CASE STUDY: Hedging Portfolio of Options (cvar_risk)

Background

This case study hedges a Portfolio of Options by a Portfolio of Stocks and Options. A similar case study was considered by Rockafellar and Uryasev, 2000.

A target portfolio consists of stock options; the hedging portfolio includes indices, stocks, and options on indices. We allow both long and short positions in the hedging portfolio. Long positions are opened at ask prices, short positions at bid prices. Long positions are closed at bid prices, and short at ask prices. The composition of the target portfolio is known. Weights of the hedging portfolio are determined by optimization.

We want to build a hedging portfolio with the lowest possible cost at the initial time $t = 0$ to hedge the target portfolio at the expiration time $t = T$. The quality of hedging at expiration is controlled through a *CVaR*-constraint, which bounds the average of $(1-\alpha)\%$ of largest underperformances of the hedging portfolio versus the target portfolio.

The target portfolio of options is hedged by the hedging portfolio on September 22, 2005. All options in the hedging portfolio and in the target portfolio expire on December 16, 2005. We solve the optimization problem on September 22, 2005 to assure hedging on December 16, 2005.

Table 1. Composition of the target portfolio.

Name of instrument	Type	Strike price (USD)	Price (USD)	Number of Contracts
Option on stock F (Ford Motor Co) (FXU.X)	Put	7.5	0.05	10
Option on stock ETR (Entergy Cp) (ETRLM.X)	Call	65	6.8	10
Option on stock Dow (DOW Chemical) (DOWLF.X)	Call	30	11.3	10
Option on stock CSC (Computer Sciences Cp) (CSCLH.X)	Call	40	6.0	10
Option on stock C (Citigroup Inc) (CXG.X)	Put	35	0.05	10
Option on stock BUD (Anheuser Busch) (BUDXH.X)	Put	40	0.2	10
Option on stock BMY (Bristol Myers Squibb) (BMYXD.X)	Put	20	0.05	10
Option on stock WMT (Wall Mart Stores) (WMTXG.X)	Put	35	0.05	10
Option on stock WMT (Wall Mart Stores) (WMTXU.X)	Put	37.5	0.2	10
Option on stock GM (Gen Motors) (GMXC.X)	Put	15	0.05	10
Option on stock GE (Gen Electric Co) (GEXY.X)	Put	27.5	0.05	10

Table 2. List of instruments in the hedging portfolio.

Name of instrument	Type	Strike price (USD)	Initial Price (USD)
Option on S&P 500 Index (SYFLT.X)	Call	500	714.1
Option on S&P 500 Index (SYGLT.X)	Call	600	615
Option on S&P 500 Index (SPZLT.X)	Call	700	516
Option on S&P 500 Index (SPZLJ.X)	Call	750	466.5
Option on S&P 500 Index (SPXLT.X)	Call	800	417
Option on S&P 500 Index (SPXLJ.X)	Call	850	367.6
Option on S&P 500 Index (SXBLT.X)	Call	900	318.3
Option on S&P 500 Index (SXBLF.X)	Call	930	288.8
Option on S&P 500 Index (SXBLJ.X)	Call	950	269.2
Option on S&P 500 Index (SXBLS.X)	Call	995	225.3
Option on S&P 500 Index (SPQLT.X)	Call	1000	220.5
Option on S&P 500 Index (SPQLA.X)	Call	1005	215.6
Option on S&P 500 Index (SPQLE.X)	Call	1025	196.4
Option on S&P 500 Index (SPQLJ.X)	Call	1050	172.5
Option on S&P 500 Index (SPQLO.X)	Call	1075	149
Option on S&P 500 Index (SPTLT.X)	Call	1100	125.9
Option on S&P 500 Index (SPTLD.X)	Call	1120	108
Option on S&P 500 Index (SPTLE.X)	Call	1125	103.6
Option on S&P 500 Index (SPTLJ.X)	Call	1150	82.2
Option on S&P 500 Index (SPTLO.X)	Call	1175	62.2
Option on S&P 500 Index (SPTLQ.X)	Call	1185	54.8
Option on S&P 500 Index (SZPLT.X)	Call	1200	44.2
Option on S&P 500 Index (SZPLE.X)	Call	1225	28.8
Option on S&P 500 Index (SZPLJ.X)	Call	1250	16.8
Option on S&P 500 Index (SZPLK.X)	Call	1255	14.8
Option on S&P 500 Index (SZPLO.X)	Call	1275	8.1
Option on S&P 500 Index (SXYLT.X)	Call	1300	3.7
Option on S&P 500 Index (SXYLE.X)	Call	1325	1.45
Option on S&P 500 Index (SXYLJ.X)	Call	1350	0.65
Option on S&P 500 Index (SXYLH.X)	Call	1375	0.5
Option on S&P 500 Index (SXZLT.X)	Call	1400	0.25
Option on S&P 500 Index (SPZXT.X)	Put	700	0.1
Option on S&P 500 Index (SPZXJ.X)	Put	750	0.5
Option on S&P 500 Index (SPXXT.X)	Put	800	0.5
Option on S&P 500 Index (SPXXJ.X)	Put	850	0.5
Option on S&P 500 Index (SXBXT.X)	Put	900	0.7
Option on S&P 500 Index (SXBXF.X)	Put	930	0.9
Option on S&P 500 Index (SXBXJ.X)	Put	950	0.9
Option on S&P 500 Index (SXBXS.X)	Put	995	1.8
Option on S&P 500 Index (SPQXT.X)	Put	1000	1.9
Option on S&P 500 Index (SPQXA.X)	Put	1005	2.05
Option on S&P 500 Index (SPQXC.X)	Put	1015	2.35
Option on S&P 500 Index (SPQXE.X)	Put	1025	2.65
Option on S&P 500 Index (SPQXJ.X)	Put	1050	3.5
Option on S&P 500 Index (SPQXO.X)	Put	1075	4.4
Option on S&P 500 Index (SPTXT.X)	Put	1100	6.2

Continuation of Table 2

Name of instrument	Type	Strike price (USD)	Initial Price (USD)
Option on S&P 500 Index (SPTXD.X)	Put	1120	8.5
Option on S&P 500 Index (SPTXE.X)	Put	1125	9
Option on S&P 500 Index (SPTXG.X)	Put	1135	10.3
Option on S&P 500 Index (SPTXJ.X)	Put	1150	12.6
Option on S&P 500 Index (SPTXO.X)	Put	1175	17.4
Option on S&P 500 Index (SPTXQ.X)	Put	1185	19.8
Option on S&P 500 Index (SZPXT.X)	Put	1200	24.3
Option on S&P 500 Index (SZPXE.X)	Put	1225	33.8
Option on S&P 500 Index (SZPXJ.X)	Put	1250	46.8
Option on S&P 500 Index (SZPXK.X)	Put	1255	49.8
Option on S&P 500 Index (SZPXO.X)	Put	1275	63.4
Option on S&P 500 Index (SXYXT.X)	Put	1300	83.5
Option on S&P 500 Index (SXYXE.X)	Put	1325	106.1
Option on S&P 500 Index (SXYXJ.X)	Put	1350	130.1
Option on S&P 500 Index (SXYXH.X)	Put	1375	154.6
Option on S&P 500 Index (SXZXT.X)	Put	1400	179.2
Option on S&P 500 Index (SXZXJ.X)	Put	1450	228.7
Option on S&P 500 Index (SXMXT.X)	Put	1500	278.2
Option on S&P 500 Index (SPBXT.X)	Put	1600	377.3
Option on Russell 2000 Index (RUWLP.X)	Call	580	79.4
Option on Russell 2000 Index (RUWLR.X)	Call	590	70.9
Option on Russell 2000 Index (RUWLT.X)	Call	600	62.6
Option on Russell 2000 Index (RUYLB.X)	Call	610	54.6
Option on Russell 2000 Index (RUYLH.X)	Call	640	33.2
Option on Russell 2000 Index (RUYLJ.X)	Call	650	27.1
Option on Russell 2000 Index (RUYLL.X)	Call	660	21.7
Option on Russell 2000 Index (RUYLN.X)	Call	670	16.9
Option on Russell 2000 Index (RUYLP.X)	Call	680	12.7
Option on Russell 2000 Index (RUYLR.X)	Call	690	9.4
Option on Russell 2000 Index (RUTLT.X)	Call	700	6.8
Option on Russell 2000 Index (RUTLB.X)	Call	710	4.7
Option on Russell 2000 Index (RUTLD.X)	Call	720	3.3
Option on Russell 2000 Index (RUTLF.X)	Call	730	2.15
Option on Russell 2000 Index (RUTLH.X)	Call	740	1.45
Option on Russell 2000 Index (RUWXP.X)	Put	580	5.9
Option on Russell 2000 Index (RUWXR.X)	Put	590	7.3
Option on Russell 2000 Index (RUWXT.X)	Put	600	8.9
Option on Russell 2000 Index (RUYXB.X)	Put	610	11
Option on Russell 2000 Index (RUYXD.X)	Put	620	13.3
Option on Russell 2000 Index (RUYXF.X)	Put	630	16
Option on Russell 2000 Index (RUYXH.X)	Put	640	19.2
Option on Russell 2000 Index (RUYXJ.X)	Put	650	23.2
Option on Russell 2000 Index (RUYXL.X)	Put	660	27.6
Option on Russell 2000 Index (RUYXN.X)	Put	670	32.7
Option on Russell 2000 Index (RUYXP.X)	Put	680	38.6
Option on Russell 2000 Index (RUYXR.X)	Put	690	45.2
Option on Russell 2000 Index (RUTXT.X)	Put	700	52.4

Continuation of Table 2

Name of instrument	Type	Strike price (USD)	Initial Price (USD)
Option on Russell 2000 Index (RUTXB.X)	Put	710	60.3
Option on Russell 2000 Index (RUTXD.X)	Put	720	68.8
Option on Russell 2000 Index (RUTXF.X)	Put	730	77.7
Option on Russell 2000 Index (RUTXH.X)	Put	740	86.9
S&P DEP RECEIPTS (SPY) long	Equity	n/a	121.34
S&P DEP RECEIPTS (SPY) short	Equity	n/a	-121.34
iShares Russell 2000 Index (IWM) long	Equity	n/a	64.74
iShares Russell 2000 Index (IWM) short	Equity	n/a	-64.73
F (Ford Motor Co) long	Equity	n/a	9.73
F (Ford Motor Co) short	Equity	n/a	-9.72
ETR (Entergy Cp) long	Equity	n/a	71.04
ETR (Entergy Cp) short	Equity	n/a	-71.01
Dow (DOW Chemical) long	Equity	n/a	41.08
Dow (DOW Chemical) short	Equity	n/a	-41.07
CSC (Computer Sciences Cp) long	Equity	n/a	45.29
CSC(Computer Sciences Cp) short	Equity	n/a	-45.28
C (Citigroup Inc) long	Equity	n/a	45.19
C (Citigroup Inc) short	Equity	n/a	-45.18
BUD (Anheuser Busch) long	Equity	n/a	44.04
BUD (Anheuser Busch) short	Equity	n/a	-44.03
BMY (Bristol Myers Squibb) long	Equity	n/a	24.18
BMY (Bristol Myers Squibb) short	Equity	n/a	-24.17
WMT (Wal Mart Stores) long	Equity	n/a	43.2
WMT (Wal Mart Stores) short	Equity	n/a	-43.19
GM (Gen Motors) long	Equity	n/a	30.44
GM (Gen Motors) short	Equity	n/a	-30.43
GE (Gen Electric Co) long	Equity	n/a	33.31
GE (Gen Electric Co) short	Equity	n/a	-33.3

The scenarios of prices of stocks and indices at expiration are modeled by bootstrapping of historical daily prices.

References

- Rockafellar, R.T., and S. Uryasev (2000): Optimization of conditional value-at-risk. Journal of Risk 2, 21–41.

Notations

$\boldsymbol{T}$ = expiration time ($\boldsymbol{T}$ **= 62** days);
$\boldsymbol{N}$ = number of stocks in the hedging portfolio; $\boldsymbol{n}$ **= {1,…,** $\boldsymbol{N}$**}** is index of stocks;
$\boldsymbol{M}$ = number of indices in the hedging portfolio; $\boldsymbol{m}$ **= {1,…,** $\boldsymbol{M}$**}** is index of stock indexes;
$\boldsymbol{S}$ = number of scenarios; $\boldsymbol{s}$ **= {1,…,**$\boldsymbol{S}$**}** is index of scenarios;

$\tilde{\boldsymbol{R}}_{nt}^{s}$ = daily return of stock $\boldsymbol{n}$ on day $\boldsymbol{t}$ under scenario $\boldsymbol{s}$;

$\boldsymbol{R}_{mt}^{s}$ = daily return of index $\boldsymbol{m}$ on day $\boldsymbol{t}$ under scenario $\boldsymbol{s}$;

$\boldsymbol{J}_{n}$ = number of options on stock $\boldsymbol{n}$; $\boldsymbol{j}$ **= {1,…,** J_n **}** is index of options on stock $\boldsymbol{n}$;

$\boldsymbol{K}_{m}$ = number of options on index $\boldsymbol{m}$**;** $\boldsymbol{k}$ **= {1,…,** K_m **}** is subscript of options on index $\boldsymbol{m}$;

O_{nj}^{S} = option j on stock n;

O_{mk}^{I} = option k on index m;

$\tilde{K}_{nj}$ = strike price of option j on stock n;

K_{mk} = strike price of option k on index m;

S_n^0 = initial price of stock n;

S_n = price of stock n at final time T;

S_n^A = ask price of stock n at initial time;

S_n^B = bid price of stock n at initial time;

RL_n^s = full return (in USD) of a long position in stock n over time T under scenario s given by formula $RL_n^s = S_n^A \cdot (\tilde{R}_{n1}^s \cdot \tilde{R}_{n2}^s \cdot \cdots \cdot \tilde{R}_{nT}^s - 1)$.

RS_n^S = full return (in USD) of a short position in stock n over time T under scenario s given by formula $RS_n^s = -S_n^B \cdot (\tilde{R}_{n1}^s \cdot \tilde{R}_{n2}^s \cdot \cdots \cdot \tilde{R}_{nT}^s - 1)$.

S_n^s = price of stock n under scenario s at final time T given by formula

$$S_n^s = S_n^0 \cdot \tilde{R}_{n1}^s \cdot \tilde{R}_{n2}^s \cdot \cdots \cdot \tilde{R}_{nT}^s ;$$

I_m^0 = initial price of index m;

I_m^s = price of index m under scenario s at final time T given by formula

$$I_m^s = I_m^0 \cdot R_{m1}^s \cdot R_{m2}^s \cdot \cdots \cdot R_{mT}^s .$$

$\tilde{C}_{nj}^s$ = value at expiration of option j on stock n under scenario s given by formula

$$\tilde{C}_{nj}^s = \begin{cases} \max(0, S_n^s - \tilde{K}_{nj}), & \text{for call option;} \\ \max(0, \tilde{K}_{nj} - S_n^s), & \text{for put option.} \end{cases}$$

C_{mk}^s = value at expiration of option k on index m under scenario s given by formula

$$C_{mk}^s = \begin{cases} \max(0, I_m^s - K_{mk}), & \text{for call option;} \\ \max(0, K_{mk} - I_m^s), & \text{for put option.} \end{cases}$$

C_{mk}^B = bid price of option k on index m at initial time;

C_{mk}^A = ask price of option k on index m at initial time;

y_{nj} = number of contracts on option j on stock n in the target portfolio ($y_{nj} > 0$ for a long position, $y_{nj} < 0$ for a short position);

x_{mk}^+ = number of long contracts on option O_{mk}^I in the hedging portfolio;

x_n^+ = number of shares of a long position in stock n in the hedging portfolio;

x_n^- = number of shares of a short position in stock n in the hedging portfolio;

x_{mk}^- = number of short contracts on option O_{mk}^I in hedging portfolio;

$P(x)$ = initial price of the hedging portfolio given by formula

$$P(x) = \sum_{m=1}^{M} \sum_{k=1}^{K_m} (x_{mk}^+ C_{mk}^A - x_{mk}^- C_{mk}^B) + \sum_{n=1}^{N} (x_n^+ S_n^A - x_n^- S_n^B) ;$$

$L(x, s)$ = hedging error at final time T under scenario s given by formula

$$L(x,s)=\sum_{n=1}^{N}\sum_{j=1}^{J_n} y_{nj}\tilde{C}^s_{nj}-\left[\sum_{m=1}^{M}\sum_{k=1}^{K_m}(x^+_{mk}C^s_{mk}-x^-_{mk}C^s_{mk})+\sum_{n=1}^{N}(x^+_n RL^s_n+x^-_n RS^s_n)\right],$$

where the first term is the value of the target portfolio at the final time T, and the second term (in square brackets) is the value of the hedging portfolio at final time T ;

γ = coefficient determining bound on CVaR of underperformances of the hedging portfolio;

U = 100 USD = available funding;

ν = 0.15 = diversification constant for exposures (see the following two formulas);

$u_m=\dfrac{\nu U}{I^0_m}$ = upper bounds on number of contracts (long or short) on option O^I_{mk} in the hedging portfolio;

$u_n=\dfrac{\nu U}{S^0_n}$ = upper bounds on number of shares of a long (short) position in stock n in the hedging portfolio.

Optimization Problem

minimizing initial portfolio value

$$\min P(x) \tag{CS.1}$$

subject to

CVaR constraint on hedging error

$$CVaR_\alpha(L(x,s))<\gamma U\ , \tag{CS.2}$$

non-negativity constraints on positions

$$u_m\ge x^+_{mk}\ge 0,\ u_m\ge x^-_{mk}\ge 0,\quad k=1,\ldots,K_m,\quad m=1,\ldots,M, \tag{CS.3}$$

$$u_n\ge x^+_n\ge 0,\ u_n\ge x^-_n\ge 0,\quad n=1,\ldots,N\ . \tag{CS.4}$$

Optimization problem is solved using two datasets each: Dataset1 for "long case study" including 45,000 scenarios and and Dataset2 for "short case study" including only 5,000 scenarios.
Calculations in the case study are conducted for γ = 5.25 for "short case study" and γ = 5.30 for "long case study", and for ν = 0.15 for both case studies.