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MATHEMATICAL METHODS TO FIND OPTIMAL CONTROL OF OSCILLATIONS OF A HINGED BEAM (DETERMINISTIC CASE)*

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Abstract. *We consider several problem statements for the optimal controlled excitation of oscillations of a hinged beam. Oscillations occur under the influence of several external periodic forces. In the simplest statement, it is assumed that the structure of the beam is homogeneous. In a more complex formulation, inhomogeneities (defects) on the beam are allowed. The goal of controlling the oscillations of the beam is to provide a predetermined shape and a predetermined pointwise phase of oscillations in a given frequency range. The problem is to determine the number of forces and their characteristics (application, amplitude, and phase of oscillations), which provide the desired waveform with a given accuracy. With the help of analytical mathematical methods, the problems in question are reduced to simpler multiextremum problems of minimizing basic functionals, which can be numerically solved using the multifunctional package AORDA PSG.*

Keywords: *vibrations, waveform, optimal actuation.*

INTRODUCTION

Let us consider a problem about optimal controlled excitation of oscillations of a hinged beam. The problem is a development of the approach proposed in [1]. By optimal control we understand providing a beam oscillation mode such that the waveform is the most close to the desired one with respect to root mean square deviation. Oscillations are subject to several external periodic forces. In the simplest problem statement, the beam structure is supposed to be homogeneous. A more complex statement takes into account inhomogeneities (defects) on the beam. Full information about parameters of the defect (its length, localization on the beam, and variation in the Young modulus) are assumed to be known. The goal of controlling the oscillations of the beam is to provide a predetermined shape and a predetermined pointwise phase of oscillations in a fixed frequency range.

1. CONTROL OF THE HOMOGENEOUS BEAM

Let us consider a simplified version of model problem about optimal controlled excitation of oscillations of a hinged elastic homogeneous beam subject to periodic lumped forces. The problem is to find the number of forces and their characteristics (application, amplitude, and phase of oscillations), which provide the desired waveform with a predetermined accuracy.

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1.1. Constitutive Equations in Dimensionless Form. According to the Kirchhoff model, the problem of excitation of a hinged elastic homogeneous beam with the use of I forces with complex amplitudes F_i, $i=1,\dots,I$, with frequency ω can be reduced to the following boundary-value problem [2, 3]:

$$\frac{\partial^2}{\partial x^2}\left(ED\frac{\partial^2 w}{\partial x^2}\right)+\rho\frac{\partial^2 w}{\partial t^2}=\sum_{i=1}^{I}F_i e^{i\omega t}\delta(x-\xi_i),\ \ x\in(0,L),$$
$$\begin{cases} w(0,t)=w(L,t)=0,\\ \dfrac{\partial^2 w(0,t)}{\partial x^2}=\dfrac{\partial^2 w(L,t)}{\partial x^2}=0, \end{cases} \tag{1}$$

where E, D, and ρ are, respectively, Young modulus, moment of inertia of cross section, and density of the beam. When considering forced oscillations, we assume that $w(x,t)=w(x)e^{i\omega t}$, where $w(x)$ is the complex amplitude of deflection.

Let us introduce the following dimensionless quantities:

$$\hat{w}=\frac{w}{w_0},\ \hat{x}=\frac{x}{L},\ \hat{k}^4=\frac{\omega^2\rho}{\pi EI},\ \hat{F}_i=\frac{F_i L^4}{EIw_0}.$$

For convenience, we omit ^ and obtain the following dimensionless boundary-value problem:

$$w^{(4)}-\pi^4k^4\mathrm{w}=\sum_{i=1}^{I}F_i\delta(x-\xi_i),\ \ i=1,\dots,I;\ \ x\in(0,1),$$
$$\begin{cases} w(0)=w(1)=0,\\ w''(0)=w''(1)=0. \end{cases} \tag{2}$$

The value $k=1$ corresponds to the first waveform. We determine the solution of Eq. (2) as

$$w(x)=\sum_{i=1}^{I}F_iG(x,\xi_i), \tag{3}$$

where $G(x,\xi)$ is the Green function for Eq. (2):

$$G_x^{(4)}-\pi^4k^4G=\delta(x-\xi),\ \ x\in(0,1),$$
$$\begin{cases} G(0,\xi)=G(1,\xi)=0, \quad x\in(0,1),\\ G_x''(0,\xi)=G_x''(1,\xi)=0. \end{cases} \tag{4}$$

1.2. Statement of the Control Problem. Let the required waveform

$$W(x)=A(x)e^{i\Phi(x)} \tag{5}$$

be specified for the residual $R(x)=w(x)-W(x)=A_Re^{i\varphi_R}$, where $w(x)$ is the solution of Eq. (2) for the given I, F_i, and ξ_i, $i=1,\dots,I$. Let $I(\cdot)$ be a positive definite convex functional in space L_2, which is used as optimality criterion in the following problems of control of the system.

Problem 1. For the given number of forces I and required waveform $W(x)$, find the values of parameters F_i and ξ_i, $i=1,\dots,I$, for which the minimum value of the functional $I(R)$ is attained.

Problem 2. Find the minimum number of forces I and respective values of parameters F_i and ξ_i, $i=1,\dots,I$, for which $I(R)\le\varepsilon^2$, where ε is a given accuracy.

Statement 1. If

$$I(f)=\int_0^1 f\bar{f}\,dx, \tag{6}$$

then minimization of the functional $I(R)$ is equivalent to minimization of the residues of amplitude and phase simultaneously.

Proof. Let $w = re^{i\varphi}$ be the solution of problem (2), $0 \le \varphi - D < \pi$. Then

$$I(R) = \int_0^1 R\overline{R}\, dx = \int_0^1 \left\{ (r-A)^2 + 4rA \sin^2\left(\frac{\varphi - D}{2}\right)\right\} dx.$$

Since r and A are positive, the functional $I(R)$ is convex with respect to $(r-A)$ and $(\varphi - D)$.

Remark 1. In what follows, we will take expression (6) as the functional $I(R)$.

1.3. Necessary Conditions of the Minimum of Functional. It is obvious that functional $I(R)$ has the form

$$I(R) = \int_0^1 |w|^2\, dx - 2\,\mathrm{Re}\int_0^1 wW dx + \int_0^1 |W|^2\, dx. \tag{7}$$

Statement 2. Let $F_j = u_j + iv_j$. Then

$$\frac{\partial I}{\partial u_j} + i\frac{\partial I}{\partial v_j} = 2\int_0^1 G_j (w - W)\, dx.$$

Proof. From expression (3) it follows that

$$\frac{\partial I}{\partial u_j} = 2\int_0^1 G_j \mathrm{Re}\,(w - W)\, dx,\ \ \frac{\partial I}{\partial v_j} = 2\int_0^1 G_j \mathrm{Im}\,(w - W)\, dx,$$

where $G_j = G(x, \xi_j)$.

Hence, functional $I(R)$ is convex with respect to F_j, $j = 1, \ldots, I$. Since minimization of root-mean-square functionals reduces to the Galerkin method [4, 5], the necessary optimality condition for the functional $I(R)$ with respect to v_j is orthogonality of R and G_j.

Let us introduce the following notation:

$$G_j = G(x, \xi_j),\ K_{ij} = \int_0^1 G_i G_j dx,\ \boldsymbol{K} = \{K_{ij}\}_{i,j=1}^{I},\ b_j = \int_0^1 G_j W dx, \tag{8}$$
$$\vec{b}^{\mathrm{T}} = \{b_j\}_{j=1}^{I},\ \vec{F}^{\mathrm{T}} = \{F_j\}_{j=1}^{I}.$$

Then it is obvious that

$$\frac{\partial I(R)}{\partial \vec{F}} = 2(\boldsymbol{K}\vec{F} - \vec{b}), \tag{9}$$

where

$$\left(\frac{\partial I(R)}{\partial \vec{F}}\right)^{\mathrm{T}} = \left\{\frac{\partial I(R)}{\partial u_j} + i\frac{\partial I(R)}{\partial v_j}\right\}_{j=1}^{I}.$$

When

$$\frac{\partial I(R)}{\partial \vec{F}} = 0, \tag{10}$$

the values of $\vec{F}$ that minimize $I(R)$ can be calculated by the formula

$$\vec{F} = \boldsymbol{K}^{-1}\vec{b}. \tag{11}$$

Let us use the notation (8) and represent (7) as

$$I(R) = \vec{F}^{\mathrm{T}}\boldsymbol{K}\vec{F} - 2\mathrm{Re}\,(\vec{F}\vec{b}^{\mathrm{T}}) + \int_0^1 |W|^2 dx. \tag{12}$$

Statement 3. Matrix $\boldsymbol{K}$ is real, symmetric, and positive definite.

Proof. The first two properties of matrix $\boldsymbol{K}$ follow from (8). Let us prove that $\boldsymbol{K}$ is a positive definite matrix. From the formula

$$w(x,t)=\sum_{j=1}^{I} G(x,\xi_j)F_j e^{i\omega t}$$

it follows that

$$\frac{\partial w}{\partial t}=\omega\sum_{j=1}^{I} G(x,\xi_j)\,(-v_j+iu_j).$$

Hence, kinetic energy of the beam, averaged over the period $2\pi/\omega$ is

$$T=\frac{1}{2}\int_0^1\left|\frac{\partial w}{\partial t}\right|^2 dx=\frac{\omega}{4\pi}\int_0^{2\pi/\omega} dt\int_0^1\left(\operatorname{Re}\frac{\partial w}{\partial t}\right)^2 dx$$

$$=\frac{\omega^2}{2}\sum_{i,j=1}^{I}\int_0^1 G(x,\xi_i)G(x,\xi_j)\,dx\,F_i\overline{F}_j=\frac{\omega^2}{2}\sum_{i,j=1}^{I}K_{ij}\,(u_iu_j+v_iv_j).$$

Since $T>0$ and F_i are arbitrary, then selecting $v_i=0$, we get $\sum_{i,j=1}^{I}K_{ij}u_iu_j>0$ for arbitrary u_i.

Statement 4. If (10) holds, then expression (12) becomes

$$I(R)=-\vec{b}^{\mathrm{T}}\boldsymbol{K}^{-1}\overline{\vec{b}}+\int_0^1|W|^2dx. \tag{13}$$

Proof. Let $\vec{F}=\boldsymbol{K}^{-1}\vec{b}$ (see (11)). Then

$$I(R)=(\boldsymbol{K}^{-1}\vec{b})^{\mathrm{T}}\,(\boldsymbol{K}\boldsymbol{K}^{-1})\,\overline{\vec{b}}-2\operatorname{Re}\,((\boldsymbol{K}^{-1}\vec{b})\,\overline{\vec{b}}^{\,\mathrm{T}})+\int_0^1|W|^2dx.$$

Since $\boldsymbol{K}$ and $\boldsymbol{K}^{-1}$ are real and symmetric matrices, we get

$$I(R)=-\vec{b}^{\mathrm{T}}\boldsymbol{K}^{-1}\overline{\vec{b}}+\int_0^1|W|^2dx.$$

Statement 5. The formula is true

$$\frac{\partial I(R)}{\partial\xi_j}=2\sum_{i\neq j}\frac{\partial K_{ij}}{\partial\xi_j}(u_iu_j+v_iv_j)+\frac{\partial K_{jj}}{\partial\xi_j}(u_j^2+v_j^2)-2\left(u_j\operatorname{Re}\left(\frac{\partial b_j}{\partial\xi_j}\right)+v_j\operatorname{Im}\left(\frac{\partial b_j}{\partial\xi_j}\right)\right). \tag{14}$$

Proof. Formula (14) can be easily derived from (12) if we take into account that the equalities hold

$$\frac{\partial K_{ki}}{\partial\xi_j}=0 \text{ for any } k,i\neq j \text{ and } \frac{\partial K_{ij}}{\partial\xi_j}=\frac{\partial K_{ji}}{\partial\xi_j} \text{ for } i\neq j.$$

Statement 6. If (10) holds, then

$$\frac{\partial I(R)}{\partial\xi_j}=-\frac{\partial\vec{b}^{\mathrm{T}}}{\partial\xi_j}\boldsymbol{K}^{-1}\vec{b}-\vec{b}^{\mathrm{T}}\boldsymbol{K}^{-1}\frac{\partial\overline{\vec{b}}}{\partial\xi_j}+\vec{b}^{\mathrm{T}}\boldsymbol{K}^{-1}\frac{\partial\boldsymbol{K}}{\partial\xi_j}\boldsymbol{K}^{-1}\overline{\vec{b}}. \tag{15}$$

Proof. Formula (15) can be easily derived from (13) if we take into account that

$$\frac{\partial\boldsymbol{K}^{-1}}{\partial\xi_j}=-\boldsymbol{K}^{-1}\frac{\partial\boldsymbol{K}}{\partial\xi_j}\boldsymbol{K}^{-1}.$$

1.4. Constructing the Green Function. Solution of Eq. (4) can be easily obtained if we present the Green function as

$$G(x,\xi)=G_1(x,\xi)+G_2(x,\xi),$$

where $G_i(x,\xi)$, $i=1,2$, is the solution of the equation $\dfrac{d^4G_i(x,\xi)}{dx^4}-k^4\pi^4G_i(x,\xi)=0$ and $G_1(x,\xi)$ at point $x=\xi$ has continuous derivatives with respect to x of orders 0, 1, 2, and 3; $G_2(x,\xi)$ at point $x=\xi$ has continuous derivatives with respect to x of orders 0, 1, and 2, and $\left.\left[\left[\dfrac{d^3G_2(x,\xi)}{dx^3}\right]\right]\right|_{x=\xi}=1$. Moreover, function $G_i(x,\xi)$ satisfies the homogeneous boundary conditions (4). It can be easily seen that

$$G(x,\xi)=\begin{cases} g(x,\xi), & x\le\xi,\\ g(\xi,x), & x>\xi, \end{cases} \tag{16}$$

where

$$g(x,\xi)=\frac{\sin k\pi x\cdot\sin k\pi(1-\xi)\cdot\operatorname{sh}k\pi-\operatorname{sh}k\pi x\cdot\operatorname{sh}k\pi(1-\xi)\cdot\sin k\pi}{2k^3\pi^3\sin k\pi\cdot\operatorname{sh}k\pi}. \tag{17}$$

2. CONTROL OF THE BEAM WITH A DEFECT

By a defect we will understand beam homogeneity violation (geometrical or physical). Let us consider the case where $E=E(x)$ and $I=\text{const}$. However, we can similarly consider that $E=\text{const}$ and $I=I(x)$.

2.1. Deriving the Equations. Assume the following:

(i) the beam is excited by one force of intensity F, which is applied at point ξ;

(ii) a defect of length 2ℓ, which is localized at point $\chi\ne\xi$, is characterized by variation in the Young modulus ΔE.

Control of oscillations is described by the equation

$$\frac{d^2}{dx^2}\left(E(x)I\frac{d^2w}{dx^2}\right)-\omega^2\rho w=F\,\delta(x-\xi), \tag{18}$$

where

$$E(x)=E_0(1-\Delta E f(x)), \tag{19}$$

$$f(x)=H(x-(\chi-\ell))-H(x-(\chi+\ell)). \tag{20}$$

Thus, we get

$$IE(x)\frac{d^4w}{dx^4}-\omega^2\rho w=F\delta(x-\xi)-I\frac{d^2E}{dx^2}\frac{d^2w}{dx^2}-2I\frac{dE}{dx}\frac{d^3w}{dx^3},$$

$$\begin{cases} \dfrac{dE}{dx}\dfrac{d^3w}{dx^3}=-E_0\Delta E f'w''',\\[2mm] \dfrac{d^2E}{dx^2}\dfrac{d^2w}{dx^2}=-E_0\Delta E f''w''. \end{cases} \tag{21}$$

Here,

$$f'=\delta(x-(\chi-\ell))-\delta(x-(\chi+\ell))=\sum_{k=0}^{\infty}\frac{\delta^{(k)}(x-\chi)}{k!}\ell^k$$

$$-\sum_{k=0}^{\infty}\frac{\delta^{(k)}(x-\chi)}{k!}(-1)^k\ell^k=2\sum_{k=0}^{\infty}\frac{\delta^{(2k+1)}(x-\chi)}{(2k+1)!}\ell^{2k+1}. \tag{22}$$

Remark 2. The rows in (22) converge weakly. The following expression can be obtained similarly to (22):

$$f''=2\sum_{k=0}^{\infty}\frac{\delta^{(2k+2)}(x-\chi)}{(2k+1)!}\ell^{2k+1}. \tag{23}$$

Carrying out obvious transformations of expression (23) and introducing dimensionless quantities yield

$$(1-\Delta E f(x))w_x^{(4)}(x,\xi)-\pi^4 k^4 w(x,\xi)=F\delta(x-\xi)$$

$$+2\Delta E w'_x(x,\xi)\sum_{k=0}^{\infty}\frac{\delta^{(2k+1)}(x-\chi)}{(2k+1)!}\ell^{2k+1}+4\Delta E w_x^{(3)}(x,\xi)\sum_{k=0}^{\infty}\frac{\delta^{(2k+2)}(x-\chi)}{(2k+1)!}\ell^{2k+1},$$

where $x, \chi, \xi,$ and F are dimensionless quantities.

We will be finding the solution in the form $w=w_0+\hat{w},$ where $\hat{w}=\bar{o}(w_0)$ for $\Delta E\sim 0$. We obtain

$$\begin{cases} w_{0x}^{(4)}(x,\xi)-\pi^4 k^4 w_0(x,\xi)=F\delta(x-\xi), \\ \hat{w}_x^{(4)}(x,\xi)-\pi^4 k^4 \hat{w}(x,\xi)=\Delta E\left\{ f(w_0+\hat{w})_x^{(4)}+2(w_0+\hat{w})_x^{(2)}\sum_{k=0}^{\infty}\frac{\delta^{(2k+1)}(x-\chi)}{(2k+1)!}\ell^{2k+1}\right. \\ \left. +4(w_0+\hat{w})_x^{(3)}\sum_{k=0}^{\infty}\frac{\delta^{(2k+2)}(x-\chi)}{(2k+1)!}\ell^{2k+1}\right\}. \end{cases} \quad (24)$$

We will be finding $\hat{w}$ in the form

$$\hat{w}=\varepsilon_1 w_1+\varepsilon_2 w_2+\dots, \quad (25)$$

where $\varepsilon_{i+1}=\bar{o}(\varepsilon_i)$ for $\Delta E\sim 0$. Note also that

$$\Phi(\ell)=\langle f,\varphi\rangle=\int_{\chi-\ell}^{\chi+\ell} f\varphi\,dx,\ \ \Phi(\ell)=2\sum_{k=0}^{\infty}\frac{\delta^{(2k)}(x-\chi)}{(2k+1)!}\ell^{2k+1}. \quad (26)$$

Hence,

$$\varepsilon_1(w_{1x}^{(4)}-\pi^4 k^4 w_1)+\varepsilon_2(w_{2x}^{(4)}-\pi^4 k^4 w_2)+\dots=2\Delta E\sum_{k=0}^{\infty}\frac{\ell^{2k+1}}{(2k+1)!}(w_{0x}^{(4)}\delta^{(2k)}(x-\chi)$$

$$+2w_{0x}^{(3)}\delta^{(2k+1)}(x-\chi)+w_{0x}^{(2)}\delta^{(2k+2)}(x-\chi))+2\Delta E\sum_{k=0}^{\infty}\frac{\ell^{2k+1}}{(2k+1)!}$$

$$\times((\varepsilon_1 w_1+\varepsilon_2 w_2+\dots)_x^{(4)}\delta^{(2k)}(x-\chi)+2(\varepsilon_1 w_1+\varepsilon_2 w_2+\dots)_x^{(3)}\delta^{(2k+1)}(x-\chi)$$

$$+(\varepsilon_1 w_1+\varepsilon_2 w_2+\dots)_x^{(2)}\delta^{(2k+2)}(x-\chi)), \quad (27)$$

$$\varepsilon_i=\frac{2\ell^{2i-1}\Delta E}{(2i-1)!},\ \ i=1,2,\dots,$$

$$\begin{cases} w_{0x}^{(4)}-\pi^4 k^4 w_0=F\delta(x-\xi), \\ w_{ix}^{(4)}-\pi^4 k^4 w_i=w_{0x}^{(4)}\delta^{(2i-2)}(x-\chi)+2w_{0x}^{(3)}\delta^{(2i-1)}(x-\chi) \\ +w_{0x}^{(2)}\delta^{(2i)}(x-\chi),\ \ i=1,2,\dots. \end{cases} \quad (28)$$

The right-hand side of Eqs. (28) for w_i is a generalized function:

$$\Psi=\frac{d^2}{dx^2}(\delta^{(2i-2)}(x-\chi)w_{0x}^{(2)}).$$

Obviously, $\forall\,\varphi\in\Omega$ ([a, b]) we get

$$\left\langle\frac{d^2}{dx^2}(\delta^{(2i-2)}(x-\chi)\,w_{0x}^{(2)}),\varphi(x)\right\rangle=\langle\delta^{(2i-2)}(x-\chi)w_{0x}^{(2)},\varphi^{(2)}\rangle$$

$$= \langle \delta^{(2i-2)}(x-\chi), w_{0x}^{(2)} \varphi^{(2)} \rangle = \left\langle \delta(x-\chi), \frac{d^{2i-2}}{dx^{2i-2}} (w_{0x}^{(2)} \varphi^{(2)}) \right\rangle$$

$$= \frac{d^{(2i-2)}}{dx^{2i-2}} (w_{0x}^{(2)}(x,\xi)\,\varphi^{(2)}(x)) \bigg|_{x=\chi} = \left\{ \sum_{s=0}^{2i-2} C_{2i-2}^{s} \frac{d^{2i-s}}{dx^{2i-s}} w_0(x,\xi) \frac{d^{2+s}}{dx^{2+s}} \varphi(x) \right\} \bigg|_{x=\chi} .$$

Therefore,

$$\Psi = \sum_{s=0}^{2i-2} C_{2i-2}^{s} \frac{d^{2i-s}}{dx^{2i-s}} w_0(x,\xi) \bigg|_{x=\chi} \delta^{(2+s)}(x-\chi). \tag{29}$$

Hence, the Green function for the beam with a defect can be defined by the solutions of the boundary-value problems:

$$G(x,\xi) = G_0(x,\xi) + \varepsilon_1 G_1(x,\xi) + \ldots ,$$

$$\begin{cases} G_{0x}^{(4)} - \pi^4 k^4 G_0 = \delta(x-\xi), \\ G_0(0,\xi) = G_0(1,\xi) = 0, \\ G_0^{(2)}(0,\xi) = G_0^{(2)}(1,\xi) = 0; \end{cases} \tag{30}$$

$$\begin{cases} G_{ix}^{(4)} - \pi^4 k^4 G_i = \sum_{s=0}^{2i-2} C_{2i-2}^{s} \frac{d^{2i-s}}{dx^{2i-s}} G_0(x,\xi) \bigg|_{x=\chi} \delta^{(2+s)}(x-\chi), \\ G_i(0,\xi) = G_i(1,\xi) = 0, \\ G_{ix}^{(2)}(0,\xi) = G_{ix}^{(2)}(1,\xi) = 0, \ \ i = 1, 2, \ \ldots . \end{cases}$$

Note that $G_0(x,\xi)$ is the Green function for the homogeneous beam (16), (17), the third derivative of this function at point $x = \xi$ has a jump (equal to one). Respectively, $G_1(x,\xi,\chi)$ will have a jump of the first derivative at point $x = \chi$.

Thus, the solution of boundary-value problems (30) is a formal weakly converging series of generalized functions and in some sense this series is not physical. It is expedient to consider the approximation $\widetilde{G}(x,\xi) = G_0(x,\xi) + \varepsilon_1 G_1(x,\xi,\chi) + \ldots + \varepsilon_k G_k(x,\xi,\chi)$ as asymptotic one. In what follows, we assume that

$$G(x,\xi) = G_0(x,\xi) + \varepsilon_1 G_1(x,\xi,\chi), \tag{31}$$

where G_0 is defined in (16), (17), $\varepsilon_1 = 2\ell \Delta E$, and G_1 is defined as follows:

$$G_{1x}^{(4)} - \pi^4 k^4 G_1 = G_{0x}^{(2)}(\chi,\xi)\delta''(x-\chi),$$
$$\begin{cases} G_1(0,\xi,\chi) = G_1(1,\xi,\chi) = 0, \\ G_{1x}^{(2)}(0,\xi,\chi) = G_{1x}^{(2)}(1,\xi,\chi) = 0. \end{cases} \tag{32}$$

Note that

$$\frac{d^4}{dx^4}\left(\frac{\partial^2 G_0}{\partial \xi^2}\right) - \pi^4 k^4 \frac{\partial^2 G_0}{\partial \xi^2} = \delta''(x-\xi).$$

Hence,

$$G_1 = \frac{\partial^2 G_0(\chi,\xi)}{\partial x^2} \frac{\partial^2 G_0(x,\chi)}{\partial \xi^2}. \tag{33}$$

The boundary conditions are satisfied automatically, for example,

$$G_1(0,\xi,\chi) = \frac{\partial^2 G_0(\chi,\xi)}{\partial x^2} \frac{\partial^2 G_0(0,\chi)}{\partial \xi^2} = 0,$$

$$\frac{\partial^2 G_1(0,\xi,\chi)}{\partial x^2} = \frac{\partial^2 G_0(\chi,\xi)}{\partial x^2} \frac{\partial^4 G_0(0,\chi)}{\partial x^2 \partial \xi^2} = 0.$$

Since $G_0(1-x, 1-\xi) = G_0(x, \xi)$, the boundary conditions at point $x = 1$ are satisfied as well.

2.2. Analyzing the Functional for the First Approximation. Let $G(x, \xi, \chi) = G_0(x, \xi) + \varepsilon_1 G_1(x, \xi, \chi)$, where G_0 is defined in (16), (17) and G_1 is defined in (33). By analogy with (8), let us introduce the notation for $\boldsymbol{K} = \{K_{ij}\}_{i,j=1}^{I}$:

$$K_{ij} = \int_0^1 G_0(x, \xi_i) G_0(x, \xi_j) dx + \varepsilon_1 \int_0^1 (G_0(x, \xi_i) G_1(x, \xi_j, \chi) + G_0(x, \xi_j) G_1(x, \xi_i)) dx$$

$$+ \varepsilon_1^2 \int_0^1 (G_1(x, \xi_i) G_1(x, \xi_j)) dx = K_{ij}^0 + \varepsilon_1 K_{ij}^1 + \varepsilon_1^2 K_{ij}^2. \tag{34}$$

Remark 3. We cannot neglect the term of order ε_1^2 in (34) since positive definiteness of the matrix $\boldsymbol{K}$ is lost in this case.

Statement 7. Matrix $\boldsymbol{K}$ is real, symmetric, and positive definite.

The proof is similar to the proof of Statement 3 with regard for Remark 3.

Thus, functional (12) is subject to minimization, where components of the matrix $\boldsymbol{K}$ are defined in (34) and $\vec{b}$ can be found as follows:

$$\vec{b}^{\mathrm{T}} = \{b_j\}_{j=1}^{I}, \ b_j = \int_0^1 G_0(x, \xi_j) w(x) dx + \varepsilon_1 \int_0^1 G_1(x, \xi_j, \chi) w(x) dx = b_j^0 + \varepsilon_1 b_j^1. \tag{35}$$

Since $I(R)$ is a convex functional, the necessary extremum existence condition coincides with (10), (11).

Let us present the computing formulas:

$$K_{ij}^0 = \int_0^1 G_0(x, \xi_i) G_0(x, \xi_j) dx, \ b_j^0 = \int_0^1 G_0(x, \xi_j) w(x) dx,$$

$$K_{ij}^1 = \frac{\partial^2 G_0(\chi, \xi_j)}{\partial x^2} \int_0^1 G_0(x, \xi_i) \frac{\partial^2 G_0(x, \chi)}{\partial \xi^2} dx + \frac{\partial^2 G_0(\chi, \xi_i)}{\partial x^2} \int_0^1 G_0(x, \xi_j) \frac{\partial^2 G_0(x, \chi)}{\partial \xi^2} dx, \tag{36}$$

$$K_{ij}^2 = \frac{\partial^2 G_0(\chi, \xi_j)}{\partial x^2} \frac{\partial^2 G_0(\chi, \xi_j)}{\partial x^2} \int_0^1 \left(\frac{\partial^2 G_0(x, \chi)}{\partial \xi^2} \right)^2 dx,$$

$$b_j^1 = \frac{\partial^2 G_0(\chi, \xi_j)}{\partial x^2} \int_0^1 \left(\frac{\partial^2 G_0(x, \chi)}{\partial \xi^2} \right)^2 w(x) dx.$$

The gradients can be calculated immediately from (36).

2.3. Multiple Defects. Let us consider the case where there are K defects in the beam and the kth defect is located at point χ_k and is characterized by the value $\varepsilon_{1(k)} = 2\ell_k \Delta E_k$. Due to problem's linearity, formulas (10)–(12) do not vary. Here,

$$\boldsymbol{K} = \sum_{k=1}^{K} \boldsymbol{K}_{(k)}, \ \vec{b} = \sum_{k=1}^{K} \vec{b}_{(k)},$$

$$\boldsymbol{K}_{(k)} = \boldsymbol{K}_{(k)}^0 + \varepsilon_{1(k)} \boldsymbol{K}_{(k)}^1 + \varepsilon_{1(k)}^2 \boldsymbol{K}_{(k)}^2, \tag{37}$$

$$\vec{b}_{(k)} = \vec{b}_{(k)}^0 + \varepsilon_{1(k)} \vec{b}_{(k)}^1.$$

3. ANALYZING THE CONVEXITY OF THE FUNCTIONAL

3.1. Scalar Case for a Homogeneous Beam. Let $w = \sum_{i=1}^{I} F_i G(x, \xi_i)$ and the values of F_i and ξ_i be defined for $i = 1, \ldots, I-1$. Assume that $\operatorname{Im} w = 0$. Then $I(R)(\xi_I, F_I) = \int_0^1 (F_I G(x, \xi_I) - f(x))^2 \, dx$, where $f(x) = w(x) - \sum_{i=1}^{I-1} F_i G(x, \xi_i)$.

Thus, to find the minimum value of the functional

$$I(R)(\xi, F) = \int_0^1 (FG(x, \xi) - f(x))^2 \, dx, \tag{38}$$

it will suffice to construct the iterative process with the use of the coordinate descent method.

Statement 8. We get

$$\max_{\xi} \min_{F} I(R) = \int_0^1 f^2(x) \, dx,$$

where optimum is attained under the following conditions:

$$\begin{cases} F = 0, \\ \int_0^1 f(x) G(x) \, dx = 0. \end{cases} \tag{39}$$

Proof. Since

$$\frac{1}{2}\frac{\partial I}{\partial F} = F \int_0^1 G^2 dx - \int_0^1 f G^2 dx, \quad \frac{1}{2}\frac{\partial^2 I}{\partial F^2} = \int_0^1 G^2 dx > 0,$$

all the minima with respect to F for $I(R)$ are attained for

$$F = \left(\int_0^1 f G \, dx \right) \Big/ \left(\int_0^1 G^2 \, dx \right). \tag{40}$$

Let us consider the ratio $\partial I / \partial \xi$. We will assume that $F = F(\xi)$ and it can be defined from expression (40):

$$\frac{1}{2}\frac{\partial I}{\partial \xi} = F^2 \int_0^1 G \frac{\partial G}{\partial \xi} dx - F \int_0^1 f \frac{\partial G}{\partial \xi} dx = \int_0^1 f G \, dx \, \frac{\left\{ \int_0^1 f G dx \int_0^1 G \frac{\partial G}{\partial \xi} dx - \int_0^1 G^2 dx \int_0^1 \frac{\partial G}{\partial \xi} f dx \right\}}{\left(\int_0^1 G^2 dx \right)^2}. \tag{41}$$

Thus, the equality $\partial I / \partial \xi = 0$ is equivalent to the following equalities:

$$\left[\begin{array}{l} \int_0^1 f G \, dx = 0, \\ \int_0^1 f G dx \int_0^1 G \frac{\partial G}{\partial \xi} dx - \int_0^1 G^2 dx \int_0^1 \frac{\partial G}{\partial \xi} f dx = 0. \end{array} \right. \tag{42}$$

The following relation holds:

$$\frac{1}{2}\frac{\partial^2 I}{\partial\xi^2}\Bigg|_{\int_0^1 fGdx=0} = -\left(\left(\int_0^1 \frac{\partial G}{\partial\xi} f\,dx\right)\Big/\left(\int_0^1 G^2 dx\right)\right)^2 < 0.$$

Hence, if (40) holds, the condition $\int_0^1 fGdx = 0$ $(F = 0)$ guarantees that $I(R)$ attains minimum at point ξ.

Statement 9. The conditions of the Statement 8 are satisfied for an arbitrary function $f(x)$ for $\xi = \{0, 1\}$.

Proof. From (16) it follows that

$$\int_0^1 fGdx = \int_0^\xi g(x,\xi)f(x)\,dx + \int_\xi^1 g(\xi,x)f(x)\,dx.$$

Due to the continuity of $g(x,\xi)$ with respect to ξ, the equality holds

$$\int_0^\xi g(x,\xi)f(x)dx\Bigg|_{\xi=0} = 0.$$

Here, $g(0,x) = 0$. Hence, the statement is proved.

Statement 10. Functional $I(R)$ has at least one minimum. All the local minima for $I(R)$ are defined by the following necessary conditions:

$$\begin{cases} F = \left(\int_0^1 fGdx\right)\Big/\left(\int_0^1 G^2 dx\right), \\ F \neq 0, \\ \int_0^1 fGdx\int_0^1 G\frac{\partial G}{\partial\xi}dx - \int_0^1 G^2 dx\int_0^1 \frac{\partial G}{\partial\xi} fdx = 0 \end{cases} \tag{43}$$

and the corresponding values of $\hat{\xi}$ are between points ξ^*, which are defined by the condition

$$\int_0^1 f(x)G(x,\xi^*)\,dx = 0, \tag{44}$$

and $\hat{\xi} \in (0,1)$.

Proof. According to Statement 9, when condition (40) is satisfied, there exist saddle points (minima with respect to F and maxima with respect to ξ). Hence, taking into account continuous differentiability of the functional $I(R)$ with respect to ξ, according to the Rolle theorem ($I(R)\big|_{\substack{\xi=0,1\\F=0}} = \int_0^1 f^2 dx$) there exists at least one local minimum (with respect to F and to ξ) for $\xi \in (0,1)$.

If (44) holds at interior points [0, 1], then Statement 10 is true on each such interval.

The last expression in (43) is a necessary condition of the extremum $I(R)$ with respect to ξ when (40) holds.

3.2. Examples of Using the Obtained Results. Let us write the Green function by using homogeneous solutions:

$$G^{(4)} - \pi^4 k^4 G = \delta(x-\xi),\ \ x \in (0,1),$$

$$\begin{cases} G(0,\xi) = G(1,\xi) = 0, \\ G^{(2)}(0,\xi) = G^{(2)}(1,\xi) = 0, \end{cases}$$

where $G(x,\xi) = \sum_{n=1}^{\infty} A_n \sin \pi n x.$

The boundary conditions are satisfied for an arbitrary A_n:

$$\sum_{n=1}^{\infty} A_n \pi^4 (n^4 - k^4) \sin \pi n x = \delta(x-\xi) \tag{45}$$

or

$$\frac{1}{2}\sum_{n=1}^{\infty} A_n \delta_{sn} \pi^4 (n^4 - k^4) = \sin \pi s \xi.$$

Thus, $A_s = \dfrac{2\sin \pi s \xi}{\pi^4 (s^4 - k^4)}$,

$$G(x,\xi) = \frac{2}{\pi^4}\sum_{n=1}^{\infty} \frac{\sin \pi n \xi}{n^4 - k^4} \cdot \sin \pi n x \equiv \sum_{n=1}^{\infty} p_n \sin \pi n \xi \cdot \sin \pi n x, \tag{46}$$

where $p_n \overset{\text{def}}{\equiv} \dfrac{2}{\pi^4 (n^4 - k^4)}$.

Let

$$f = \sum_{n=1}^{\infty} f_n \sin \pi n x. \tag{47}$$

Then condition (39) becomes

$$\sum_{r,\, n=1}^{\infty} p_n f_r \sin \pi n \xi \int_0^1 \sin \pi n x \cdot \sin \pi r x dx = 0$$

or

$$\begin{cases} F(\xi) = \displaystyle\sum_{n=1}^{\infty} \frac{f_n}{n^4 - k^4} \sin \pi n \xi, \\ F(\xi) = 0. \end{cases} \tag{48}$$

From (48) it follows that minima of the functional $I(R)$ should be found between points ξ^*: $F(\xi^*) = 0$.

Let $f_n = \delta_{ns}$ (i.e., $f(x) = \sin \pi s x$). Then $F(\xi) = \dfrac{1}{s^4 - k^4} \sin \pi s \xi$ and $\xi_h^* = n/s,\ s = 0, \ldots, S$. In this case, $I(R)$ has at least S local minima.

Let us verify condition (43):

$$\begin{cases} \displaystyle\int_0^1 fG dx = \frac{1}{2}\sum_{n=1}^{\infty} f_n p_n \sin \pi n \xi, \\ \displaystyle\int_0^1 G^2 dx = \frac{1}{2}\sum_{n,\, r=1}^{\infty} p_n p_r \sin \pi n \xi \cdot \sin \pi r \xi \delta_{hr} = \frac{1}{2}\sum_{n=1}^{\infty} p_n^2 \sin^2 \pi n \xi \\ \displaystyle = \frac{1}{4}\left\{\sum_{n=1}^{\infty} p_n^2 - \sum_{n=1}^{\infty} p_n^2 \cos 2\pi n \xi\right\}, \\ \displaystyle\int_0^1 G \frac{\partial G}{\partial \xi} dx = \frac{1}{2}\frac{\partial}{\partial \xi}\int_0^1 G^2 dx = \frac{1}{8} 2\pi \sum_{n=1}^{\infty} p_n^2 h \sin 2\pi n \xi, \\ \displaystyle\int_0^1 \frac{\partial G}{\partial \xi} f dx = \frac{\partial}{\partial \xi}\int_0^1 fG dx = \frac{\pi}{2}\sum_{n=1}^{\infty} f_n p_n h \cos \pi n \xi. \end{cases} \tag{49}$$

Thus, the necessary extremum condition has the form

$$\begin{cases} M(\xi) = \sum_{h=1}^{\infty} f_n p_n \sin \pi n\xi \sum_{n=1}^{\infty} np_n^2 \sin 2\pi n\xi - \left\{ \sum_{h=1}^{\infty} p_h^2 - \sum_{n=1}^{\infty} p_n^2 \cos 2\pi n\xi \right\} \sum_{n=1}^{\infty} nf_n p_n \cos \pi n\xi, \\ M(\hat{\xi}) = 0, \\ \sum_{n=1}^{\infty} p_n^2 - \sum_{n=1}^{\infty} p_n^2 \cos 2\pi n\hat{\xi} \neq 0. \end{cases} \quad (50)$$

If $f_n = \delta_{ns}$, then

$$M(\xi) = p_s \sin \pi s\xi \sum_{n=1}^{\infty} np_n^2 \sin 2\pi n\xi - sp_s \cos \pi s\xi \sum_{n=1}^{\infty} p_n^2 (1 - \cos 2\pi n\xi) . \quad (51)$$

If ξ is a zero of function $M(\xi)$, then $1-\xi$ is also a zero of $M(\xi)$. Indeed,

$$M(1-\xi) = p_s \sin \pi s\xi (-1)^s \sum_{n=1}^{\infty} np_n^2 \sin 2\pi n\xi - sp_s (-1)^s \cos \pi s\xi \sum_{n=1}^{\infty} p_n^2 (1 - \cos 2\pi n\xi) = (-1)^s M(\xi). \quad (52)$$

If $s = 2r-1$, then $M\left(\frac{1}{2}\right) = 0$.

Indeed,

$$M\left(\frac{1}{2}\right) = p_s \cdot 0 - sp_s \cos \pi \left(r - \frac{1}{2}\right) \sum_{n=1}^{\infty} p_n^2 (1 - (-1)^n) = 0. \quad (53)$$

It can be easily shown that $\xi = 1/2$ for a fixed s not for all frequencies (k) will define minimum of the functional $I(R)$. Indeed, the sign for $(\partial^2 I(R) / \partial \xi^2)\big|_{\xi=1/2}$ depends on the sign of function $S(s, k) = \sum_{n=1}^{\infty} p_n(k)^2 ((-1)^n (2n^2 - s) + s)$. It is possible to show that for all $s = 2r+1$, $k \in (0, 2)$, $S(s, k) > 0$ holds. For the values of $k \in (r, r+1)$, $r > 2$, function $S(s, k)$ changes its sign.

It can be easily seen that

$$I(R) = \frac{1}{2} - \frac{p_s^2 \sin^2 s\pi\xi}{\sum_{n=1}^{\infty} p_n^2 (1 - \cos 2\pi n\zeta)}, \quad (54)$$

$$I(R)\big|_{s=2r+1, \xi=1/2} = \frac{1}{2} - \frac{p_s^2}{\sum_{n=1}^{\infty} p_n^2 (1 - (-1)^n)}$$

$$= \frac{1}{2} - \frac{32k^7}{\pi(k^4 - s^4)\left(\sec^2 \frac{k\pi}{2} (k\pi - 3\sin k\pi) + \sec h^2 \frac{k\pi}{2} (-k\pi + 3\sin hk\pi)\right)}, \quad (55)$$

where $s = 2r+1$.

It is obvious that

$$I(R)\big|_{s=2r+1, \xi=1/2, k=s} = 0, \quad (56)$$

i.e., minimum is attained for these values of variables. At the same time, for $s = 2r+1$, $\xi = \frac{1}{2}$, $k = 2n+1$, $n \neq r$

$$I(R) = \frac{1}{2},$$

i.e., maximum is attained.

Let force F_ξ be applied at point ξ and force $F_{1-\xi}$ at point $1-\xi$. In this case,

$$w = F_\xi G(x, \xi) + F_{1-\xi} G(x, 1-\xi).$$

From (46) it follows that

$$G(x, 1-\xi) = \sum_{n=1}^{\infty} (-1)^{n+1} p_n \sin n\pi\xi \cdot \sin n\pi x. \tag{57}$$

Let us introduce the notation:

$$\begin{cases} G_1(x, \xi) = \dfrac{G(x, \xi) + G(x, 1-\xi)}{2} = \displaystyle\sum_{n=1}^{\infty} p_{2n-1} \sin(2n-1)\pi\xi \cdot \sin(2n-1)\pi x, \\ G_2(x, \xi) = \dfrac{G(x, \xi) - G(x, 1-\xi)}{2} = \displaystyle\sum_{n=1}^{\infty} p_{2n} \sin 2n\pi\xi \cdot \sin 2n\pi x, \\ F_1 = F_\xi + F_{1-\xi}, \\ F_2 = F_\xi - F_{1-\xi}. \end{cases} \tag{58}$$

Then

$$I(R) = \int_0^1 (F_1 G_1 + F_2 G_2 - w)^2 \, dx. \tag{59}$$

Conditions $\partial I / \partial F_i = 0$, $i = 1, 2$, reduce to the system of linear equations

$$\begin{cases} \hat{\boldsymbol{K}} \vec{F} = \overline{\vec{b}}, \\ \hat{\boldsymbol{K}} = \{a_{ij}\}_{i,j=1}^{r}, \ \vec{F}^{\mathrm{T}} = \{F_i\}_{i=1}^{r}, \ \vec{b}^{\mathrm{T}} = \{\hat{b}_i\}_{i=1}^{r}, \\ a_{ij} = \displaystyle\int_0^1 G_i G_j \, dx, \ b_i = \int_0^1 w G_i \, dx. \end{cases} \tag{60}$$

It can be easily seen that $a_{12} = a_{21} = \int_0^1 G_1 G_2 \, dx = 0$. Hence,

$$F_i = \left(\int_0^1 w G_i \, dx \right) \Big/ \left(\int_0^1 G_i^2 \, dx \right), \ i = 1, 2. \tag{61}$$

If $w = \sum_{h=1} w_h \sin h\pi x$, then

$$\begin{cases} F_1 = \dfrac{\displaystyle\sum_{n=1}^{\infty} w_{2n-1} p_{2n-1} \sin(2n-1)\pi\xi}{\displaystyle\sum_{n=1}^{\infty} p_{2n-1}^2 \sin^2 (2n-1)\pi\xi}, \\ F_2 = \dfrac{\displaystyle\sum_{n=1}^{\infty} w_{2n} p_{2n-1} \sin 2n\pi\xi}{\displaystyle\sum_{n=1}^{\infty} p_{2n}^2 \sin^2 2n\pi\xi}. \end{cases} \tag{62}$$

If $w_h = \delta_{hs}$, then for $s = 2k-1, k \in N$, we get $w_{2n} = 0$, and hence $F_2 = 0 \ (F_\xi = F_{1-\xi})$.

In case of $w_h = \delta_{hs}$ and $s = 2k, k \in N$, we get $w_{2n-1} = 0$ and $F_1 = 0 \ (F_\xi = F_{1-\xi})$. The analysis is similar to that considered above.

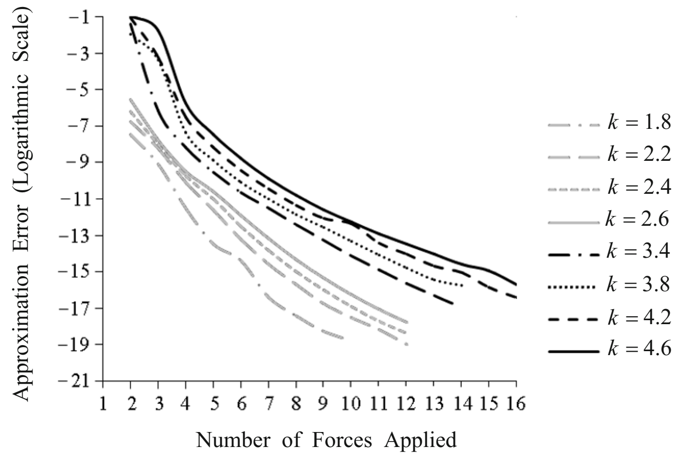

Fig. 1. Dependence of root mean square deviation (approximation error) on the applied forces and frequency (wave number k).

4. RESULTS OF NUMERICAL EXPERIMENTS OF THE ANALYSIS OF MODELS OF OPTIMAL CONTROL OF OSCILLATIONS OF THE HINGED BEAM

To determine the dependence of approximation accuracy of the given waveform on the number of forces and their characteristics (application, amplitude and phase of oscillations), a series of numerical experiments was carried out with the use of the multifunctional package PSG (Matlab Interface) provided by American Optimal Decision, USA [6]. Two cases were considered: oscillation of a homogeneous beam and oscillation of a beam with a defect.

4.1. Results of Modeling of Oscillation of a Homogeneous Beam. Let us present the results of numerical solution of the problems of optimal control of oscillations of a homogeneous beam under different numbers of applied forces and different frequencies (wave number k). The mathematical apparatus presented in Sec. 1 and the multifunctional package PSG (Matlab Interface) were used.

As the objective function, we used root mean square deviation of the solution from the given waveform on the frequency corresponding to the given wave number k. As is seen from Fig. 1, for any fixed frequency of oscillations, increase in the number of applied forces reduces the approximation error. Moreover, for any fixed number of applied forces, the approximation error increases with increase in the wave number k. For example, for any fixed number of applied forces, approximation error corresponding to $k = 1.8$ is less than approximation error corresponding to greater values of wave number k.

For wave numbers $k = 1.8$ and $k = 4.6$ and different numbers of forces, their optimal characteristics were determined (applications, real and imaginary parts of amplitudes) that ensured the best approximation of the given form and phase of oscillations of the homogeneous beam.

Figure 2 shows optimal characteristics of forces that correspond to wave number $k = 1.8$. As follows from the figure, for any number of forces from the considered range, optimal amplitudes are regularly distributed. If the number of applied forces does not exceed four, then the values of amplitudes are mutually commensurable. For five and more forces, the values of the real parts of amplitudes of forces applied along beam's edges are much greater than those for forces applied to its interior points. The signs of imaginary parts of amplitudes (if) alternate when passing from one point of application of forces to another. Prevailing values of imaginary parts of amplitudes correspond to extreme points of application of forces on the beam.

Figure 3 shows optimal characteristics of forces corresponding to wave number $k = 4.6$. Optimal characteristics of the forces applied to the beam for $k = 4.6$ substantially differ from those shown in Fig. 2 and corresponding to the case $k = 1.8$. These graphs have sine curve shapes: the graph of imaginary part of amplitudes is associated with a full wave sine curve, and the graph of real part with a semiperiodic sine curve.

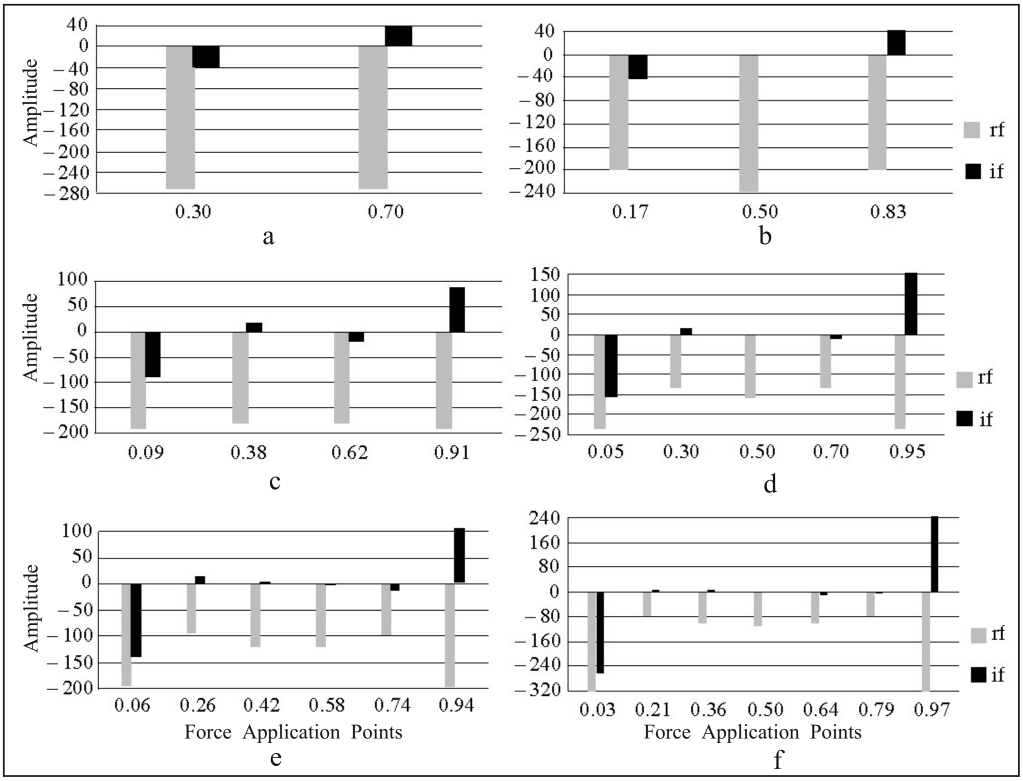

Fig. 2. Optimal characteristics of the applied forces (application, real (rf) and imaginary (if) parts of amplitudes) for $k = 1.8$: for two forces (a), three forces (b), four forces (c), five forces (d), six forces (e), seven forces (f).

4.2. Results of Modeling of Oscillation of the Beam with a Defect. Let us consider the case of oscillation of a beam with one defect, which is characterized by the following parameters: location on the beam at point χ, geometrical dimension $2\ell \in (0, 0.1)$, and relative variation in the Young modulus ΔE. The defect can be at any section of the beam at point $\chi \in (0,1)$. Relative variation in the Young modulus ΔE can be within the interval $(-0.5, 0.5)$. The negative value ΔE corresponds to strengthening of the cross section and positive to its weakening. The value $\Delta E = 0$ means absence of a defect. Parameter $Delta = 2\ell \cdot \Delta E$ is used as the characteristics of the defect.

On the assumption that all the information about defect's parameters is known, a series of optimal control problems for oscillations of a beam with one defect was solved for fixed values of the wave number 1.8 and four applied forces under different values of the defect *Delta* and its location on the beam at point χ. The mathematical apparatus presented in Secs. 2.1 and 2.2 and the multifunctional package PSG (Matlab Interface) provided by American Optimal Decision, USA [6] were used. As a result, for each fixed pair of values *Delta* and χ, optimal characteristics of forces and optimal (minimum) value of root mean square deviation (approximation error) corresponding to them were obtained. In what follows, for simplicity, by approximation error we will mean the minimum value of approximation error obtained as a result of solution of the optimization problem. The results are graphically presented in Figs. 4, 5. Each curve shown in Fig. 4 describes the dependence of approximation error on the value of the defect *Delta* in case where the wave number is equal to 1.8, the number of forces is four, and the unique defect on the beam is located at point χ. In the case under study, all the curves decrease on the interval $-0.05 < Delta < 0$ and increase on the interval $0 < Delta < 0.05$. The minimum value of approximation error for all the curves equal to 1.0E−5 is attained when there are almost no defects on the beam ($|Delta| \le 5.0\text{E}{-6}$).

The behavior of the curve when the defect is located at beam's center ($\chi = 0.5$) substantially differs from the behavior of all other curves. In this case, the greatest value of the approximation error for $Delta = -0.05$ is smaller than for $Delta = 0.05$. In other cases, the greatest value of approximation error is attained for $Delta = -0.05$, and the maximum value equal to 5.2E−4 is attained when the defect is located at point $\chi = 0.2$ or at point $\chi = 0.8$.

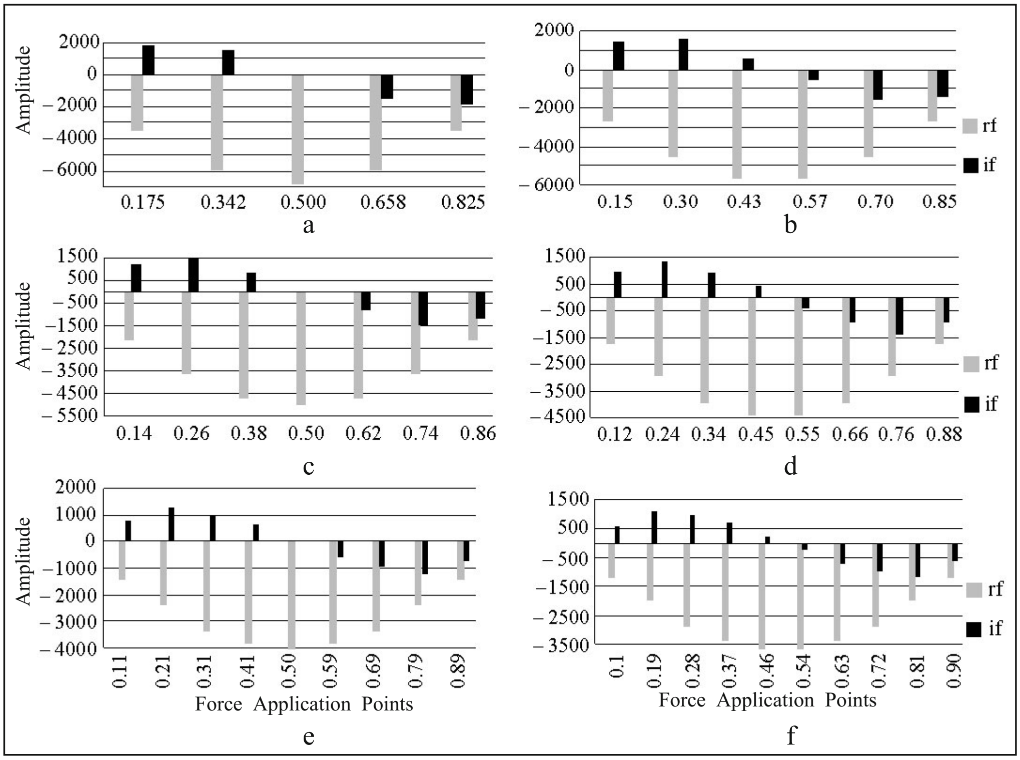

Fig. 3. Optimal characteristics of the applied forces (application, real (rf) and imaginary (if) parts of amplitudes) for $k = 4.6$: for five forces (a), six forces (b), seven forces (c), eight forces (d), nine forces (e); ten forces (f).

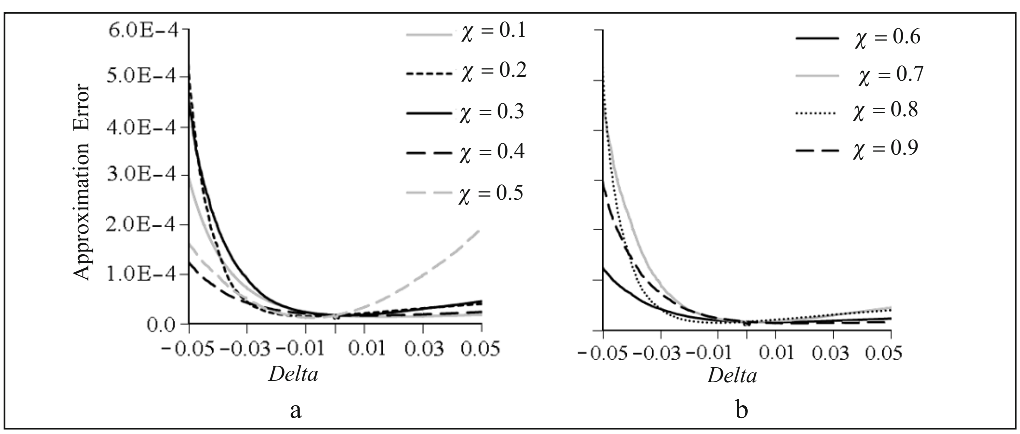

Fig. 4. Dependence of approximation error on *Delta* for locations of the defect on the beam $\chi = 0.1$, 0.2, 0.3, 0.4, 0.5 (a); $\chi = 0.6$, 0.7, 0.8, 0.9 (b) for the wave number 1.8 and number of forces equal to 4.

Since optimal value of the approximation error for fixed values of the parameters *Delta* and χ corresponds to the minimum of the objective function when the wave number is equal to 1.8, the number of forces is four, and the value of the defect is $Delta = -0.05$ and the defect is located at point $\chi = 0.2$ or at point $\chi = 0.8$, optimal value of the objective function (approximation error) cannot be smaller than 5.2E−4, which exceeds 52 times the optimal value of approximation error in case of absence of defects on the beam.

Figure 5 shows four curves, each of them describing the dependence of approximation error on the value of the defect (*Delta*) for a fixed location of the defect on the beam at point $\chi = 0.2$. Despite different rates of the curves, there is some similarity. All of them attain their minimum values for $Delta = 0$ (when defects on the beam are absent). For any fixed value

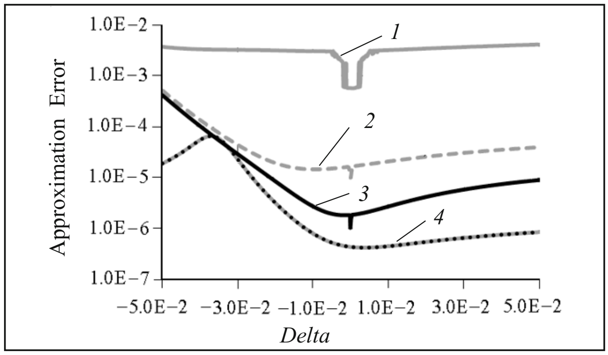

Fig. 5. Dependence of approximation error on *Delta* when two forces (*1*), four forces (*2*), six forces (*3*), eight forces (*4*) are used, position of the defect on the beam is fixed at point $\chi = 0.2$, and wave number is 1.8.

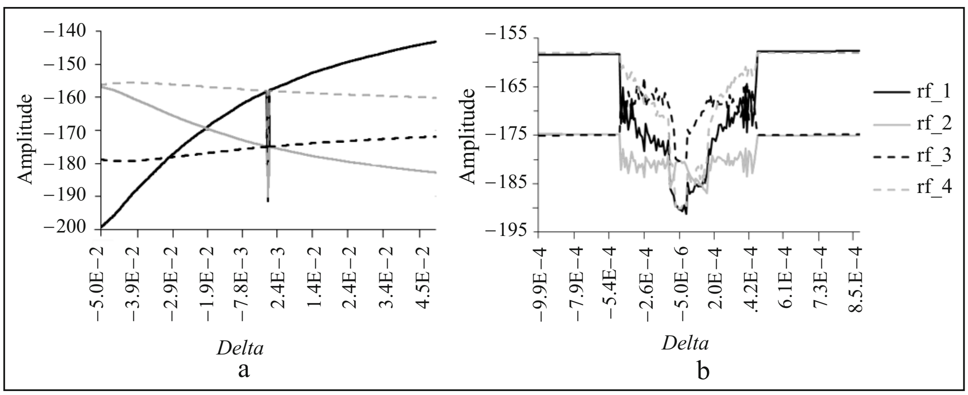

Fig. 6. Dependence of the real parts of amplitudes of four applied forces (rf_1, rf_2, rf_3, rf_4) on *Delta* for a fixed position of the defect on the beam at point $\chi = 0.4$, wave number 1.8 in case of variation of *Delta* within the limits $(-0.05, 0.05)$ (a), $(-0.001, 0.001)$ (b).

of the defect (*Delta*), the curve corresponding to the greater number of forces has smaller values of approximation error. The greatest values of approximation error are shown by the curve corresponding to the least number of forces (two forces).

As follows from the results presented in Fig. 5, to reduce the optimal value of approximation error for fixed values of *Delta* and χ, it is necessary to increase the number of forces applied to the beam.

In Fig. 6a, we can see variation in the value of the real parts of the amplitudes as the value of the defect (*Delta*) localized in the left-hand side of the beam ($\chi = 0.4$) increases from -0.05 to 0.05. As one can see, for any value of *Delta* on the interval $(-0.05, 0.05)$, the sign of these characteristics remains negative. The real (rf_1, rf_2) parts of the amplitudes of forces applied to the left-hand side of the beam vary greater than respective components of the forces (rf_3, rf_4) applied to its right-hand side. At the left end of the interval of values of the defect *Delta*, $|\text{rf\_1}| > |\text{rf\_2}|$, and at the right end $|\text{rf\_1}| < |\text{rf\_2}|$. Here, $|\text{rf\_4}| > |\text{rf\_3}|$ for any value of *Delta* on the interval $(-0.05, 0.05)$. Hence, if the defect is located on the left-hand part of the beam, variation in its value renders a greater influence on the forces applied on the same part of the beam. The values of rf_1 and rf_3 increase on the intervals of variation of *Delta* $(-5\text{E}-2, -4.6\text{E}-4)$ and $(0, 5\text{E}-2)$, and the values of rf_2 and rf_4 decrease on these intervals. For $Delta = 0$, the real parts of all amplitudes attain their minimum, which corresponds to oscillation of a defect-free beam. Figure 6b shows (in close up) variation of all the characteristics on a small interval $(-0.001, 0.001)$ of the values of *Delta* near the minimum point.

Figure 7a shows variation in the value of imaginary parts of the amplitudes as the value of the defect (*Delta*) localized in the left-hand side of the beam ($\chi = 0.4$) increases from -0.05 to 0.05. As one can see, for any value of *Delta* on the interval $(-0.05, 0.05)$, the sign of each characteristics does not vary. As well as in the case of no defects (see Fig. 2), signs of imaginary parts of the amplitudes alternate when passing from one point of application of forces to

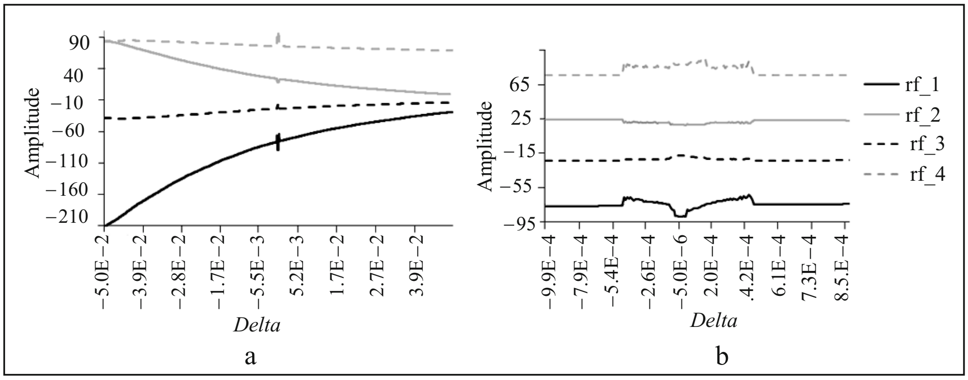

Fig. 7. Dependence of the imaginary part of the amplitudes of four applied forces (if_1, if_2, if_3, if_4) on *Delta* for a fixed position of a defect on the beam at point $\chi = 0.4$, wave number 1.8 under variation of *Delta* within the limits (−0.05, 0.05) (a), (−0.001, 0.001) (b).

another. Imaginary parts of the amplitudes of forces (if_1, if_2) applied at points in the left-hand side of the beam vary greater than respective components of forces (if_3, if_4) applied at points in its right-hand side. Figure 7b shows (in close up) variation of all the characteristics on a small interval (−0.001, 0.001) of the values of *Delta*.

CONCLUSIONS

We have considered the optimal control problem for oscillations of a hinged beam under the influence of external periodic forces. We have analyzed two deterministic statements of this problem on the assumption that all the parameters are known exactly. In the simplest statement of the problem, it is supposed that the beam structure is homogeneous. In the more complex statement, presence of inhomogeneities (defects) on the beam is assumed. The problem is to find the number of forces and their characteristics (application, amplitude and phase of oscillations) that ensure the required waveform with a given accuracy. By means of analytical methods, we have reduced the problems under study to simpler optimization problems. We have obtained analytical expressions for the objective functions and their gradients.

We have used the analytical results and carried out a series of numerical experiments in order to investigate the dependence of the accuracy of approximation of the given waveform on the number of forces and their characteristics (application, amplitude and phase of oscillations). We have established that in case of a homogeneous beam (no defects), for any fixed frequency of oscillations, increase in the number of applied forces reduces the approximation error. Moreover, for any fixed number of applied forces, the approximation error increases as wave number increases. We have analyzed the dependences of optimal characteristics of applied forces on the number of applied forces. When modeling the oscillation of the beam with a defect, we have analyzed the dependence of approximation error on the value of the defect, its position on the beam, and numbers of forces applied to the beam. We have shown that for fixed parameters of the defect, approximation error can be reduced by increasing the number of forces applied to the beam. We have also shown that optimal characteristics of forces depend on location of the defect and parameters.

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