Optimal Allocation of Retirement Portfolios

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Abstract

A retiree with a savings account balance, but without a pension is confronted with an important investment decision that has to satisfy two conflicting objectives. Without a pension the function of the savings is to provide post-employment income to the retiree. At the same time, most retirees will want to leave an estate to their heirs. Guaranteed income can be acquired by investing in an annuity. However, that decision necessarily takes funds away from investment alternatives that might grow the estate. The decision is made even more complicated because one does not know how long one will live. A long life expectancy would suggest more annuities, and short life expectancy will immediately promote more risky investments. However there are very mixed opinions about either strategy. A framework has been developed to assess consequences and the trade-offs of alternative investment strategies. We propose a stochastic programming model to frame this complicated problem. The objective is to maximize expected terminal net worth (the estate) subject to CVaR constraints on target income shortfalls. Objective is calculated using probabilities of scenarios of returns of invested instruments and mortality probabilities. The CVaR constraints are applied each year of the portfolio investment horizon. We consider that the investment strategy is running for the whole investment horizon and the CVaR constraints should be satisfied for each year (to guaranty need cash flows for survived individuals). We use kernel functions to build position adjustment functions that control how much is invested in each asset. These adjustments nonlinearly depend upon on asset returns in previous years. Case study was conducted using two variations of the model. The parameters used in this case study correspond to typical retirement situation. The results of the case study show that if the market forecasts are pessimistic, it is optimal to invest in annuity. The case study results, codes, and data are posted at the website.

1 Introduction

The problem of selecting optimal portfolios for retirement has unique features that are not addressed by more commonly used portfolio selection models used in trading. One distinct feature of a retirement portfolio is that it should incorporate the life span of the investor. The planning horizon depends on the age of investor, or more specifically, on a conditional life expectancy of the investor. Another important feature is to guarantee, in some sense, that the individual will be able to withdraw some amount of money every year from a portfolio by selling some predefined amount of assets without injecting external funds. Finally, one of the questions that the models tries to answer is, in what situation is it beneficial to invest in annuity instead of more risky assets.

Most of the literature around portfolio optimization considers generic portfolios that focus either on expected profit maximization with some risk constraints or other way around, risk minimization with budget constraints. The famous mean-variance (or Markowitz) portfolio Markowitz

[1952] maximizes the expected return of a portfolio while constraining the variance of the portfolio. The original paper by Markowitz was published in 1952 and since that time portfolio optimization has been a subject of active research. There are a couple of directions that extend the original mean-variance portfolio and deal with its shortcomings. One direction has been to substitute variance with some other measure that captures a nature of risk better. Variance measures both positive and negative deviations of the portfolio returns, however investors are concerned only about negative deviations. In papers Rockafellar and Uryasev [2000, 2002] and Krokhmal et al. [2002] authors use Conditional Value at Risk instead of standard deviation as a risk measure. CVaR is a convex functional and therefore problems involving CVaR can be solved efficiently in many cases. Another frequently used risk measure is drawdown, which also leads to tractable problems (for more on drawdown see Chekhlov et al. [2003]). Other direction of extending the portfolio theory focuses on multistage models. In multistage models the decision to invest is made on multiple time points in the future. Multistage models can be formulated as the stochastic optimization problems, however the number of decision variable increases very fast with the number time periods considered. Therefore, frequently this type of models can not be solved for practical time horizons. In order to avoid ”curse of dimensionality” Calafiore [2008] models the investment decisions as linear functions that remain same across all scenarios and produce the investment decision based on previous performance of the asset. A paper by Takano and Gotoh [2014] models the decision function as the Kernel method, that results in a nonlinear control functions.

The model that is developed in this paper assumes that the investor wishes to maximize the terminal wealth while maintaining predefined cash outflows each year. The mortality tables are used to weight the portfolio value in each year in the objective function. The cash outflow requirements are formulated as the Conditional Value at Risk (CVaR) constraints. The model developed in this paper follows the ideas in Takano and Gotoh [2014] and models multistage portfolio using kernel methods. The investment horizon is 35 years, starting from the retirement of the investor at the age of 65. The objective is to maximize the discounted terminal wealth. We model terminal wealth as the weighted average of the discounted expected portfolio values in each scenario, where the probabilities of death are used as weights. The probability of death is calculated from the U.S. mortality table. Along with the maximum terminal wealth requirement, the investor wants to have predetermined and stable cash outflows from the portfolio that are the result of selling a portion of the portfolio. In this paper we develop 2 versions of the portfolio model, that differ in the way they treat the requirement for the cash outflow from the portfolio. The first version imposes CVaR constraints on the difference between required cash outflow and actual cash outflow (portfolio shrinkage). This constraint allows portfolio model to not provide the required cash flows on a small number of scenarios if necessary. The second model puts monotonicity constraint on the cash outflows from the portfolio. The monotonicity constraint forces the model to provide required cash continuously until the end of investment horizon or until the portfolio value drops to 0. The benefit of having monotonic cash outflows comes with the cost of smaller terminal wealth, however some retirees might prefer this strategy. We conducted a case study that corresponds to a typical investment decision upon retirement, in order to see the conditions in which it is preferable to invest in annuity. The results show that if the required cash outflows are kept low (compared to initial investment) and market scenarios are not pessimistic then both models can easily provide necessary funds without investing anything in the annuities. In the pessimistic scenario we simulate the future market evolution as in the ordinary case, however in this simulation we subtract 12% from all growth rates in the scenarios. As a result both models invest heavily in the annuities.

2 Notations

- $N :=$ number of assets available for investment,
- $S :=$ number of scenarios,
- $T :=$ portfolio investment horizon,
- $\mathcal{N} :=$ the index set of all assets considered in the portfolio, $\mathcal{N} = \{1, \ldots, N\}$,
- $\mathcal{S} :=$ the index set of all scenarios of the market movements, $\mathcal{S} = \{1, \ldots, S\}$,
- $\mathcal{T} :=$ the index set of all time periods, starting from the retirement until the end of investment horizon, $\mathcal{T} = \{1, \ldots, T\}$,
- $r_{i,t}^s :=$ growth rate of asset $i \in \mathcal{N}$ during period $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$, the vector form of returns is denoted as $\boldsymbol{r}_t^s = (r_{1,t}^s, \ldots, r_{N,t}^s)$,
- $\boldsymbol{v}_t^s = \{\boldsymbol{r}_1^s, \ldots, \boldsymbol{r}_t^s\} :=$ the set of all previous growth rates observed until the end of period t in scenario s,
- $d_t^s :=$ discount factor at time t in scenario s; discounting is done using inflation rate ρ_t^s, $d_t^s = 1/(1+\rho_t^s)^t$
- $p_t :=$ probability that a person will die at the age $65 + t$ (conditional that he is alive at the age of 65),
- $\boldsymbol{y}_{i,t} :=$ vector of control variables for investment adjustment function.
- $f(\boldsymbol{v}_t^s, \boldsymbol{y}_{i,t}) :=$ investment adjustment function. This function controls how much investment is made in each scenario s in asset i at the end of period t,
- $G(\boldsymbol{y}_{i,t}) :=$ regularization function of control parameters,
- $K(\boldsymbol{v}_t^s, \boldsymbol{v}_t^k) :=$ positive definite kernel function, $k \in \mathcal{S}$,
- $x_{i,t}^s :=$ investment amount to i-th asset at time t in scenario s,
- $x_i :=$ investments to i-th asset at time $t = 0$,
- $u_{i,t}^s :=$ adjustment for asset i at the beginning of period t in scenario s,
- $u_{i,t} :=$ adjustment for asset i at period t calculated with information available at $t = 0$,
- $R_t^s :=$ total change in the portfolio value from asset adjustments at time t in scenarios s,
- $L_{i,t} :=$ lower bound on position in asset i at time t as a fraction of portfolio value ($L_{i,t} \in [0, 1]$),
- $U_{i,t} :=$ upper bound on position in asset i at time t as a fraction of portfolio value ($U_{i,t} \in [0, 1]$),
- $V_0 :=$ value of the portfolio at time $t = 0$ (initial investment),
- $V_t^s :=$ value of the portfolio at time t in scenario s,

- z := investment in an annuity (in dollars),
- A_t^s := Yield of the annuity at time t in scenario s,
- l_t := amount of money that the portfolio holder is planing to withdraw as each time t,
- α_t := confidence level of CVaR at time t, $\alpha_t \in [0, 1)$,
- λ := regularization coefficient, $\lambda > 0$.

3 Model Formulation

This section develops a general model for a retirement portfolio selection. We consider a portfolio including stocks indexes, bonds indexes, and an annuity. The annuity pays amount $A_t^s z$ at each period t and does not contribute funds to the terminal wealth. Annuity is bought at time $t = 0$ and can not be bought or sold after that moment. Given initial investments in the assets x_i, the dynamics of investments in stocks and bonds are as follows

$$\begin{aligned} x_{i,1}^s &= (1 + r_{i,1}^s)(x_i + u_{i,1}^s), \\ x_{i,t}^s &= (1 + r_{i,t}^s)(x_{i,t-1}^s + u_{i,t}^s). \end{aligned} \tag{1}$$

Variables $u_{i,t}^s$ control how much is invested at the end of each period in each asset. The variable $u_{i,t}^s$ is defined as

$$u_{i,t}^s = u_{i,t} + f(\boldsymbol{v}_t^s, \boldsymbol{y}_{i,t}), \tag{2}$$

where $\boldsymbol{v}_t^s$ is set of all growth rates for all assets i, until time t, in scenario s, and $\boldsymbol{y}_{i,t}$ are some parameters controlling the shape of the function f. Therefore $u_{i,t}^s$ are some nonlinear transformations of the previous growth rates of assets. The explicit form of function f is unspecified in this section. The only requirement on function f is that it should be linear in $\boldsymbol{y}_{i,t}$, or

$$f(\boldsymbol{v}_t^s, \beta\hat{\boldsymbol{y}}_{i,t} + \gamma\bar{\boldsymbol{y}}_{i,t}) = \beta f(\boldsymbol{v}_t^s, \hat{\boldsymbol{y}}_{i,t}) + \gamma f(\boldsymbol{v}_t^s, \bar{\boldsymbol{y}}_{i,t}),$$

where $\beta, \gamma \in \mathbb{R}$ and $\hat{\boldsymbol{y}}_{i,t}, \bar{\boldsymbol{y}}_{i,t}$ are some control variables. The linearity requirement for function f is necessary so that the portfolio model is solvable using convex programming.

By R_t^s we denote the total change in the portfolio value at time t in scenario s, resulting from buying and selling assets (alternatively, portfolio value can change due to the growth of individual assets value). R_t^s is equal to the sum total of the adjustments for a given period t and scenario s

$$\sum_{i=1}^{N} u_{i,t}^s = R_t^s, \tag{3}$$

The value of the portfolio at time t and scenario s equals

$$V_t^s = \sum_{i=1}^{N} x_{i,t}^s. \tag{4}$$

We consider upper and lower bounds on investment in each asset i at time t. The value of the portfolio at the time of re-balancing in scenario s is $V_t^s = V_{t-1}^s + R_t^s$. The investment in asset i at time t must satisfy the following lower and upper bounds

$$\begin{aligned} x_{i,t-1}^s + u_{i,t}^s &\geq L_{i,t}(V_{t-1}^s + R_t^s), \\ x_{i,t-1}^s + u_{i,t}^s &\leq U_{i,t}(V_{t-1}^s + R_t^s). \end{aligned} \tag{5}$$

The objective is to maximize terminal wealth of a portfolio. Terminal wealth of the portfolio is the weighted average of the discounted expected portfolio values in each scenario, where the probabilities of death are used as weights. The portfolio value at t is discounted to time 0 using inflation as the discount rate. In order to avoid over-fitting the data, we included the regularization term $G(\boldsymbol{y}_{i,t})$ in the objective function. The objective function is

$$-\frac{1}{S}\sum_{t=1}^{T}\sum_{s=1}^{S} p_t d_t^s V_t^s + \lambda \sum_{i=1}^{N}\sum_{t=1}^{T} G(\boldsymbol{y}_{i,t}). \tag{6}$$

The function G is a convex function.

Because $G(\boldsymbol{y}_{i,t})$ is a convex function by assumption then (6) is also a convex function in $\boldsymbol{y}_{i,t}$ and linear in V_t^s.

Let X be some random variable. We measure risk of X using Conditional Value at Risk (CVaR) defined as

$$\mathrm{CVaR}_\alpha(X) = \min_\zeta \left(\zeta + \frac{1}{1-\alpha}\mathbb{E}[X-\zeta]^+\right) \quad \text{for } \alpha \in [0,1),$$

where $[x]^+ = \max(x,0)$, $\alpha \in [0,1)$ and $\zeta \in \mathbb{R}$. For a fixed number S of equally probable scenarios and corresponding random variable realizations X_s the CVaR_α equals

$$\mathrm{CVaR}_\alpha(X) = \min_\zeta \left(\zeta + \frac{1}{S(1-\alpha)}\sum_{s=1}^{S}[X_s-\zeta]^+\right) \quad \text{for } \alpha \in [0,1).$$

For a comprehensive analysis of the $\mathrm{CVaR}_\alpha(X)$ risk measure see Rockafellar and Uryasev [2002, 2000].

The cash outflow from the portfolio occurs when a portion of the portfolio is sold. Because R_t^s is the sum of all adjustments, the outflow from portfolio in dollars, equals $-R_t^s$. Therefore, the amount of money that the investor receives from the portfolio and annuity at time t, in scenario s, is equal to

$$A_t^s z - R_t^s. \tag{7}$$

If this number is less than l_t then there is a shortage of money, meaning that the investor did not receive the amount of money he wishes from the portfolio. We impose CVaR constraint on (7) with confidence level α_t at time t,

$$\min_{\zeta_t}\left(\zeta_t + \frac{1}{S(1-\alpha_t)}\sum_{s=1}^{S}[R_t^s - A_t^s z - \zeta_t]^+\right) \leq -l_t. \tag{8}$$

Note that the CVaR constraint is imposed on $-(A_t^s z - R_t^s)$, this formulation defines a convex feasible region.

The objective is to maximize regularized portfolio value (6) while satisfying the constraints (1) to (5) and (8). Finally we arrive to the following optimization problem

$$\min_{\substack{u_{i,t},u^s_{i,t},R^s_t,\\ V_0,V^s_t,\boldsymbol{y}_{i,t},\\ x^s_i,x^s_{i,t},z,\zeta_t}} \quad -\frac{1}{S}\sum_{t=1}^{T}\sum_{s=1}^{S} p_t d^s_t V^s_t + \lambda \sum_{i=1}^{N}\sum_{t=1}^{T} G(\boldsymbol{y}_{i,t}) \tag{9}$$

$s.t.$

$$\begin{aligned}
&\zeta_t + \frac{1}{S(1-\alpha_t)}\sum_{s=1}^{S}[R^s_t - A^s_t z - \zeta_t]^+ \leq -l_t \qquad t \in \mathcal{T}\\
&x^s_{i,1} = (1+r^s_{i,1})(x_i + u^s_{i,1}) \qquad i \in \mathcal{N}; \quad s \in \mathcal{S}\\
&x^s_{i,t} = (1+r^s_{i,t})(x^s_{i,t-1} + u^s_{i,t}) \qquad i \in \mathcal{N}; \quad t \in \mathcal{T}\backslash\{1\}; \quad s \in \mathcal{S}\\
&\sum_{i=1}^{N} x_i = V_0 - z\\
&V^s_t = \sum_{i=1}^{N} x^s_{i,t} \qquad t \in \mathcal{T}; \quad s \in \mathcal{S}\\
&\sum_{i=1}^{N} u^s_{i,t} = R^s_t \qquad t \in \mathcal{T}; \quad s \in \mathcal{S}\\
&L_{i,1}(V_0 + R^s_1) \leq x_i + u^s_{i,1} \leq U_{i,1}(V_0 + R^s_1) \qquad i \in \mathcal{N}\\
&L_{i,t}(V^s_{t-1} + R^s_t) \leq x^s_{i,t-1} + u^s_{i,t} \leq U_{i,t}(V^s_{t-1} + R^s_t) \quad i \in \mathcal{N};\ t \in \mathcal{T}\backslash\{1\};\ s \in \mathcal{S}\\
&u^s_{i,t} = u_{i,t} + f(\boldsymbol{v}^s_t, \boldsymbol{y}_{i,t}) \qquad i \in \mathcal{N}; \quad t \in \mathcal{T}; \quad s \in \mathcal{S}\\
&z \geq 0\\
&x_i \geq 0 \qquad i \in \mathcal{N} \qquad\qquad (10)\\
&x^s_{i,t} \geq 0 \qquad i \in \mathcal{N}; \quad t \in \mathcal{T}; \quad s \in \mathcal{S}
\end{aligned}$$

In model (9), only CVaR constraints control the cash outflow from the portfolio. We will refer to this model as "CVaR-only" model. The CVaR constraints allow the cash outflows to be smaller then predefined amount l_t, however this will happen on a very small number of scenarios (approximately on 1/2 α% of the scenarios). The numerical experiments showed that the model with only CVaR constraints exhibits an interesting behavior. When the model is loosing money, due to the unfortunate market movements, on certain scenarios, the adjustment functions do not pay the required amount l_t and reinvest it in the portfolio (if CVaR constraint allows it). As a result of this behavior, there is a higher chances that the investor will be able to withdraw the necessary amount l_t in the future time moments. This behaviour might not be ideal for investors who prefer having continuous cash outflows until the portfolio value shrinks to 0.

In order to account for investors that prefer continuity in their cash outflows, we add monotonicity constraint on R^s_t values, in the model (9). The monotonicity constraint,

$$R^s_t - R^s_{t-1} \leq 0, \tag{11}$$

will not allow the reinvestment behavior mention previously. We will refer to this model as "CVaR plus monotonicity" model. The constraint (11) forces each cash outflow at time t, from the portfolio, to be no grater than the previous cash outflow at time $t-1$. So, if the model does not provide full amount l_t at some time moment t, then it can not provide full amount in any subsequent moment in that scenario, even if there is enough equity in the portfolio. Therefore, this constraint invalidates the reinvestment strategy and forces continuity on the cash outflows.

4 Specific Formulation

In this section we choose a specific form for the functions $G(\boldsymbol{y}_{i,t})$ and $f(\boldsymbol{r}_t^s, \boldsymbol{y}_{i,t})$. This model is similar to the model developed in Takano and Gotoh [2014]. Let $K(\boldsymbol{v}_t^s, \boldsymbol{v}_t^k)$ be the kernel function defined as follows

$$K(\boldsymbol{v}_t^s, \boldsymbol{v}_t^k) = \exp\Bigg(-\sigma \sum_{j=1}^{N} \sum_{l=1}^{t-1} (r_{j,l}^k - r_{j,l}^s)^2 \Bigg), \tag{12}$$

where $\sigma > 0$ is some constant. Using (12) the control function has the form

$$f(\boldsymbol{v}_t^s, \boldsymbol{y}_{i,t}) = \sum_{j=1}^{S} y_{i,t}^j K(\boldsymbol{v}_t^s, \boldsymbol{v}_t^j), \text{ where } \boldsymbol{y}_{i,t} = (y_{i,t}^1, \ldots, y_{i,t}^S). \tag{13}$$

This function is linear in $\boldsymbol{y}_{i,t}$ as required. By substituting (13) in constraint (2), we get the following adjustment functions

$$u_{i,t}^s = u_{i,t} + \sum_{j=1}^{S} y_{i,t}^j K(\boldsymbol{v}_t^s, \boldsymbol{v}_t^j) \qquad i \in \mathcal{N}; \quad t \in \mathcal{T}; \quad s \in \mathcal{S}. \tag{14}$$

We choose L2 norm as the regularization function $G(\boldsymbol{y}_{i,t})$,

$$G(\boldsymbol{y}_{i,t}) = ||\boldsymbol{y}_{i,t}||_2^2 = \sum_{s=1}^{S} (y_{i,t}^s)^2. \tag{15}$$

Substituting (15) in the objective, we get

$$-\frac{1}{S} \sum_{t=1}^{T} \sum_{s=1}^{S} p_t d_t^s V_t^s + \lambda \sum_{i=1}^{N} \sum_{t=1}^{T} ||\boldsymbol{y}_{i,t}||_2^2. \tag{16}$$

Using the above formulations, problem (9) becomes a convex quadratic programming problem. Other formulations are also possible. For example using L1 norm instead of L2 norm in (15) leads to linear programming formulation. Another variation of of this model could be to use linear functions instead of the kernel function for the investment adjustments. Linear investment adjustments will lead to lower terminal wealth, at least on the scenarios that are used to fit the model. However the dimensionality of the problem will be reduced significantly, because the problem size (number of parameters that need to be estimated) will increase linearly with the number of scenarios, instead of quadratically, which is the case with kernel functions.

5 Market Scenarios, Inflation, and Mortality Tables

We simulate the scenarios of market evolution for T years in the future. This simulations are based on end-of-year data of N assets over $\bar{T}$ years. Let $\bar{t} \in \{1, \ldots, \bar{T}\}$ be a year index for the historical

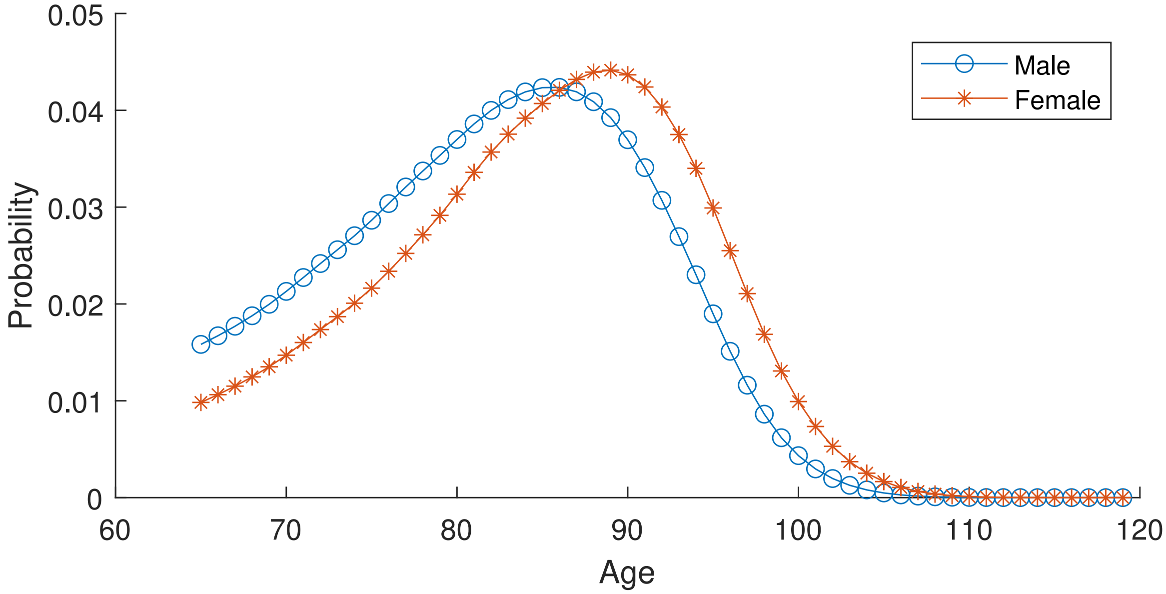


Figure 1: Probabilities $\bar{p}(x)$ that person dies at the given age x, conditional that he/she is alive at the age 65.

dataset and $\bar{r}_{i,\bar{t}}$ be the historical return of asset i. The returns of the indexes are represented as $N \times \bar{T}$ matrix

$$R = \begin{bmatrix} \bar{r}_{1,1} & \bar{r}_{2,1}, & \dots, & \bar{r}_{N,1} \\ \bar{r}_{1,2} & \bar{r}_{2,2}, & \dots, & \bar{r}_{N,2} \\ \dots & \dots & \dots & \dots \\ \bar{r}_{1,\bar{T}} & \bar{r}_{2,\bar{T}}, & \dots, & \bar{r}_{N,\bar{T}} \end{bmatrix} \tag{17}$$

We generate the sample paths (scenarios) with the historical simulation method also known as "Bootstrap" method. Historical simulation method samples a random row from matrix (17) and uses this row as a possible future realization of the market. Therefore the future simulations of the market are just sampling of matrix (17) with replacement. Each such sample is a random time series that represent a future dynamics of return of the assets. Note that the simulation method samples entire row from matrix (17), therefore the correlations among assets are maintained in the random sample.

The model requires probabilities that the investor (retiree) will pass away at a given time t. Obviously this probabilities depend on many factors, however for this model we chose to use average probabilities based on demographic data. We use mortality table of USA (Table 1) for year 2017, in order to estimate this probabilities. Mortality table give probability that a person who is x years old will die within a year, more specifically $\hat{p}(x) = \mathbb{P}(\text{age of death} \leq x + 1 \mid \text{age} = x)$. We calculate the probability that a person dies at the given age conditional that he/she is 65 years old, or $\bar{p}(x) = \mathbb{P}(\text{age of death} = x \mid \text{age} = 65)$. The formula for $\bar{p}(x)$ is following

$$\bar{p}(x) = \hat{p}(x) \prod_{i=65}^{x} (1 - \hat{p}(i-1)) \tag{18}$$

Figure 1 shows the the function $\bar{p}(x)$.

	$\hat{p}$(age)	
Age	Male	Female
65	0.0158	0.0098
66	0.0170	0.0107
...	...	...
119	0.8820	0.8820

Table 1: USA Mortality table . This table shows the probabilities of death for Male and Female US citizens. Specifically this table shows the probability of death for a person who is x years old, during the next one year, or $\hat{p}(x) = \mathbb{P}(\text{age of death} \leq x + 1 \mid \text{age} = x)$

6 Case Study

This case study considers following investment scenario

- The retiree is 65 years old.
- Investment horizon is 35 years.
- The retiree is a male.
- The retiree has $500,000 available for investment .
- Yearly inflation rate is 3%.
- Yearly rate of return on annuity is 3%.
- The retiree wishes to have $ 15,000 at the end of each year.
- CVaR confidence level (α) is 90% during the entire investment horizon.
- Kernel functions are used as the adjustment rules, $\sigma = 1$ is chosen as a kernel parameter.

10 stock and bond indexes are considered for investment, these are

Index Name	Index Abbreviation
Barclays Muni	FI-MUNI
Barclays Agg	FI-INVGRD
Russell 2000	USEQ-SM
Russell 2000 Value	USEQ-SMVAL
Russell 2000 Growth	USEQ-SMGRTH
S&P 500	USEQ-LG
S&P 400 Mid Cap	USEQ-MID
S&P Citi 500 Value	USEQ-LGVAL
S&P Citi 500 Growth	USEQ-LGGRTH
MSCI EAFE	NUSEQ

Table 2: The list of assets that are considered for the retirement portfolio.

The performances of both, CVaR-only model (model (9)) and the model with monotonicity constraints are evaluated. Both models are fitted for λ values of 1, 0.1 and 0.01. The performance is calculated for the out-of-sample datasets. The out-of-sample dataset is generated in the same way as the training set, however it is not used to fit the model. Additionally, the model is fitted for a stress scenario, where for each asset, on each scenario we subtract 12% from the original growth rate. The stress scenario is generated in order to see how the model allocates funds into the annuity. The average yield of the assets that we consider for the portfolio, is more than 3%, that the annuity offers. If the market is expected to grow more than 3% annually, over the investment horizon, it is not beneficial to invest in the annuity. Therefore, if the same growth that was observed previously for the assets, will continue in the future, the annuity is not an optimal choice. It makes sense to buy the annuity, only if the market outlooks are pessimistic.

The model is fitted on 100 scenarios of stock movements, generated using the historical simulation method. We used PSG version 2.3 to solve the resulting quadratic optimization problem.

In order to test the performance of the model on external data, we generate new set of 100 samples using the Bootstrap procedure. These new simulations are then plugged into the kernel functions of CVaR-only and CVaR plus monotonicity models. Evaluating the kernel functions on the new dataset gives the adjustment values for each asset at each time moment t and each scenario s. From these adjustments, the evolution of the portfolio value, as well as the cash outflows are constructed. Tables 3, 4 and 5 show the average (over scenarios) investments in assets over time for the CVaR-only model for the out-of-sample dataset. Tables 6, 7 and 8 show average investments, for λ values of 1, 0.1 and 0.01 respectively for the CVaR plus monotonicity model, on the out-of-sample dataset. If we compare the average investment values of CVaR-only and CVaR plus monotonicity models,we will see that the investments are not significantly different. This is because of the low outflow requirements, both portfolio models have no problem providing necessary funds on each scenario and time moment.

Figures 9 and 10 show the average (by scenarios) portfolio value in each year for different regularization parameters, for CVaR only and CVaR plus monotonicity models respectively.

From tables 3 to 6 we observe that the kernel functions sometimes invest in short positions. In fact the smaller the regularization parameter bigger the short positions. During the fitting process the positions are constrained to be positive (long only).

The results for the stress scenario are given in tables 11 and 12 for the CVaR-only and CVaR plus monotonicity models respectively. Evidently both CVaR-only and CVaR plus monotonicity formulations give the same answer in terms of average portfolio value. In both cases, 84% of the initial \$500 000 is invested in the annuities. Therefore \$12 580 out of required \$15 000 is financed with annuities each year.

All models were able to provide required funding of \$15 000 in each scenario.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	27	24	35	70	72	99	562
FI-INVGRD	0	-1	-1	-3	-4	-5	-7	1
USEQ-SM	0	1	1	1	2	4	7	36
USEQ-SMVAL	0	1	0	0	1	2	5	61
USEQ-SMGRTH	0	29	77	117	197	339	550	1318
USEQ-LG	0	0	-1	-2	-5	-8	-13	-18
USEQ-MID	0	1	2	2	5	11	23	152
USEQ-LGVAL	0	38	-3	-4	-10	-13	-35	-59
USEQ-LGGRTH	500	733	1415	2581	4870	9521	16857	29064
NUSEQ	0	2	2	2	2	4	6	31

Table 3: Average investment (given in thousand dollars) in assets over time for the CVaR-only model with regularization $\lambda = 1$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	119	134	196	303	417	528	751
FI-INVGRD	0	-4	-4	-18	-28	-41	-58	-83
USEQ-SM	0	2	4	-1	-3	-6	-13	-18
USEQ-SMVAL	0	3	1	-5	-15	-29	-51	-96
USEQ-SMGRTH	0	54	205	302	479	854	1465	2437
USEQ-LG	0	-3	-1	-18	-39	-71	-124	-287
USEQ-MID	0	15	18	19	35	62	127	265
USEQ-LGVAL	0	226	-9	-3	-27	-38	-159	-292
USEQ-LGGRTH	500	369	1015	1906	3678	7090	12724	22923
NUSEQ	0	20	25	26	41	62	85	199

Table 4: Average investment (given in thousand dollars) in assets over time for the CVaR-only model with regularization $\lambda = 0.1$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	358	193	312	555	834	1150	1559	2221
FI-INVGRD	0	10	51	-14	-44	-68	-105	-151
USEQ-SM	0	0	5	-37	-85	-159	-283	-497
USEQ-SMVAL	0	-3	-9	-60	-148	-283	-504	-985
USEQ-SMGRTH	0	171	363	620	1116	1955	3431	5839
USEQ-LG	0	-7	26	-60	-155	-275	-491	-1238
USEQ-MID	0	50	47	14	-16	-75	-78	-161
USEQ-LGVAL	0	337	234	402	724	1193	1836	3746
USEQ-LGGRTH	142	-117	20	228	437	947	1771	2974
NUSEQ	0	90	77	17	-3	-56	-89	3

Table 5: Average investment (given in thousand dollars) in assets over time for the CVaR-only model with regularization $\lambda = 0.01$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	27	24	35	68	71	99	562
FI-INVGRD	0	-1	-1	-3	-4	-5	-7	-2
USEQ-SM	0	1	1	1	1	3	7	24
USEQ-SMVAL	0	1	0	0	1	2	5	29
USEQ-SMGRTH	0	29	76	116	197	339	550	1735
USEQ-LG	0	-1	-1	-2	-5	-8	-13	-22
USEQ-MID	0	1	2	2	5	11	22	94
USEQ-LGVAL	0	37	-3	-4	-10	-13	-35	-64
USEQ-LGGRTH	500	733	1416	2582	4873	9524	16860	28761
NUSEQ	0	2	2	2	3	4	7	27

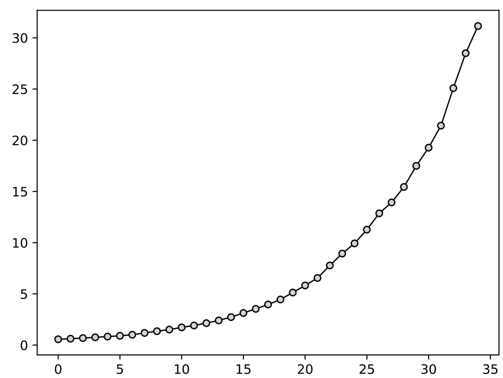
Table 6: Average investment (given in thousand dollars) in assets over time for the CVaR plus monotonicity model with regularization $\lambda = 1$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	119	133	196	304	418	529	750
FI-INVGRD	0	-4	-4	-18	-27	-40	-57	-81
USEQ-SM	0	2	4	-1	-3	-4	-10	-13
USEQ-SMVAL	0	3	1	-5	-14	-28	-50	-93
USEQ-SMGRTH	0	54	206	304	481	858	1470	2445
USEQ-LG	0	-2	-1	-18	-39	-70	-121	-281
USEQ-MID	0	15	18	18	34	61	126	263
USEQ-LGVAL	0	226	-9	-4	-28	-40	-161	-297
USEQ-LGGRTH	500	368	1014	1904	3673	7080	12707	22890
NUSEQ	0	20	25	26	40	61	84	197

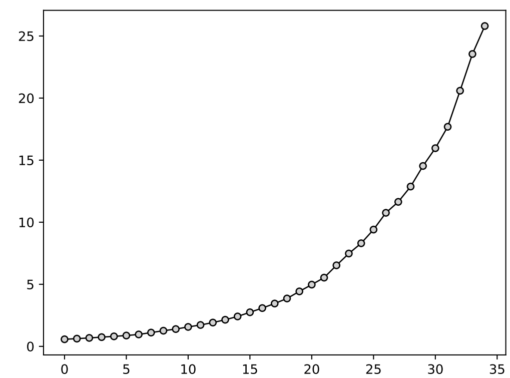
Table 7: Average investment (given in thousand dollars) in assets over time for the CVaR plus monotonicity model with regularization $\lambda = 0.1$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	357	193	312	555	834	1149	1557	2216
FI-INVGRD	0	10	51	-15	-45	-68	-107	-151
USEQ-SM	0	0	6	-36	-83	-155	-275	-481
USEQ-SMVAL	0	-2	-9	-59	-147	-280	-498	-973
USEQ-SMGRTH	0	171	363	620	1115	1952	3425	5825
USEQ-LG	0	-7	26	-60	-154	-273	-487	-1228
USEQ-MID	0	49	46	12	-20	-81	-89	-184
USEQ-LGVAL	0	337	234	401	723	1190	1832	3737
USEQ-LGGRTH	143	-116	21	230	441	954	1783	2987
NUSEQ	0	90	77	17	-3	-56	-87	9

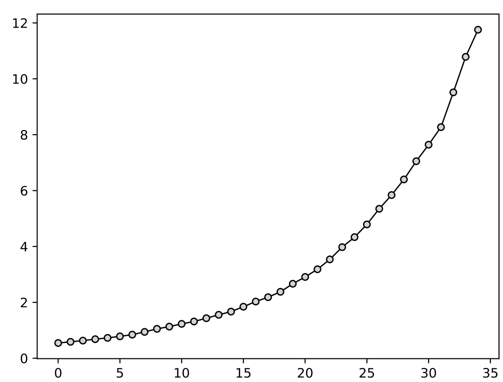
Table 8: Average investment (given in thousand dollars) in assets over time for the CVaR plus monotonicity model with regularization $\lambda = 0.01$. Average is taken over scenarios.



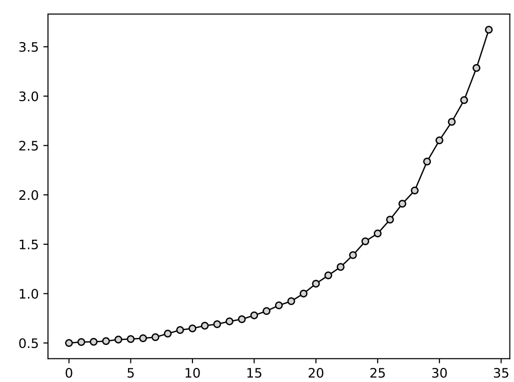
Average portfolio value with $\lambda = 1$



Average portfolio value with $\lambda = 0.1$

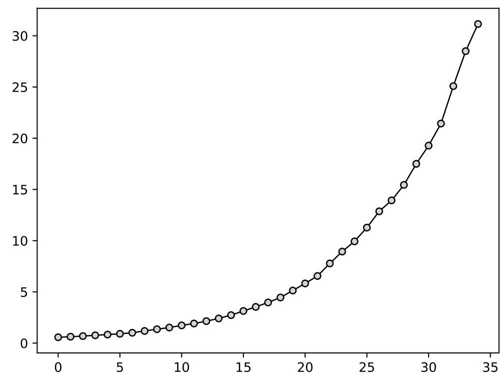


Average portfolio value with $\lambda = 0.01$

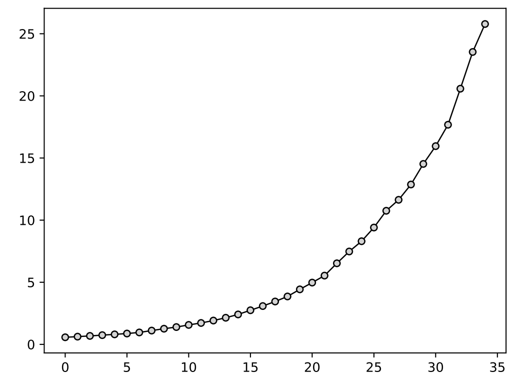


equally weighted (market) portfolio

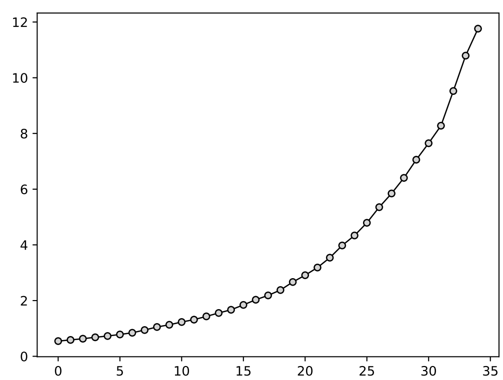
Table 9: Average portfolio value of CVaR only model, evaluated on the out-of-sample scenarios. Average is taken over scenarios.



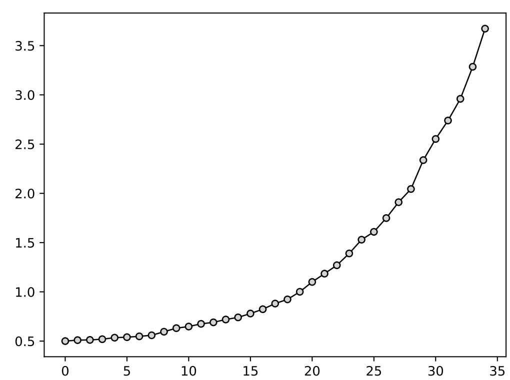
Average portfolio value with $\lambda = 1$



Average portfolio value with $\mathbb{E}_S V_t$ with $\lambda = 0.1$



Average portfolio value with $\lambda = 0.01$



equally weighted (market) portfolio

Table 10: Average portfolio value of CVaR plus monotonicity model, evaluated on the out-of-sample scenarios. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	30	31	41	56	74	101	141
FI-INVGRD	16	6	7	11	17	26	34	48
USEQ-SM	0	0	1	2	4	7	13	24
USEQ-SMVAL	0	0	1	2	4	8	14	28
USEQ-SMGRTH	0	6	22	24	42	72	118	196
USEQ-LG	0	-3	-5	-9	-16	-26	-44	-92
USEQ-MID	0	2	5	9	18	37	67	139
USEQ-LGVAL	0	2	2	2	4	6	8	8
USEQ-LGGRTH	64	74	139	264	498	972	1739	3136
NUSEQ	0	1	1	2	3	5	8	16

Table 11: Average investment (given in thousand dollars) in assets over time for the CVaR plus monotonicity model with regularization $\lambda = 0.01$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
FI-MUNI	0	30	31	41	56	74	101	141
FI-INVGRD	16	6	7	11	17	26	34	48
USEQ-SM	0	0	1	2	4	7	13	24
USEQ-SMVAL	0	0	1	2	4	8	14	28
USEQ-SMGRTH	0	6	22	24	42	72	118	196
USEQ-LG	0	-3	-5	-9	-16	-26	-44	-92
USEQ-MID	0	2	5	9	18	37	67	139
USEQ-LGVAL	0	2	2	2	4	6	8	8
USEQ-LGGRTH	64	74	139	264	498	972	1739	3136
NUSEQ	0	1	1	2	3	5	8	16

Table 12: Average investment (given in thousand dollars) in assets over time for the CVaR plus monotonicity model with regularization $\lambda = 0.01$. Average is taken over scenarios.

7 Summary

In this paper we have developed 2 variations of the retirement portfolio selection model. These 2 models differ only in the monotonicity constraint, that forces the cash outflows from the portfolio to remain constant until the end of the horizon or until the portfolio value drops to 0. Both model produce nonlinear investment rules that are implemented using kernel functions. The case study showed the performance of both models, in case of different regularization parameters, as well as different scenarios. This paper illustrates the case when it is beneficial for a retiree to invest in annuities. If the average growth rate of the stock/bond indexes (that were considered for investment) is expected to remain relatively unchanged, it is not beneficial to invest in the annuities, for the given risk aversion parameters, that we believe corresponds to a typical investment case for majority of the retirees. However if the market forecasts are pessimistic, then it is optimal to invest a considerable portion of initial capital in annuities.

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