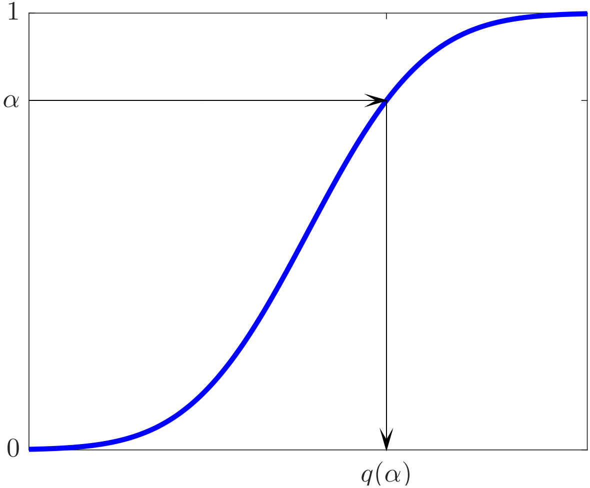
- Material Science:
  material should withstand the load x with high probability

Quantile (VaR in finance)

Quantile $q(\alpha)$ is inverse of CDF.

Quantile is a solution of equation $F(x) = \alpha$, i.e. $F(q(\alpha)) = \alpha$.

Quantile is a solution of equation $p(x) = 1 - \alpha$, i.e., $p(q(\alpha)) = 1 - \alpha$.



Risk Management with Quantiles (VaR)

Requirement: Quantile with confidence α is less than some threshold

$$q(\alpha) \leq x$$

- Finance: e.g., VaR for daily loss is below $1 billion

Equivalence of POE and Quantile Constraints

Some engineering areas use POE other areas use Quantiles.

Constraints on POE and quantiles are equivalent. It is a matter of convenience.

Finance uses quantiles (Value-at-Risk or VaR) specified in USD.

Nuclear engineering uses POE, maybe because probabilities are more understandable to people than radiation dosages.

$\text{POE}(x) \leq 1 - \alpha \quad \Longrightarrow \quad \text{quantile}(\alpha) \leq x$

Continuous and strictly increasing CDF

$$p(x) \leq 1 - \alpha$$

$$\Longrightarrow \quad F(x) \geq \alpha$$

$$\Longrightarrow \quad F^{-1}(F(x)) \geq F^{-1}(\alpha) = q(\alpha)$$

$$\Longrightarrow \quad x \geq q(\alpha)$$

$$\Longrightarrow \quad q(\alpha) \leq x$$

$\text{quantile}(\alpha) \leq x \quad \Longrightarrow \quad \text{POE}(x) \leq 1 - \alpha$

Continuous and strictly increasing CDF

$$q(\alpha) = F^{-1}(\alpha) \leq x$$

$$\Longrightarrow \quad F(F^{-1}(\alpha)) \leq F(x)$$

$$\Longrightarrow \quad \alpha \leq F(x)$$

$$\Longrightarrow \quad \alpha \leq 1 - p(x)$$

$$\Longrightarrow \quad p(x) \leq 1 - \alpha$$

POE and Quantiles: Poor Properties

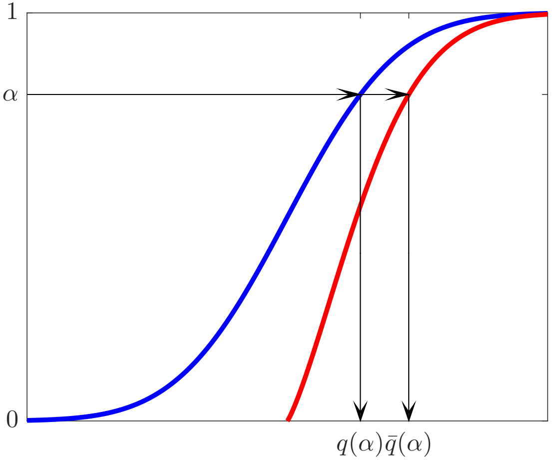
POE and Quantile have poor mathematical properties:

- nonconvex in random variable
- discontinuous for discrete distributions w.r.t. parameters
- difficult to manage (optimize)
- are not conservative: do not take into account the values of outcomes in the tail of the distribution

Superquantile (CVaR) vs Quantile (VaR)

Superquantile $\bar{q}(\alpha)$ = average of the tail in excess of quantile (VaR)

$\bar{q}(\alpha)$ = inverse of $\bar{F}(x)$ which is CDF of Superdistribution (red curve)



Superquantile (CVaR) Properties

Formal Superquantile (CVaR) definition:

continuous distributions:

$$\bar{q}(\alpha) = \mathbb{E}\{X | X > q(\alpha)\}$$

general (including discrete) distributions:

$$\bar{q}(\alpha) = \frac{1}{1-\alpha} \int_{\alpha}^{1} q(\alpha)\, d\alpha = \min_{C} \{C + \frac{1}{1-\alpha} \mathbb{E}[X - C]^{+}\},$$

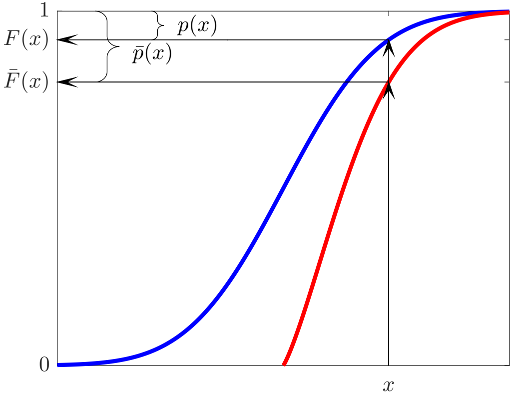
where $[X - C]^{+} = \max\{0, X - C\}$

- takes into account values of outcomes in the tail of the distribution
- coherent risk measure (the best from theoretical perspective)
- convex in random variable
- continuous w.r.t. parameters
- easy to manage and optimize with convex and linear programming, (Rockafellar & Uryasev (2000))

bPOE vs POE

Buffered Probability of Exceedance (bPOE) $= 1 - \bar{F}(x) = 1 - \alpha$, where α satisfies equation $\bar{q}(\alpha) = x$.

Superdistribution $\bar{F}(x)$ (Rockafellar & Royset (2013)). Special case of bPOE with $x = 0$ (Rockafellar & Royset (2010)). General bPOE case and optimization representation (Norton & Uryasev (2014), Mafusalov & Uryasev (2014)).

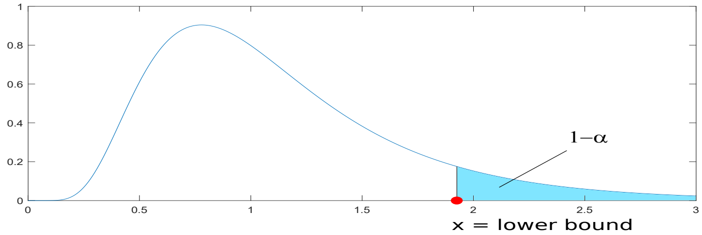


bPOE properties

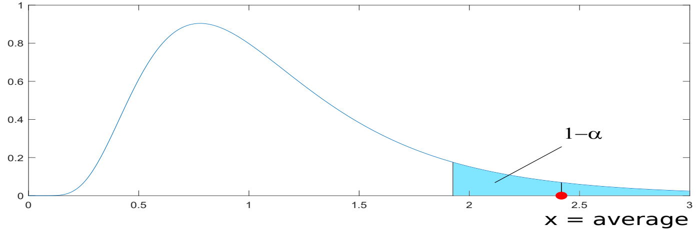
bPOE: will be a new hit in risk management, similar to CVaR

- optimization representation: $\bar{p}(x) = \min_{a \geq 0} \mathbb{E}[a(X - x) + 1]^+$
- takes into account values of outcomes in the tail of the distribution
- quasi-convex in random variable X
- lowest quasi-convex (in X) upper bound of POE
- bPOE is about twice bigger than POE
- continuous w.r.t. parameters
- easy to manage (optimize with convex and linear programming)
- $\bar{q}(\alpha) \leq x \quad \Longleftrightarrow \quad \bar{p}(x) \leq 1 - \alpha$

Low Bound vs. Average of Tail



$$q(\alpha) \leq x \quad \Longleftrightarrow \quad p(x) \leq 1 - \alpha$$



$$\bar{q}(\alpha) \leq x \quad \Longleftrightarrow \quad \bar{p}(x) \leq 1 - \alpha$$

Risk Management in Different Fields

$p(x) \le 1 - \alpha$	nuclear, material, finance
$q(\alpha) \le x$	finance
$\bar{q}(\alpha) \le x$	finance
$\bar{p}(x) \le \alpha$	optimization of large physical systems

Example: bPOE Minimization

- $L(z) = c_0 + \sum_{i=1}^{n} c_i z_i$ is a linear function w.r.t. $z = (z_1, ...z_n)$ with random coefficients $(c_0, c_1, ..., c_n)$

- minimize bPOE of $L(z)$ w.r.t. z

$$\min_{z} \bar{p}(x, L(z)) = \min_{z} \min_{a \geq 0} \mathbb{E}\,[\, a(L(z) - x) + 1]^{+}$$

$$= \min_{z, a \geq 0} \mathbb{E}\,[\, a(c_0 + \sum_{i=1}^{n} c_i z_i - x) + 1]^{+}$$

$$= \min_{z, a \geq 0} \mathbb{E}\,[\, (c_0 - x)a + \sum_{i=1}^{n} c_i a z_i + 1]^{+}$$

$$= \min_{y, a \geq 0} \mathbb{E}\,[\, (c_0 - x)a + \sum_{i=1}^{n} c_i y_i + 1]^{+}$$

- change of variables $az -> y$ reduces the problem to convex and linear proramming w.r.t. variables y, a