A Unified Framework for Default Modeling

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Introduction

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2 Survival curve processes

3 Properties of survival processes

4 Predictability

5 Examples

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Default modeling

- Explicitly model defaults to enable pricing of risky instruments

Classically, two types of models:

- Reduced form models
- Structural models
Classical reduced form models:

- Directly model statistics of the default time, not the structure of the firm
- Exponentially distributed arrival time
- Default time is unpredictable (totally inaccessible)

Artzner and Delbaen [AD95], Jarrow and Turnbull [JT95], ...
Classical structural models:

- Model assets and liabilities
- Assets follow a continuous process
- Default when assets cross a barrier (become too small relative to liabilities)
- Default time is a consequence
- Default time is predictable

Merton [Mer74], Black and Cox [BC76], Avellaneda and Zhu [AZ01], Longstaff and Schwartz [LS95], Duffie and Singleton [DS99], Hull and White [HW01], ...
Contemporary default modeling

Today the definitions have changed

Reduced form:

- Any model where default time is directly modeled, even if it involves modeling assets and liabilities

Structural:

- Any model where default occurs when a process crosses a barrier, even if not directly modeling assets and liabilities
Modern classification

Modern approach: Information based classification.

Reduced form:

- Exponentially distributed arrival time
- Insufficient information to predict default
- Default time is unpredictable (totally inaccessible)

Structural:

- Continuous process hitting barrier
- Sufficient information to predict default
- Default time is predictable

Two ends of a continuum
Adjust between them by modeling and adding uncertainty
Informational approach

Informational approach characteristics:

- Very elegant characterization of differences between models
- Lacks tools for manipulating and working directly with filtrations
  - Filtrations are typically generated by processes

As a result, approaches tend to be ad hoc

References:

- “Modeling credit risk with partial information,” Çetin, Jarrow, Protter, and Yildirim [Çet+04]
- “Default and information,” Giesecke [Gie06]
- “Structural versus reduced form models: a new information based perspective,” Jarrow and Protter [JP04]
- ...
Philosophy

The philosophy of this work – three separate but interrelated components:

- Default time model
- Recovery model
- Valuation model

Important to understand and model each component
Don’t muddy the waters by mixing them together
This work

This work focuses on step 1 – default modeling

Default time modeling general framework:

- **All** default models can be expressed and understood in terms of their hazard rate and survival processes

Benefits:

- Easy to combine models and modify predictability by directly modifying these processes
- The stopping time itself can be largely be relegated to the background
- Default modeling resembles interest rate modeling
- Appears that degree of predictability can be directly read off of these processes

Related work:

- “The instantaneous and forward default intensity of structural models,” Chen [Che07]
- “Dirac Processes and Default Risk,” Kenyon and Green [KG15]
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Setting

We work in a filtered probability space \((\Omega, \sigma, P, \mathcal{F})\):

- \(\Omega\): Sample space
- \(\sigma\): Sigma algebra
- \(P\): Real world probability measure
- \(\mathcal{F}\): Filtration

In all default models, the time at which default occurs is a stopping time. Let this be \(\tau\). Then \(\tau: \Omega \rightarrow \mathbb{R}^+\) is measurable and satisfies

\[
\{\tau \leq t\} \in \mathcal{F}_t
\]  

(1)

We consider \(\mathcal{F}_t\) to specify the entire world state at time \(t\)

Whether or not individuals know \(\mathcal{F}_t\) is a different matter
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Classical reduced form

Classical reduced form models are based on the statistics of a default stopping time:

\[ \tau \] Default stopping time
\[ s(t) \] Survival probability
\[ p(t) \] Default time probability density
\[ \lambda(t) \] Hazard rate – instantaneous rate at which default events occur

Relationships:

\[ s(t) = \text{Prob}(\tau > t) = E[1_{\tau > t}] = 1 - \text{CDF}(\tau) \] (2)

\[ p(t) = -\frac{ds(t)}{dt} \] (3)

\[ s(t) = 1 - \int_0^t p(x)dx = \int_t^\infty p(x)dx = e^{-\int_0^t \lambda(x)dx} \] (4)

\[ \lambda(t) = \frac{p(t)}{s(t)} = -\frac{d \log(s(t))}{dt} \] (5)

\[ E[\tau] = \int_0^\infty tp(t)dt = \int_0^\infty s(t)dt \] (6)
To extend classical reduced form models to a general framework:

- Consider the distribution of \( \tau \) at future times conditional on the state of the world
- Leads to considering survival and hazard rate processes

The **survival curve process** (for all \( t \leq T \)):

\[
s_t(T) = E_t[1_{\tau > T}]
\]  

The **hazard curve process** \( \lambda_t(T) \):

\[
s_t(T) = e^{-\int_t^T \lambda_t(u) du}
\]  

We will see that all default models can be expressed in this framework.
The definition

\[ s_t(T) = E_t[1_{\tau > T}] \]  

(9)

only defines \( s_t(T) \) up to sets of measure zero

- Need point-wise limits of \( s_t(T) \)
- Need \( S_t(T) \) as a function of \( T \) to be an explicit curve

So, we select a particular version (up to indistinguishability). For each \( T \),

- Take \( s_T(T) = 1_{\tau > T} \)
- Take the process \( s_t(T) \) to be an RCLL version

Each of these is then the unique (up to indistinguishability) RCLL process satisfying \( s_t(T) = E_t[1_{\tau > T}] \)
The hazard rate curve process satisfies:

\[ s_t(T) = e^{-\int_t^T \lambda_t(u)du} \]  

(10)

Because \( s_t(T) \) can have jumps:

- \( \lambda_t(T) \) cannot be an ordinary function

\( \lambda_t(T) \) must be the distributional derivative of \(-\log s_t(T)\)

- Differs from Chen [Che07], where \( \lambda_t(u) = p_t(u)/s_t(u) \), where \( p \) is the default probability density
**Default and HJM**

This is analogous to interest rate modeling:

- \( Z_t(T) \): Price process of zero coupon bond maturing at time \( T \).
- \( f_t(T) \): Time \( T \) forward rate observed at time \( t \).

<table>
<thead>
<tr>
<th>HJM</th>
<th>Credit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq Z \leq 1 )</td>
<td>( 0 \leq s \leq 1 )</td>
</tr>
<tr>
<td>( Z_t(T) ) is monotonically decreasing in ( T )</td>
<td>( s_t(T) ) is monotonically decreasing in ( T )</td>
</tr>
<tr>
<td>( Z_t(T) = e^{-\int_t^T f_t(u)du} )</td>
<td>( s_t(T) = e^{-\int_t^T \lambda_t(u)du} )</td>
</tr>
<tr>
<td>( Z_T(T) = 1 )</td>
<td>( s_T(T) = 1_{\tau &gt; T} ) is 1 or 0</td>
</tr>
<tr>
<td>( Z_t(T) ) generally smooth in ( T )</td>
<td>( s_t(T) ) can be discontinuous in ( T )</td>
</tr>
<tr>
<td>( Z_t(T) ) is a martingale in the ( T )-forward measure</td>
<td>( \forall T, \ s_t(T) ) is a martingale in the real world measure</td>
</tr>
</tbody>
</table>
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Theorem (Stopping times and survival processes)

Stopping times satisfy the following properties:

1. \( \tau(\omega) = T \) iff \( T = \min_{s_t(t)(\omega) = 0} t \)
2. \( \{ \omega \mid \tau \leq t \} = \{ \omega \mid s_t(t) = 0 \} \)
3. \( \{ \omega \mid \tau > t \} = \{ \omega \mid s_t(t) = 1 \} \)

Proof.

For item (1), \( \tau(\omega) = T \) iff \( 1_{\tau > T}(\omega) = 0 \) and \( 1_{\tau > T'}(\omega) = 1 \) for all \( T' < T \).

Items (2) and (3) follow.
Theorem (Properties of Survival Processes)

The survival curve processes have the following properties:

1. \( s_t(T) \) is monotonically decreasing as a function of \( T \) and is RCLL
2. \( 0 \leq s_t(T) \leq 1 \)
3. For each \( t \) and \( T \), \( S_t(u) \) is of bounded variation on \( u \in [t, T] \)
4. \( s_t(T) = E_t[s_{t'}(T)] \), for \( t \leq t' \leq T \) (i.e. for each \( T \), \( s_t(T) \) is a martingale)
5. The conditional CDF of \( \tau \) at time \( t \) is \( 1 - s_t(T) \)
From survival to stopping

It’s not surprising that a stopping time produces a set of survival curve processes.

We also have the following converse:

**Theorem (Stopping times from survival processes)**

Consider an RCLL family of processes $s_t(T)$ adapted to $\mathcal{F}_t$ that satisfy:

\[
\begin{align*}
    s_t(t) &\in \{0, 1\} \\
    s_t(t) &= \sup_{t' > t} s_{t'}(t') \\
    s_0(0) &= 1 \\
    s_t(T) &= E_t[s_t(T)], \text{ for } t \leq T
\end{align*}
\]

Then there exists an $\mathcal{F}$ stopping time $\tau$ satisfying:

\[
    s_t(T) = E_t[1_{\tau > T}]
\]
From survival to stopping

Constructing stopping times from survival processes is straightforward:

- $s_t(t)$ are indicator functions
- $s_t(t) = \sup_{t' > t} s_{t'}(t')$ insures that these indicator functions nest
- Thus, there’s a stopping time $\tau$ for which $1_{\tau > t} = s_t(t)$
- $s_0(0) = 1$ ensures we start undefaulted
- $s_t(T) = E_t[s_T(T)]$, for $t \leq T$ then ensures that the system of processes correspond to the survival processes of $\tau$

Conclusions:

- We have a bijection between stopping times and survival processes
- We can define a default model by specifying the survival processes instead of the stopping time
- We can interchangeably work with stopping times and survival processes
Rewriting the survival curve conditions in terms of hazard rates yields:

**Theorem (Stopping times from hazard rate curves)**

The map from survival curve processes to hazard rate curve processes given by mapping a survival curve process \( s \) to the hazard rate curve process \( \lambda \) satisfying

\[
s_t(u) = e^{-\int_t^u \lambda_t(x)dx}
\]

(16)

is a bijection of the set of survival curves to the set of distributional derivatives \( \lambda_t(u) \) adapted to \( \mathcal{F}_t \) satisfying:

1. \( \lambda_t(t) \) is never a finite weight point mass
2. \( 1_{\lambda_t(t)=\infty} = \sup_{t'>t} 1_{\lambda_t'(t')=\infty} \)
3. \( \lambda_0(0) \) is not a point mass
4. \( e^{-\int_t^u \lambda_t(x)dx} = \mathbb{E}[e^{-\int_t^u \lambda_t(x)dx} | \mathcal{F}_t] \), for \( t \leq u \)
Drift condition

Survival curves are non-negative martingales, so the hazard rate curves must satisfy the HJM condition.

Consider a continuous survival curve process of the form

\[ ds_t(T) = s_t(T)\sigma_t(T) \cdot dW \]  \hspace{1cm} (17)

**Theorem (Drift condition, continuous case)**

For such a continuous survival curve process, the hazard rate curve process is of the form

\[ d\lambda_t(T) = a^\lambda_t(T)dt + \sigma^\lambda_t(T) \cdot dW \]  \hspace{1cm} (18)

where

\[ \sigma^\lambda = -\frac{\partial \sigma_t(T)}{\partial T} \]  \hspace{1cm} (19)

\[ a^\lambda_t(T) = \left( \int_t^T \sigma^\lambda_t(x)dx \right)^t \rho \sigma^\lambda_t(T) \]  \hspace{1cm} (20)
Implication:

**Theorem**

*If* \( \lambda^1 \) *and* \( \lambda^2 \) *are hazard rate curve processes given by*

\[
d\lambda^i_t(T) = a^i_t(T)dt + \sigma^i_t(T)dW_i
\]

\[
dW_1dW_2 = \rho dt
\]

*then* \( \lambda = \lambda^1 + \lambda^2 \) *is a hazard rate curve process only if* \( \rho = 0 \) *or one of the* \( \sigma^i \) *is zero.*

Intuition:

- \( \tau \) *should be* \( \min(\tau^1, \tau^2) \)
- \( \min(\tau^1, \tau^2) \) *only has hazard rate* \( \lambda^1 + \lambda^2 \) *if* \( \lambda^1 \) *and* \( \lambda^2 \) *are uncorrelated*
Predictability

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Predictability

The key difference between classical reduced form models and classical structural models is the extent to which default can be predicted.

Recall:

**Definition (Predictable stopping times)**

The default time $\tau$ is

- **predictable** if there exists a sequence of stopping times $\{\tau_n\}$ such that $\tau_n < \tau$ and $\lim_{n \to \infty} \tau_n = \tau$ almost surely.
- **accessible** if there exists a sequence of predictable stopping times $\{\tau_n\}$ such that

$$\mathbb{P}[\bigcup_n \{\omega : \tau(\omega) = \tau_n(\omega) < \infty\}] = \mathbb{P}[\tau < \infty].$$

(23)

Such a sequence $\{\tau_n\}_{n \geq 1}$ is said to *envelop* $\tau$.

- **totally inaccessible** if for every predictable stopping time $T$,

$$\mathbb{P} [\{\omega : \tau(\omega) = T(\omega) < \infty\}] = 0.$$  

(24)
Predictability intuition

The predictability definitions are complicated. The intuition is simple:

- **predictable** – We can see the default coming – we can predict with increasing certainty the times and states at which default occurs
- **accessible** – We know the times at which defaults occur, but not the states
- **totally inaccessible** – We don’t know when the defaults occur and we don’t know which states will default. We only know the rate at which defaults occur

In other words:

- **predictable** – Default times and default states are “known”
- **accessible** – Times are “known” but states are not
- **totally inaccessible** – Times and states are “unknown”
Examples:

- **predictable** – Continuous process hitting a barrier
- **accessible and not predictable** – Flipping a coin at times $t_i$ (deterministic or predictable stopping times) to determine default
- **totally inaccessible** – Continuously flipping coins

So, flipping a coin when a continuous process hits a barrier yields an accessible (but not predictable) stopping time
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Reduced form example

Reduced form model:

- Default arrival time is exponentially distributed
- Instantaneous hazard rate $h_t$ is deterministic
- $s^R_t(T) = e^{-\int_t^T h_u du}$

Unified version:

- $s^R_t(T) = e^{-\int_t^T h_u du} 1_{\tau > t}$
- This is same as above $s$ when default has not yet occurred
- Indicator needed to make $s$ zero after default has occurred, so that $s^R$ is a survival curve process, a martingale, etc
- Can sometimes do away with the indicator (to the extent that only the distribution up to default is needed)

Properties:

- Totally inaccessible
- Defaults only occur when $h_t > 0$
- $\lim_{t \uparrow T} s^R_t(T) = 1$ on $\{\tau \geq T\}$
Merton example

Merton:

- Assets of firm follow GBM: $dV = \mu V dt + \sigma V dW$
- Default at $T$ if $V < B$

Unified version:

$$s^M_t(u) = \begin{cases} 
(1 - \Phi(-d_2)1_{u \geq T}) & \text{if } t \leq T \\
1_{V_T > B} & \text{if } t > T 
\end{cases} \quad (25)$$

Properties:

- Predictable
- $\lim_{t \uparrow u} s^M_t(u) = s^M_u(u)$
Adding inaccessibility to Merton:

- \( s^{M'} = s^M s^R \)
- \( \lambda^{M'} = \lambda^M + \lambda^R \)
- \( \tau^{M'} = \min(\tau^M, \tau^R) \)

Properties:

- Some defaults are predictable and some are totally inaccessible.
- For \( \tau^M < \tau^R \), \( \lim_{t \uparrow u} s^M_t(u) = 1 \) at \( \tau^{M'} = u \)
- For \( \tau^R < \tau^M \), \( \lim_{t \uparrow u} s^M_t(u) = 0 \) at \( \tau^{M'} = u \)
Forgetful Merton

Forgetful Merton, as in Giesecke [Gie06] or Çetin, Jarrow, Protter, and Yildirim [Çet+04].

- Know $V_0$, don’t know $V_t$ for $t < T$
- I.e. take the filtration of $V$ out for $0 < t < T$

Unified version:
Let $p = P(V_T < B|V_0)$

$$s^f_t(u) = \begin{cases} 
1 - p 1_{u \geq T} & \text{if } t < T \\
1_{V_T > B} & \text{if } t \geq T 
\end{cases}$$

The model becomes like flipping a biased coin at time $T$ to determine default

- Accessible, but not predictable
- $\lim_{t \uparrow T} s^f_t(T) = 1 - p$, telling us that $p$ of the states die off at time $T$, and we don’t know which ones
Black-Cox model

- Like Merton, except default when $V$ crosses barrier $K$

Unified version:

$$s_t^{BC}(u) = P(V_s > K, \forall t \leq s \leq u \mid \mathcal{F}_t)$$

$$= \left( \Phi[d(V_t/K, u - t)] - \left( \frac{K}{V_t} \right)^{2r_2 - 1} \Phi[d(K/V_t, u - t)] \right) 1_{\tau > t}$$

(27)

(28)

Properties:

- Predictable
- $\lim_{t \uparrow u} s_t^{BC}(u) = s_u^{BC}(u)$
Black-Cox model with coin flip

- Black-Cox, but when $V$ crosses barrier $K$, flip coin for default

Properties:

- Accessible, not predictable
- $\lim_{t \uparrow u} s^{BCC}_t(u) = 1/2$ on $\tau = u$
Conjectures
Predictability and survival processes

From the above examples, we expect the survival processes to simply express predictability:

**Conjecture**

*With an appropriate definition of $s_t(T)$, the following are equivalent:*

- $\tau$ is predictable
- $\tau(\omega) = T \implies \lim_{t \uparrow T} s_t(T)(\omega) = 0$
- $\lim_{t \uparrow T} s_t(T) = s_T(T)$

**Intuition:**

- When approaching a predictable default time, the probability of surviving beyond it has to go to zero
- When approaching a time of no default, the probability of surviving it has to go to one

**Appropriate definition of $s_t(T)$:**

- We will see that it’s insufficient to only know $s_t(T)$ up to indistinguishability
Continuity $\iff$ predictability

One direction is not difficult.

Suppose $\lim_{t\uparrow T} s_t(T) = s_T(T)$. Then $\tau$ is announced by

$$
\tau_n(\omega) = \inf\left\{ u \mid s_u(u)(\omega) = 1 \text{ and } s_u(u + 1/n)(\omega) \leq 1/n \right\}
$$

(29)
Predictability $\iff$ Continuity

Almost surely a proof.
Suppose $\tau_n$ announces $\tau$. Then

$$\{\tau > T\} = \bigcup_n \{\tau_n > T\} = \bigcup_n \{\tau_n \geq T\} \quad (30)$$

$$\quad = \bigcup_n \bigcap_k \{\tau_n > T - 1/k\} \quad (31)$$

For all stopping times $\nu$,

$$\{\nu \geq T\} = \bigcap_k \{\nu > T - 1/k\} \quad (32)$$

so

$$\{\tau > T\} = \bigcup_n \{\tau_n \geq T\} \quad (33)$$

$$\quad = \bigcup_n \bigcap_k \{\tau_n > T - 1/k\} \quad (34)$$

Thus

$$\lim_{t \uparrow T} E_t[1_{\tau > T}] = 1_{\tau > T} \quad (35)$$

This is insufficient because $P(\tau = T)$ can be zero, and the last equality is only a.s.
The above examples also point to reading total inaccessibility from the survival curves:

**Conjecture**

*With an appropriate definition of $s_t(T)$, the following are equivalent:*

- $\tau$ is totally inaccessible
- $\lim_{t \uparrow T} s_t(T) = 1$ on $\{\tau \geq T\}$

**Intuition:**

- When approaching a totally inaccessible default time, the probability of surviving beyond it can’t reflect that it’s coming up.
Inaccessibility and survival

Another proof that’s almost there

- \( X_t = 1_{\tau > t} \)
- \( A \) be the compensator of \( 1 - X \)

Then

- \( s_t(T) = E_t[X_T] \)
- \( A \) is an increasing, continuous process [Jea17].
- \( 1 - X_t - A_t = E_t[1 - X_T - A_T] \)
- \( X_t + A_t = s_t(T) + E_t[A_T] \)
- \( \lim_{t \uparrow T} s_t(T) = \lim_{t \uparrow T} X_t + A_t - E_t[A_T] \)

But

- \( \lim_{t \uparrow T} X_t = 1_{\tau \geq T} \)
- \( \lim_{t \uparrow T} E_t[A_T] = A_T \) [KS98]

Thus

\[
\lim_{t \uparrow T} s_t(T) = 1 \text{ on } \{\tau \geq T\} \quad (36)
\]

But again, only a.s.
Accessibility and survival

As a result,

**Theorem**

*If the above conjectures hold, the following are equivalent:*

- \( \tau \) is accessible
- \( \lim_{t \uparrow T} s_t(T) < 1 \) on \( \{ \tau = T \} \)
Up to indistinguishability is not enough

Consider

• Random variable \( Y \)
• Process \( X_t = E_t[Y] \)

We think of \( X \) as a process, but

• If \( X'_t = E_t[Y] \), then only have \( X_t = X'_t \) a.s. for each \( t \): different measure zero sets for each \( t \)
• If \( X \) and \( X' \) are RCLL modifications, then \( X_t = X'_t \) a.s. for all \( t \).

But

• For \( s_t(u) \), still have different measure zero sets for each \( u \).
• Not an issue for HJM because only concerned with pricing, i.e., expected values.

Need the right version of \( E_t[1_{\tau > u}] \)
Pinning down $s$

Consider:

- $\tau$ can always be given by a process $X$ hitting zero
- Let $Y_t = \inf_{t' \leq t} X_{t'}$
- Then $1_{\tau > t} = 1_{Y_t > 0}$

If $X$ is Markovian, then

- $E[1_{Y_u > 0} \mid \mathcal{F}_t] = E[1_{Y_u > 0} \mid X_t, Y_t]$
- PDF of $Y_u$ given $\mathcal{F}_t$ is a function of $X_t$ and $Y_t$, say $f(y; t, u, X_t, Y_t)$

So, define

- $s_t(u)(X_t, Y_t) = 1 - \int_{-\infty}^{0} f(y; t, u, X_t, Y_t)dy$

If any version of $s$ works, it should be this one
More HJM inspiration

In interest rate modeling, the short rate $r$ and the spot rate $R$ are important:

$$R_t(T) = \frac{-\log Z_t(T)}{T - t}$$

$$r_t = \lim_{T \downarrow t} R_t(T)$$

Define the **spot hazard rate** $\bar{\lambda}$ and the **instantaneous hazard rate process** $h_t$:

$$\bar{\lambda}_t(T) = \frac{-\log s_t(T)}{T - t}$$

$$h_t = \lim_{T \downarrow t} \bar{\lambda}_t(T)$$
## Properties

**Spot rate and instantaneous hazard rate properties**

- \( R_t(T) \) exists for \( T > t \)
- \( h_t \) exists iff \( s_t(T) \) is differentiable at \( T = t \)

### HJM

\[
Z_t(T) = e^{-(T-t)R_t(T)} \quad \forall t \leq T
\]

### Credit

\[
s_t(T) = e^{-(T-t)\lambda_t(T)} \quad \forall t < T
\]

\[
Z_t(T) = E_t \left[ e^{-\int_t^T r_u du} \right]
\]

\[
s_t(T) = E_t \left[ e^{-\int_t^T h_u du} \right]
\] if \( \tau \) is totally inaccessible

\[
h_u = 0 \text{ if } \tau \text{ is predictable}
\]
Predictability with hazard rates

We also believe predictability can also be read from the spot hazard rate

**Theorem**

*If the above conjectures hold, the following are equivalent:*

- $\tau$ is predictable
- $\tau(\omega) = u \implies \lim_{t \uparrow u} s_t(u)(\omega) = 0$
- $\lim_{t \uparrow u} s_t(u) = s_u(u)$
- $\tau(\omega) = u \implies \lim_{t \uparrow u} \bar{\lambda}_t(u)(\omega) = \infty$
Instantaneous hazard rates and inaccessibility

Consider stopping times $\tau$ for which $h_t$ exists

Conjecture

If $\tau$ is predictable then $h_t = 0$ a.s. on $\tau > t$.

Proven under certain conditions in Giesecke [Gie06], Duffie and Lando [DL01] and Jeanblanc [Jea17]

Conjecture

$\tau$ is totally inaccessible iff

$$s_t(u) = \mathbb{E}_t\left[e^{-\int_t^u h_x \, dx}\right] 1_{\tau > t}. \quad (41)$$

Intuition:

- A positive instantaneous hazard rate means default can occur instantaneously (i.e. without warning).
- Factor out the positive $h_t$ and if that’s all of $s$, then $\tau$ must be totally inaccessible.
One way a default can be predictable:
Survival example 2

Another way a default can be predictable (or maybe just accessible):

![Survival curves graph]

-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
0  2  4  6  8  10
Probability
Year
s(t)
s4(t)
Building models

Survival processes allow us to build models:

**Theorem**

If $\tau^1$ and $\tau^2$ are independent stopping times with survival processes $s^1$ and $s^2$ and conditional CDFs $F^1$ and $F^2$, then the survival processes for $\tau^u = \min(\tau^1, \tau^2)$ is given by

$$s^1 s^2$$

and the conditional CDF of $\tau^l = \max(\tau^1, \tau^2)$ is given by

$$F^1 F^2$$

Conversely, by multiplying survival processes, we create new stopping times.

Multiplying survival processes is adding hazard rate processes, so this is closely related to the fixed income practice of adding spreads to rates.
Decomposition

Survival processes also sometimes allow us to decompose stopping times into pieces by their predictability.

Definition
The totally inaccessible component of a stopping time $\tau$ is given on $\tau > t$ by the proto-survival processes

$$s^i_t(u) = E[e^{-\int_t^u h_x \, dx} \mid F_t] \quad (44)$$

The predictable component of a stopping time

$$\tau^p(\omega) = u \text{ iff } s_t(u)(\omega) > 0 \ \forall t < u \text{ and } \lim_{t \uparrow u} s_t(u)(\omega) = 0 \quad (45)$$

The (purely) accessible component of a stopping time

$$\tau^a(\omega) = u \text{ iff } \lim_{t \uparrow u} s_t(u)(\omega) > \epsilon > 0 \text{ and } \lim_{t \uparrow u} \Delta s_t(u)(\omega) > \epsilon > 0$$
If the above conjectures hold, then

**Theorem**

$\tau^a$ *is accessible and “nowhere predictable”.*

**Theorem**

$\tau^p$ *is predictable.*

**Theorem**

$\tau = \min(\tau^a, \tau^p, \tau^i)$
Thank you!
References I


<table>
<thead>
<tr>
<th>Reference</th>
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References IV


