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APPLICATION OF BUFFERED PROBABILITY OF EXCEEDANCE IN RELIABILITY OPTIMIZATION PROBLEMS*

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Abstract. *We propose an approach to solving the problem of optimizing the reliability of complex systems using Buffered Probability of Exceedance (bPOE). As a research subject, we consider the model of optimal control of oscillations of a hinged beam with random defects. This example shows that minimizing bPOE in reliability optimization problems is more preferable than minimizing the classical probability of exceedance.*

Keywords: *bPOE, reliability optimization, waveform, defects, approximation error.*

INTRODUCTION

This paper describes an approach to solving reliability optimization problems for complex systems using buffered probability of exceedance (bPOE). The subject of research considered is an optimal control model for oscillations of a hinged beam with random defects proposed in [1, 2].

At the stage of structural design of a mechanical device for excitation and formation of undulation, there is the issue of optimal selection of the device parameters. Such devices can be used for generation, transformation, and transfer of information (and, more generally, for transfer of wave energy). Excitation of oscillations is caused by several external periodic forces. The main objective of the design stage is to select structural parameters such that the oscillatory behavior is as similar to the given one as possible for a long time of operation of this device.

Paper [1] considers the simplest deterministic mathematical model of such a device, which describes controlled excitation of oscillations of a hinged homogenous beam without any defects, and elaborates on a semianalytic method for solving the optimization problem of determining the number and characteristics of forces providing the desired modal behavior to the specified accuracy. The target function used is in the form of an approximation error, which is determined as a root-mean-square deviation of the simulated modal behavior from the desired one. Paper [2] proposes a more general deterministic mathematical model taking into account the presence of defects on the beam surface.

While modeling the oscillation of a beam with a defect, [2] analyses the type of approximation error dependence on the defect magnitude, its localization on the beam, and the number of forces applied. It was demonstrated that, depending on the parameters of the defect, the optimal value of the approximation error may significantly (by up to two orders) exceed that obtained without regard to defects. Thus, underestimation of the effect of defects on the structure performance can result in substantial losses during its commercial operation. Since information about the number and the

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parameters of future defects that will occur in the course of the structure operation is not available at the design stage, a need arose for elaboration of a stochastic version of the model taking into account that onset of the defects is random.

In the classical statement, the stochastic optimal control problem for oscillations of a hinged beam with random defects is to select parameters of control actions such that the probability of the approximation error exceeding a certain set threshold (probability of failure) is minimized. As we know, the probability of failure does not account for large values of the approximation error located in the tail of the distribution function. Furthermore, minimization of the probability of failure entails certain mathematical difficulties. This article proposes minimizing the bPOE rather that the probability of failure, since the former measure of risk takes account of the average value of the distribution function tail and has a number of useful properties that simplify the reliability optimization process [3–7].

Section 1 presents the statement of the optimal control problem for excitation of oscillations of a hinged beam. Subsection 1.1 considers the deterministic statement of the problem. Subsection 1.2 outlines the classical statement of the reliability optimization problem. Subsection 1.3 provides an overview of the best-known measures of risk, as well as defines the optimization problem using bPOE. Section 2 analyses the obtained results of solving this problem.

1. STATEMENT OF THE OPTIMAL CONTROL PROBLEM FOR EXCITATION OF OSCILLATIONS OF A HINGED BEAM

Paper [1] addressed steady-state oscillations of a hinged elastic homogenous beam of unit length without any defects with a frequency of ω under I point forces applied at points ξ_j with complex magnitudes $F_j = u_j + iv_j$, $j = 1, \ldots, I$. Let us introduce the following notations: $\vec{F} = (F_1, \ldots, F_I)^{\mathrm{T}}$ and $\vec{\xi} = (\xi_1, \ldots, \xi_I)^{\mathrm{T}}$. According to the Kirchhoff's model, the problem of exciting such oscillations is reduced to the following boundary-value problem [8, 9]:

$$\frac{\partial^2}{\partial x^2}\left(ED\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho\frac{\partial^2 w}{\partial t^2} = \sum_{i=1}^{I} F_i e^{i\omega t}\delta(x-\xi_i),\ x \in (0, L),$$
$$\begin{cases} w(0,t) = w(L,t) = 0, \\ \dfrac{\partial^2 w(0,t)}{\partial x^2} = \dfrac{\partial^2 w(L,t)}{\partial x^2} = 0, \end{cases} \tag{1}$$

where E, D, and ρ are the Young's modulus, the moment of inertia of the cross-section, and the beam density, respectively, $w(x,t)$ is a deflection of the beam at time t at a point x, $x \in (0,1)$. Solution of problem (1) was sought in the form $w(x,t) = w(x)e^{i\omega t}$, where $w(x)$ is the complex magnitude of deflection.

1.1. Deterministic Statement of the Problem. Let $w(\vec{F}, \vec{\xi}, x)$ be the complex magnitude of deflection obtained when solving problem (1). The desired modal behavior is defined as $W(x) = A(x)e^{i\Phi(x)}$. Deviation of the simulated modal behavior from the desired one at fixed $\vec{F}$ and $\vec{\xi}$ is a complex function $R(\vec{F}, \vec{\xi}, x) = w(\vec{F}, \vec{\xi}, x) - W(x)$. Root mean square deviation is calculated using the formula

$$H(\vec{F}, \vec{\xi}) = \int_0^1 R(\vec{F}, \vec{\xi}, x)\overline{R}(\vec{F}, \vec{\xi}, x)\,dx = \int_0^1 |\,w(\vec{F}, \vec{\xi}, x) - W(x)\,|^2\,dx,$$

where $\overline{R}(\vec{F}, \vec{\xi}, x)$ is the complex conjugate of the function $R(\vec{F}, \vec{\xi}, x)$. In the deterministic statement, the optimal control problem for excitation of oscillations of a hinged homogenous beam is to find the values of parameters F_i, ξ_i, $i = 1, \ldots, I$, at which the minimum of the root mean square deviation of the simulated modal behavior from the desired $H(\vec{F}, \vec{\xi})$ is achieved:

$$H^* = H(\vec{F}^*, \vec{\xi}^*) = \min_{\vec{F}, \vec{\xi}} H(\vec{F}, \vec{\xi}). \tag{2}$$

Paper [2] considered a more complicated problem statement taking into account the presence of inhomogeneities (defects) on the beam. There, it was assumed that there is comprehensive information available on the parameters of defects, which are presented as components of a vector $\vec{V}^{\mathrm{T}} = (N_{\mathrm{def}}, \vec{\chi}_{\mathrm{def}}, \vec{\ell}_{\mathrm{def}}, d\vec{E}_{\mathrm{def}})$, where N_{def} is the number of defects; $\vec{\chi}_{\mathrm{def}} = (\chi_1, \ldots, \chi_{N_{\mathrm{def}}})$ is the vector of defect localization points on the beam; $\vec{\ell}_{\mathrm{def}} = (\ell_1, \ldots, \ell_{N_{\mathrm{def}}})$ is the vector of geometric dimensions of the defects; $d\vec{E}_{\mathrm{def}} = (\Delta E_1, \ldots, \Delta E_{N_{\mathrm{def}}})$ is the vector of relative variations of the Young's modulus. It was assumed that the kth defect with a length ℓ_k, localized at point $x_k = \chi_k \neq \xi_i$, $i = 1, \ldots, I$, is described by variation of the Young's modulus ΔE_k, $k = 1, \ldots, N_{\mathrm{def}}$. With such a statement, the target function is calculated using the formula [2]

$$L(\vec{F}, \vec{\xi}, \vec{V}) = \int_0^1 | w(\vec{F}, \vec{\xi}, \vec{V}, x) - W(x) |^2 \, dx,$$

where $w(\vec{F}, \vec{\xi}, \vec{V}, x)$ is the complex magnitude of deflection obtained with regard for the presence of defects (see [2]), and the deterministic optimal control problem for excitation of oscillations of a hinged beam with a known vector of defect parameters $\vec{V}$ is defined as follows:

$$L^*(\vec{V}) = L(\vec{F}^*, \vec{\xi}^*, \vec{V}) = \min_{\vec{F}, \vec{\xi}} L(\vec{F}, \vec{\xi}, \vec{V}).$$

1.2. Classical Statement of the Reliability Optimization Problem. The stochastic statement of the optimization problem assumes that a random number of defects described by random parameters emerge on the beam. Let $\vec{\theta} = (N_{\mathrm{def}}, \vec{\chi}_{\mathrm{def}}, \vec{l}_{\mathrm{def}}, d\vec{E}_{\mathrm{def}})$ denote a random vector, in which N_{def} is a random number of defects limited to a given maximum value; $\vec{\chi}_{\mathrm{def}}$ is the vector of random defect locations on the beam; $\vec{l}_{\mathrm{def}}$ is the vector of random geometric dimensions of the defects; $d\vec{E}_{\mathrm{def}}$ is the vector of random relative variations of the Young's modulus. In the stochastic case, the root mean square deviation of the simulated modal behavior from the desired one at the given values of components of vectors $\vec{F}$ and $\vec{\xi}$ is a random variable $L(\vec{F}, \vec{\xi}, \vec{\theta})$, which depends on the random vector $\vec{\theta}$.

The quantitative indicator of proper system performance used in the reliability theory is the probability of failure, i.e., the probability that, at fixed values of control variables $\vec{F}$ and $\vec{\xi}$, value of a random quantity $L(\vec{F}, \vec{\xi}, \vec{\theta})$ will exceed a certain given threshold *h*. Values of the random vector $\vec{\theta}$, at which $L(\vec{F}, \vec{\xi}, \vec{\theta}) > h$, correspond to failure of the system. According to the classical reliability theory, the stochastic problem of optimal parameter selection for a mechanical device used for excitation and formation of undulation is to minimize the probability that root mean square deviation of the simulated modal behavior from the desired one will exceed the given threshold *h*:

$$\min_{\vec{F}, \vec{\xi}} P\{L(\vec{F}, \vec{\xi}, \vec{\theta}) > h\}. \tag{3}$$

1.3. Generalized Statement of the Stochastic Problem Using the Measure of Risk. Probability of failure is one of possible measures of risk. With a more general statement, the problem of optimal parameter selection for a mechanical device used for excitation and formation of undulation can be defined as

$$\min_{\vec{F}, \vec{\xi}} \rho(L(\vec{F}, \vec{\xi}, \vec{\theta})), \tag{4}$$

where ρ is a certain measure of risk.

Along with the probability $P\{Y > h\}$ of exceedance (POE) by the random quantity Y of the given threshold h, there is another commonly used measure of risk, namely, α-quantile, $q_\alpha(Y)$. In financial applications, the quantile is called VaR (Value at Risk). These measures of risk, i.e., POE and the quantile, take into account only minimum values of the random quantity Y, which are located in the distribution function tail, and do not contain any information on the values located at the end of the tail, which the random quantity Y may adopt with a very low probability.

More useful properties are inherent to other measures of risk, such as CVaR (Conditional Value at Risk, that is, conditional VaR) and bPOE. A measure of risk called CVaR describes expected values of a random quantity that are located in the tail of its distribution function. Unlike POE and VaR, CVaR takes account of the magnitude of scatter in the distribution function tail. Instead of the term CVaR, which is commonly used in financial engineering publications, papers [3, 10] suggest a more general term "superquantile". At that, instead of the conventional designation $\mathrm{CVaR}_\alpha(Y)$, $\bar{q}_\alpha(Y)$ is used, pointing to association with the quantile. The measure of risk CVaR is determined as follows.

If, for a given confidence level α, the distribution function $F_Y(y)$ of the random quantity Y does not demonstrate a jump at point $q_\alpha(Y)$, then CVaR, i.e., $\bar{q}_\alpha(Y)$, is calculated using the formula [11]: $\bar{q}_\alpha(Y) = E[Y : Y \geq q_\alpha(Y)]$. In the general case, when the distribution function $F_Y(y)$ demonstrates a jump at point $q_\alpha(Y)$, the measure of risk CVaR is calculated using a more complicated formula proposed in [12]:

$$\bar{q}_\alpha(Y) = \min_{v} \left(v + \frac{E[Y - v]^+}{1 - \alpha} \right),$$

where $[\cdot]^+ = \max\{\cdot, 0\}$. This formula allows simultaneous calculation of both risk indicators, VaR and CVaR, using linear programming techniques and non-smooth optimization.

Paper [3] discusses an auxiliary random quantity $\bar{Y}$, which is generated by CVaR values of a random quantity Y on change of α from 0 to 1, and its distribution function called superdistribution, $\bar{F}_Y = F_{\bar{Y}} = \bar{q}_Y^{-1}$. Thus, at a fixed value of α the value of CVaR, i.e., $\bar{q}_\alpha(Y)$, is interpreted as quantile of superdistribution $\bar{F}_Y$.

Paper [3] proposes a new measure of reliability with a threshold $h = 0$; an alternative to POE is a buffered probability of failure. This indicator is more conservative than POE. Along with the probability of threshold exceedance, it allows taking into account the degree of that exceedance, which is impossible when using the classical probability of failure.

Paper [6] proposes adopting the average value of the distribution tail as the threshold, as well as introducing a new measure bPOE, which is a function of the threshold value y and the random variable Y and is determined as follows:

$$\bar{p}_Y(y) = \begin{cases} 0 & \text{if } \; y \geq \sup Y; \\ 1 - \bar{q}_Y^{-1}(y) & \text{if } \; E[Y] < y < \sup Y; \\ 1 & \text{otherwise.} \end{cases}$$

In [7], the following formula for calculating bPOE has been obtained:

$$\bar{p}_Y(y) = \begin{cases} \min\limits_{b \geq 0} 0, \;\; E[b(Y - y) + 1]^+ & \text{if } \; y \neq \sup Y; \\ 0 & \text{if } \; y = \sup Y. \end{cases}$$

Since bPOE is the upper limit of POE, the minimization of bPOE helps to reduce the POE value. Although bPOE and CVaR are interrelated, there is a fundamental difference between the bPOE and CVaR minimization problems. Minimization of bPOE reduces the probability of an unwanted event, while the CVaR minimization problem provides for minimizing the average value of the distribution tail at a fixed value of this probability. Thus, these problems are complementary.

Since bPOE has many useful properties, it is reasonable to use bPOE as the measure of risk ρ in problem (4). In order to increase the algorithm stability, target functional (4) should also include a regularizing functional $D(\vec{F}, \vec{\xi})$ with a penalty coefficient λ.

The paper further considers the following generalized statement of the stochastic problem of optimal parameter selection for a mechanical device used for excitation and formation of undulation in the presence of defects:

$$\min_{\vec{F}, \vec{\xi}} bPOE_h(L(\vec{F}, \vec{\xi}, \vec{\theta})) + \lambda D(\vec{F}, \vec{\xi}). \tag{5}$$

2. THE SIMULATION RESULTS FOR OSCILLATION OF A BEAM WITH RANDOM DEFECTS

The optimal control problem for oscillation of a non-homogenous beam with random defect parameters at different values of frequency (wave number k) and number I of forces applied was solved using the multifunction package PSG (Matlab Interface) provided by American Optimal Decision, USA [13].

It was assumed that the number of defects N_{def} does not exceed the given value $N_{\max}$, and other components of the defect parameter vector $\vec{\theta}$ are independent random variables, which are uniformly distributed within the given limits.

The numerical simulation was performed as follows.

At the first stage, given input parameters (the desired modal behavior $W(x) = \text{Re}\, W(x) + i\text{Im}\, W(x)$, wave number k, and number of forces I) a deterministic optimal control problem was solved for oscillations of an elastic beam without any defects (2). As a result, the following optimum characteristics of the forces $\vec{F}^{\,\text{det}}$ and $\vec{\xi}^{\,\text{det}}$ have been obtained: their points of application $\vec{\xi}^{\,\text{det}} = (\xi_1^{\text{det}}, \dots, \xi_I^{\text{det}})^{\text{T}}$, as well as real and imaginary parts of their magnitudes $\vec{F}^{\,\text{det}} = (F_1^{\text{det}}, \dots, F_I^{\text{det}})^{\text{T}}$ $= (u_1^{\text{det}} + iv_1^{\text{det}}, \dots, u_I^{\text{det}} + iv_I^{\text{det}})^{\text{T}}$, which provide best approximation of the given modal behavior and pointwise phase of the beam oscillation at a frequency corresponding to the given wave number k. In addition, an optimal value $H^* = H(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}})$ of the root mean square deviation of the simulated modal behavior from the desired one has been obtained, which corresponds to the optimal solution of the deterministic problem $\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}$.

At the second stage, a sample of values of a random variable $L(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta})$ was generated with a size of $M = 20000$ under the assumption, that there are defects described by random parameters on the beam. Here, the following procedure was used:

1) a number of defects N_{def} was randomly selected, so that it does not exceed the set value $N_{\max}$;

2) for $n = 1, \dots, N_{\text{def}}$, the values of parameters describing the nth defect were randomly generated, i.e., its location on the beam $\chi_n \in (0, 1)$, geometrical dimension $\Delta L_n \in (0, 0.1)$, and variation of the Young's modulus ΔE_n;

3) components of the vector $\vec{\theta}^{\text{T}} = (N_{\text{def}}, \vec{\chi}_{\text{def}}, \vec{l}_{\text{def}}, d\vec{E}_{\text{def}})$: $\vec{\chi}_{\text{def}} = (\chi_1, \dots, \chi_{N_{\text{def}}})$, $\vec{l}_{\text{def}} = (l_1, \dots, l_{N_{\text{def}}})$ and $d\vec{E}_{\text{def}} = (\Delta E_1, \dots, \Delta E_{N_{\text{def}}})$ were determined;

4) the root mean square deviation of the simulated modal behavior from the desired

$$L(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}) = \int_0^1 | w(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}, x) - W(x) |^2 \, dx$$

(the approximation error) was calculated.

The resulting sample consisting of M values $L_1(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}_1), \dots, L_M(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}_M)$ can be regarded as equally probable realizations of a discrete random variable $L(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta})$ with the distribution function $F(x) = P\{L(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}) \le x\}$.

Four samples of the same size $M = 20000$ were generated, corresponding to values of the wave number $k = 1.8$ and the number $I = 5$ of forces applied at $N_{\max} = 1, 3, 5, 7$, and the corresponding distribution functions were plotted for each of them (Fig. 1).

In Fig. 1, G_i is the conditional distribution function, $G_i(x) = P\{L(\vec{F}^{\,\text{det}}, \vec{\xi}^{\,\text{det}}, \vec{\theta}) \le x \mid N_{\max} = i\}$. According to the reliability theory, $G_i(x)$ is the system survival probability distribution function in the presence of i defects at most. Figure 1 shows that with the increase of value $N_{\max}$ the tails of the system survival probability distribution become more "heavy".

The effect of the number of forces applied to the beam on the probability of exceedance by the approximation error of the set threshold was studied at fixed values of the parameters and $N_{\max}$. The simulation was performed for wave numbers $k = 3.6$ and 4.6, with number of defects not exceeding $N_{\max} = 5$. For each of the values k a series of deterministic optimal control problems was solved for oscillation of an elastic beam without any defects (2) at different

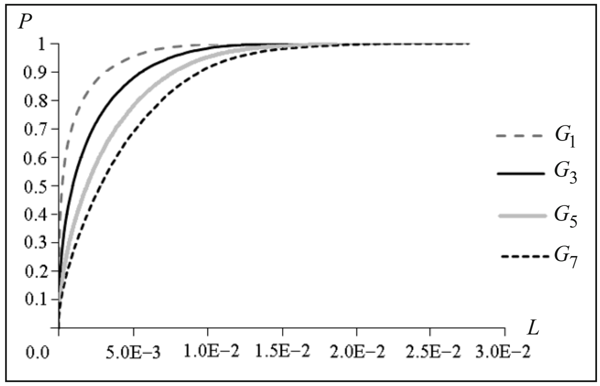

Fig. 1. Approximation error distribution functions $L(\vec{F}^{\,\det}, \vec{\xi}^{\,\det}, \vec{\theta})$ at different values $N_{\max}$.

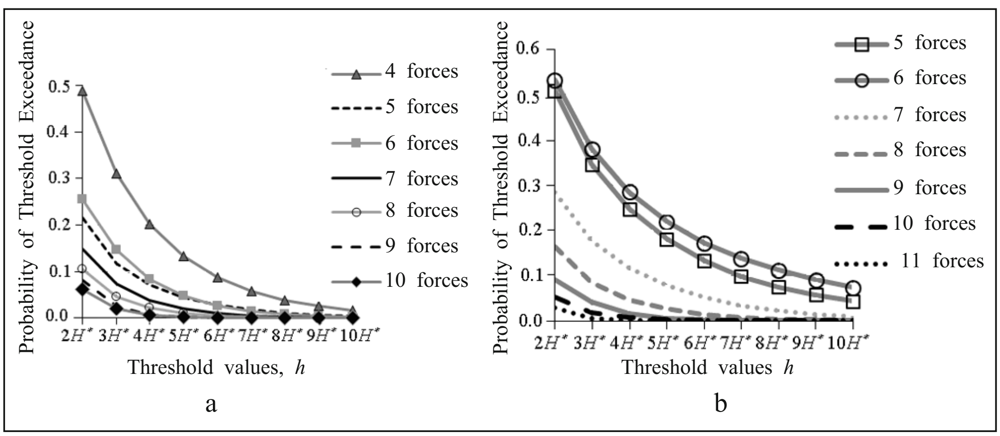

Fig. 2. Probability of exceedance by random variable $L(\vec{F}^{*}_{kI}, \vec{\xi}^{*}_{kI}, \vec{\theta}_{kI})$ of threshold h versus its values and the number I of forces: at $k = 3.6$ (a) and at $k = 4.6$ (b).

number I of forces applied. At the same time, optimal characteristics of the forces $\vec{F}^{*}_{kI}$ and $\vec{\xi}^{*}_{kI}$ and optimal value $H^{*}_{kI} = H(\vec{F}^{*}_{kI}, \vec{\xi}^{*}_{kI})$ of the root mean square deviation of the simulated modal behavior from the desired one were determined. After that, a random number of defects not exceeding $N_{\max}$ was generated $M = 2000$ times, for which random values of their parameters were generated (length, location on the beam, and variation of the Young's modulus). Thus, a random vector $\vec{\theta}_{kI}$ was formed. As a result, a sample has been obtained from the distribution of root-mean-square deviations of the simulated modal behavior from the desired one $L(\vec{F}^{*}_{kI}, \vec{\xi}^{*}_{kI}, \vec{\theta}_{kI})$ with a size of M, from which probabilities $P\{L(\vec{F}^{*}_{kI}, \vec{\xi}^{*}_{kI}, \vec{\theta}_{kI}) > h\}$ of exceedance by the random variable $L(\vec{F}^{*}_{kI}, \vec{\xi}^{*}_{kI}, \vec{\theta}_{kI})$ of the given threshold values h were calculated. For each wave number k, a set of threshold values $\{2H^{*}_{k}, \ldots, 10H^{*}_{k}\}$ was formed, where $H^{*}_{k} = H^{*}_{kI_{\min}}$ and $I_{\min}$ is the minimum number of forces corresponding to the wave number k. At $k = 3.6$, the values $I_{\min} = 4$ and $H^{*}_{3.6} = 4.38\text{E–}4$; at $k = 4.6$, values $I_{\min} = 5$ and $H^{*}_{4.6} = 5.99\text{E–}4$. The results obtained are presented in Fig. 2.

Fig. 2 shows that at any fixed threshold h, as the number of forces applied increases, the probability of threshold exceedance tends to decrease. From separate consideration of the results obtained for odd and even number of the forces, the probability of threshold exceedance is readily seen to be strictly decreasing with increase of the number of forces. With increase of the threshold value h, there is a decrease in the scatter of these probabilities.

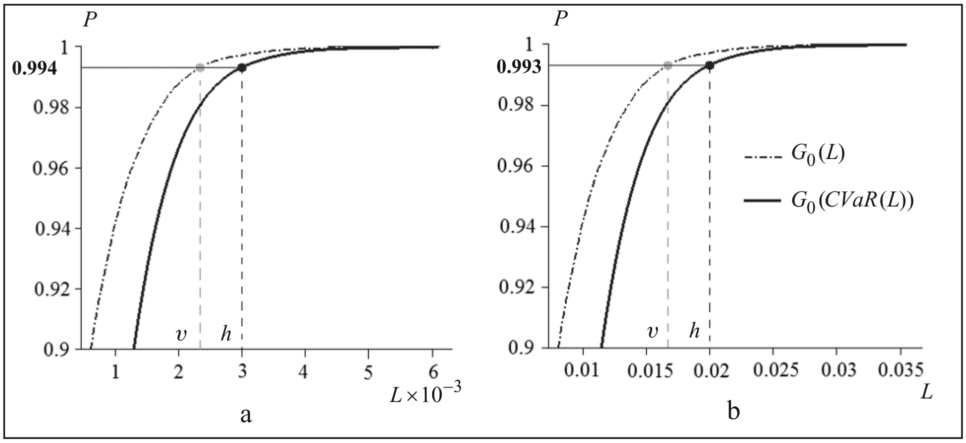

Fig. 3 Tails of distribution functions $G_0(L)$ and $G_0(CVaR(L))$ before optimization: at $k = 3.6$, $I = 8$ (a) and at $k = 1.8$, $I = 4$ (b).

The paper further examines the case when the number of forces and the threshold value are fixed. Reliability of a system is commonly described by the value of POE, i.e., by the probability $P(X > h)$ that the random quantity X exceeds a certain given threshold h. If X simulates the losses, this indicator does not reflect the losses far above the threshold h, which can arise with very low probabilities. Selection of a new threshold value v such that $E[X \mid X > v] = h$ allows taking account of the losses located at the end of the tail of the random variable distribution function X. Therefore, instead of minimizing POE (problem (3)), bPOE is minimized (problem (5)).

Let $u_1, \dots, u_M$ denote the values that the random variable $L(\vec{F}^*_{kI}, \vec{\xi}^*_{kI}, \vec{\theta}_{kI})$ takes. For simplicity, in what follows, we will denote this random variable as L, and its distribution function as $G_0(L)$. For probability levels $\alpha_i = G_0(u_i) = P\{L \le u_i\}$, $i = 1, \dots, M$, let us calculate the values $w_i = CVaR_{\alpha_i}(L)$, $i = 1, \dots, M$. Let $CVaR(L)$ denote the random variable taking the values $w_1, \dots, w_M$ with equal probabilities, and let $G_0(CVaR(L))$ denote its distribution function. Figure 3a shows the tails of distribution functions $G_0(L)$ and $G_0(CVaR(L))$ in case when $k = 3.6$ and $I = 8$, while Fig. 3b shows them in case when $k = 1.8$ and $I = 4$.

Suppose a threshold h is set for values of a random quantity $CVaR(L)$, and a threshold v is chosen for values of a random quantity L, so that the following equality holds: $E[L \mid L > v] = h$. We selected α-quantile of the distribution function $G_0(L)$, $\alpha = P\{CVaR(L) \le h\}$, for v. For the case of $k = 3.6$ and $I = 8$ (see Fig. 3a), the value selected is $h = 3.0\text{E}-3$ and $v = 2.5\text{E}-3$, $\alpha = 0.994$. For the case of $k = 1.8$ and $I = 4$ (see Fig. 3b), the value selected is $h = 0.02$ and $v = 0.0167$, $\alpha = 0.993$.

Figure 4 shows how the tail of distribution function $G_1(CVaR(L))$ of the random quantity $CVaR(L)$ obtained after the optimization differs from its initial distribution function $G_0(CVaR(L))$; it also indicates the ranges of bPOE_0 and bPOE_1 of the probability values $P\{CVaR(L) > h\}$ corresponding to these distribution functions. In case of $k = 3.6$ and $I = 8$ (see Fig. 4a), the value $\text{bPOE}_{0.003}$ has decreased by 29.43% from $5.76\text{E}-3$ to $4.07\text{E}-3$, while the average value of the tail of distribution function $CVaR(L)$ corresponding to the probability level $\alpha = 0.994$ has decreased by 4.99% from $3.00\text{E}-3$ to $2.85\text{E}-3$.

In case of $k = 1.8$ and $I = 4$ (see Fig. 4b), the value $\text{bPOE}_{0.02}$ has decreased by 30.15% from $6.60\text{E}-3$ to $4.61\text{E}-3$, while, the average value of the tail of distribution function $CVaR(L)$c orresponding to the probability level $\alpha = 0.993$ has decreased by 5.5% from 0.02 to 0.0189.

Figure 5 shows how much the tail of distribution function $G_1(L)$ of the random variable L obtained after the optimization differs from its initial distribution function $G_0(L)$. In case of $k = 3.6$ and $I = 8$ (see Fig. 5a), due to the minimization of bPOE, the value $\text{POE}_{0.025}$ has decreased by 20.14% from $5.76\text{E}-3$ to $4.60\text{E}-3$. In case $k = 1.8$ and $I = 4$ (see Fig. 5b), the value $\text{POE}_{0.0167}$ has decreased by 32.58% from $6.60\text{E}-3$ to $4.45\text{E}-3$.

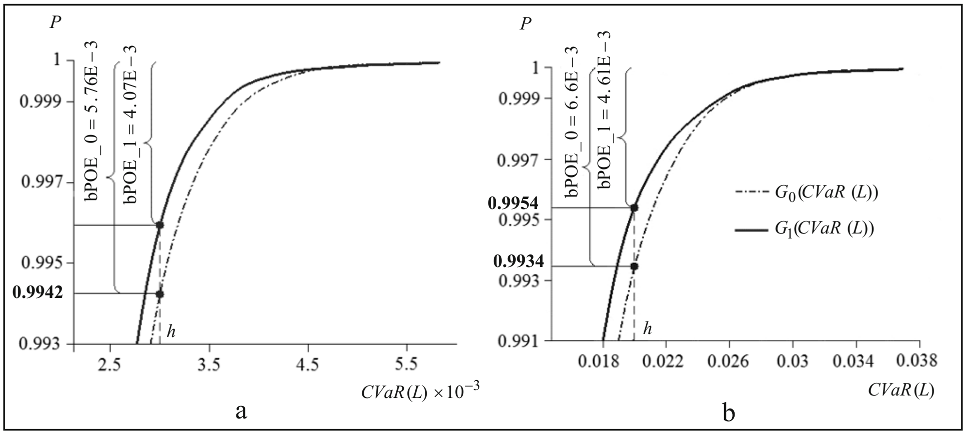

Fig. 4. Tails of distribution functions $G_0(CVaR(L))$ and $G_1(CVaR(L))$: at $k = 3.6$, $I = 8$ (a) and at $k = 1.8$, $I = 4$ (b).

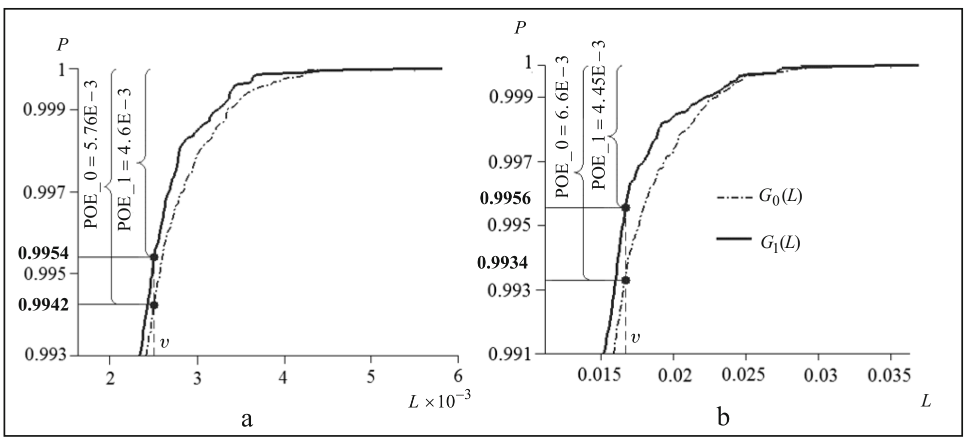

Fig. 5. Tails of distribution functions $G_0(L)$ and $G_1(L)$: at $k = 3.6$, $I = 8$ (a) and at $k = 1.8$, $I = 4$ (b).

CONCLUSIONS

The article demonstrates that minimization of bPOE can simultaneously decrease the value of POE and the average value of the distribution function tail. In case of $k = 3.6$ and $I = 8$, with minimization of bPOE the value of this indicator has decreased by 29.43%, while the value of POE has decreased by 20.14%, and the average value of the tail has decreased by 4.99%. In case of $k = 1.8$ and $I = 4$, with minimization of bPOE its value has decreased by 30.15%, the value of POE has decreased by 32.58%, and the average value of the tail has decreased by 5.5%.

The paper describes numerical experiments for determining how the number of forces applied to the beam influences the probability of exceedance by the approximation error of the set threshold. The results of simulation have shown that at any fixed threshold h, as the number of forces applied increases, the probability of threshold exceedance tends to decrease. From separate consideration of the results obtained for odd and even number of the forces, the probability of threshold exceedance is readily seen to be strictly decreasing with increase of the number of forces. The decrease in scatter of these probabilities becomes noticeable with increase of the threshold value h.

REFERENCES

1. G. M. Zrazhevsky, "Determination of the optimal parameters of the beam waveform actuation," Bulletin of Taras Shevchenko National University of Kyiv, Ser. Physics & Matematics, Issue 3, 138–141 (2013).
2. G. Zrazhevsky, A. Golodnikov, and S. Uryasev, "Mathematical methods to find optimal control of oscillations of a hinged beam (Deterministic case)," Cybern. Syst. Analysis, Vol. 55, No. 6, 1009–1026 (2019).
3. R. T. Rockafellar and J. O. Royset, "On buffered failure probability in design and optimization of structures," J. of Reliability Engineering and System Safety, Vol. 95, Iss. 5, 499–510 (2010).
4. R. T. Rockafellar, "Convexity and reliability in engineering optimization," in: Proc. of the 9th Intern. Conf. on Nonlinear Analysis and Convex Analysis (Chiangrai, Thailand) (2015), pp. 1–10.
5. R. T. Rockafellar and J. O. Royset, "Random variables, monotone relations, and convex analysis," Mathematical Programming, Vol. 148, Iss. 1–2, 297–331 (2014).
6. A. Mafusalov and S. Uryasev, "Buffered probability of exceedance: Mathematical properties and optimization algorithms," SIAM J. on Optimization, Vol. 28, Iss. 2, 1077–1103 (2018).
7. M. Norton and S. Uryasev, "Maximization of AUC and buffered AUC in binary classification," Mathematical Programming, Vol. 174, 575–612 (2019).
8. L. H. Donnell, Beams, Plates, and Shells, McGraw-Hill Book Company, New York (1976).
9. S. Timoshenko asnd S. Woinowsky-Krieger, Theory of Plates and Shells, McGraw-hill Book Company, New York (1959).
10. R. T. Rockafellar and S. Uryasev, "The fundamental risk quadrangle in risk management, optimization and statistical estimation," Surveys in Operations Research and Management Science, Vol. 18, Iss. 1–2, 33–53 (2013).
11. R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," The J. of Risk. Vol. 2, Iss. 3, 21–41 (2000).
12. R. T. Rockafellar and S. Uryasev, "Conditional Value-at-Risk for general loss distributions," J. of Banking and Finance, Vol. 26, No. 7, 1443–1471 (2002).
13. AORDA Portfolio Safeguard (PSG). URL: http://www.aorda.com/html/PSG_Help_HTML/index.html?bpoe.htm.