Case Study: CoCDaR and mCoCDaR Approaches for Systemic Risk Contribution Measure

1 Background

This Case Study considers a new systemic risk CoCDaR, the Conditional Drawdown-at-Risk (CDaR) of the financial system conditional on an institution being in financial distress in terms of its drawdown levels, and its multiple regressio version mCoCDaR. This measure is similar to Adrian and Brunnermeier's (2008) CoVaR and Huang and Uryasev's (2014) CoCVaR, but we changed the systemic risk from CVaR to CDaR which puts focus on risk contribution in drawdown scenarios. This measure considers continual distress events. We defined the systemic risk contribution of an institution as the difference between financial system's CoCDaR conditional on the institution being under distress (drawdown at its Drawdown-at-Risk level) and the CoCDaR conditional on the intitution's drawdown being at its median level. We estimated the systemic risk contributions of 10 largest publicly traded banks in the United States for the sample period of February 2000 to January 2015. Alternatively, using mCoCDaR we can define the systemic risk contribution of an institution as the difference in CoCDaR of the financial system conditional on different levels of drawdowns of the institution. We compared CoCDaR(mCoCDaR) to CoCVaR(mCoCVaR) risk contributions for this period.

The CoCDaR is estimated with the linear regression minimizing the CVaR (Superquantile error), as defined by Rockafellar et al (2014), of drawdowns instead of returns. This error is called CVaR2_err in PSG. Problem 1 and 2 minimizes CVaR2_err function. An institution distress DaR level is estimated with quantile regression by minimizing Koenker and Basset error on drawdowns of that institution, see Problem 3. Similarly, we perform the mCoCDaR regression using drawdowns of all the institutions together, and obtain another set of coefficient estimates. We then calculate risk contributions using these coefficient estimates.

10 largest publicly traded banks in the United States by total assets as of December 31, 2014: 1) JP Morgan Chase & Co. (JPM), 2) Bank of America (BAC), 3) Citigroup Inc. (C), 4) Wells Fargo & Company (WFC), 5) The Bank of New York Mellon Corporation (BK), 6) U.S. Bancorp (USB), 7) Capital One Financial Corporation (COF), 8) PNC Financial Services Group, Inc. (PNC), 9) State Street Corporation (STT), 10) The BB&T Corporation (BBT). Time period is from 2/18/2000 to 1/30/2015 including 754 weeks. The data cover two recessions (2001 and 2007- 09) and two financial crisis (2000 and 2008).

2 Statistical Problems

2.1 Drawdown Definition

Suppose $X_1, \dots, X_T$ are the rates of return of a risky instrument coming from a distribution of return random variable X. Let ξ_t be the cumulative rate of return of the instrument for time t, which can be either uncompounded and defined by $\xi_t = \sum_{k=1}^{t} X_k$. The drawdown of the instrument at time t is defined by

$$Y_t = \max_{1 \le k \le t} \xi_k - \xi_t, \qquad t = 1, \dots, T \tag{1}$$

2.2 Estimation of CoCDaR

The Conditional Drawdown-at-Risk (CDaR) for return random variable X is equivalent to the Conditional Value-at-Risk (CVaR) of its drawdown random variable Y:

$$CVaR_\alpha(Y) = CDaR_\alpha(X) \ .$$

Where CVaR of random variable X, see Rockafellar and Uryasev (2002), can be calculated as:

$$CVaR_\alpha(X) = \min_C \{C + \frac{1}{1-\alpha} E[(X-C)^+]\} ,$$

and $A^+ = \max\{0, A\}$.

Let X^{sys} define the return random variable of a financial system and let the return variables of financial institutions $i = 1, \ldots, I$ be denoted as X^i. Given a sample path of data $\{X_t^{sys}, X_t^1, \ldots, X_t^I\}_{1\le t\le T}$, we can obtain the drawdown observations for the financial system as well as all the institutions and denote them by $\{Y_t^{sys}, D_t^1, \ldots, D_t^I\}_{1\le t\le T}$. Similar to Huang and Uryasev (2017), we can define CoCDaR as:

$$CoCDaR_\alpha^{sys|i} = CDaR_\alpha(X^{sys}|X^i, M_1, ..., M_n) = CVaR_\alpha(Y^{sys}|D^i, M_1, ..., M_n) .$$

Here $M_1, ..., M_n$ are state factor variables which are the lagged system variables to provide more explanatory powers in the later considered conditional drawdown regressions and drawdown regressions.

Consider the following regression which is also used by Adrian and Brunnermeier (2008) and Huang and Uryasev (2017):

$$Y_t^{sys} \sim \beta_0 + \beta_1 D_t^i + \omega_1 M_{t-1,1} + ... + \omega_n M_{t-1,n} .$$

We define the residual random variable as:

$$L = Y^{sys} - (\beta_0 + \beta_1 D^i + \omega_1 M_1 + ... + \omega_n M_n) .$$

The estimate of the α-CVaR of Y^{sys} can be obtained by minimizing the CVaR (superquantile) error function from Rockafellar et al (2014):

$$\mathcal{E}_\alpha^{CVaR}(L) = \frac{1}{1-\alpha}\int_0^1 CVaR_\gamma^+(L)\,d\gamma - E[L] . \tag{2}$$

It has been proved by Golodnikov et al (2019) that minimization of error (2) for CVaR regression can be reduced to the minimization of the Rockafellar error (convex and liner programming formulations are in Appendix A, Golodnikov et al (2019)). The Rockafellar error belongs to the Mixed Quantile Quadrangle, as defined by Rockafellar and Uryasev (2013). For given confidence levels $\alpha_k \in (0,1)$ and wights $\lambda_k > 0$, $k = 1, \ldots, K$ such that $\sum_{k=1}^{K} \lambda_k$=1, the Rockafellar error equals:

$$\mathcal{E}^{ROC}(L) = \min_{C_1,\ldots,C_K} \left\{ \sum_{k=1}^{K} \lambda_k \mathcal{E}_{\alpha_k}^{KB}(L - C_k) \mid \sum_{k=1}^{K} \lambda_k C_k = 0 \right\} , \tag{3}$$

where the rescaled Koenker-Bassett (KB) error equals:

$$\mathcal{E}_\alpha^{KB}(L) = E\left[\frac{\alpha}{1-\alpha} L^+ + (-L)^+\right] . \tag{4}$$

Denote by $\hat\beta_0^\alpha, \hat\beta_1^\alpha, \hat\omega_1^\alpha, ..., \hat\omega_n^\alpha$ the regression coefficients obtained by minimizing Rockafellar error (3). The CoCVaR of the system drawdown, which is equivalent to the CoCDaR of the financial system is estimated by:

$$CoCDaR_{t,\alpha}^{sys} = \hat\beta_0^\alpha + \hat\beta_1^\alpha D_t^i + \hat\omega_1^\alpha M_{t-1,1} + ... + \hat\omega_n^\alpha M_{t-1,n} .$$

This regression estimation is done for every institution $i = 1, \ldots, I$.

2.3 Institutional Drawdown-at-Risk

To calculate system CoCDaR at some risk level conditioned on institution i being in distress, we should set an institutional distress level D_t^i.

We define α-Drawdown-at-Risk (α-DaR) as α-quantile (VaR) for the drawdown random variable for $\alpha \in [0,1]$. We can use quantile regression for estimation of α-DaR:

$$D_t^i \sim \gamma_0^i + \gamma_1^i M_{t-1,1}^i + ... + \gamma_n^i M_{t-1,n}^i .$$

Let the residual term be denoted as:

$$G^i = D^i - (\gamma_0^i + \gamma_1^i M_1^i + ... + \gamma_n^i M_n^i)\ .$$

For residual variable L the rescaled Koenker-Bassett (KB) error is given be (3).

By minimizing KB-error, $\mathcal{E}_\alpha^{KB}(G^i)$, we can find coefficients $\hat{\gamma}_0^i, ..., \hat{\gamma}_n^i$ and estimate conditional quantile of D_t^i:

$$DaR_{t,\alpha}^i = \hat{\gamma}_0^i + \hat{\gamma}_1^i M_{t-1,1}^i + ... + \hat{\gamma}_n^i M_{t-1,n}^i\ .$$

2.4 Sytemic Risk Contribution

We first perform the quantile regression in Section 2.2 to estimate $DaR_{t,\alpha'}^i$ for all t for two particular levels: $\alpha' = 0.9$ and $\alpha' = 0.5$. The level $\alpha' = 0.9$ corresponds to the distress level of the institution in terms of its drawdowns and $\alpha' = 0.5$ corresponds to the median (normal) state of the institution.

Next we perform the CVaR regression from Section 2.3 and obtain the estimate of the financial system CDaR conditioned on the drawdowns of institution i and state factors. For every time step t, we can calculate:

$$CoCDaR_{t,\alpha}^{sys|D_t^i = DaR_{t,\alpha'}^i} = \hat{\beta}_0^\alpha + \hat{\beta}_1^\alpha DaR_{t,\alpha'}^i + \hat{\omega}_1^\alpha M_{t-1,1} + ... + \hat{\omega}_n^\alpha M_{t-1,n}\ .$$

By choosing $\alpha' = 0.9$ and $\alpha' = 0.5$ for the DaR distress level for an individual institution and selecting another quantile α for system conditional DaR we get:

$$\Delta CoCDaR_{t,\alpha}^{sys|i} = CoCDaR_{t,\alpha}^{sys|D_t^i = DaR_{t,0.9}^i} - CoCDaR_{t,\alpha}^{sys|D_t^i = DaR_{t,0.5}^i}\ .$$

This difference is the systemic drawdown risk contribution of institution i to the financial system at the selected level α.

2.5 Estimation of mCoCDaR

We define mCoCDaR as:

$$mCoCDaR_\alpha^{sys|1,..,I} = CDaR_\alpha(X^{sys}|X^1, ..., X^I, M_1, ..., M_n) = CVaR_\alpha(Y^{sys}|D^1, ..., D^I, M_1, ..., M_n)\ .$$

Consider the following regression using the same set of state factors as CoCDaR regression:

$$Y_t^{sys} \sim \beta_0 + \beta_1 D_t^1 + ... + \beta_I D_t^I + \omega_1 M_{t-1,1} + ... + \omega_n M_{t-1,n}\ ,$$

with residual:

$$L = Y^{sys} - (\beta_0 + \beta_1 D^1 + ... + \beta_I D^I + \omega_1 M_1 + ... + \omega_n M_n)\ .$$

This regression problem uses I institutions drawdowns and state variables as factors to model the drawdown of the financial system. We have a number of T observations for the system drawdown random variable, the I institutions' drawdown random variables, and the state factors' random variables. We next perform a CVaR regression of the above model to find the superquantile of the system drawdown conditioned on all I institutions drawdown. Denote the coefficients $\hat{\beta}_0^\alpha, \hat{\beta}_1^\alpha, ..., \hat{\beta}_I^\alpha, \hat{\omega}_1^\alpha, ..., \hat{\omega}_n^\alpha$ obtained by minimizing Rockafellar error (3).

The multiple-CoCVaR of the system drawdown, which is equivalent to the CoCDaR of the financial system, is estimated by:

$$mCoCDaR_{t,\alpha}^{sys} = \hat{\beta}_0^\alpha + \hat{\beta}_1^\alpha D_t^1 + ... + \hat{\beta}_I^\alpha D_t^I + \hat{\omega}_1^\alpha M_{t-1,1} + ... + \hat{\omega}_n^\alpha M_{t-1,n}$$

2.6 Sytemic Risk Contribution using mCoCDaR

We first perform the quantile regression in Section 2.2 to estimate $DaR_{t,\alpha'}^i$ for all t and for all i for two particular levels: $\alpha_i' = 0.9$ and $\alpha_i' = 0.5$. Level $\alpha_i' = 0.9$ corresponds to the distress level of the respective institution in terms of its drawdowns and $\alpha_i' = 0.5$ corresponds to the median (normal) state of that institution.

Next we perform the CVaR regression from Section 2.5 and estimate the financial system conditional DaR conditioned on the drawdown levels of all I institutions and state factors. For every time step t, we can calculate:

$$mCoCDaR_{t,\alpha}^{sys|D_t^1=DaR_{t,\alpha'_1}^1,\ldots,D_t^I=DaR_{t,\alpha'_I}^I} = \hat{\beta}_0^\alpha + \hat{\beta}_1^\alpha DaR_{t,\alpha'_1}^1 + \ldots + \hat{\beta}_I^\alpha DaR_{t,\alpha'_I}^I + \hat{\omega}_1^\alpha M_{t-1,1} + \ldots + \hat{\omega}_n^\alpha M_{t-1,n}\,.$$

Now to analyze the effect of a single institution i on the financial system we choose: $\alpha'_i = 0.9$ and $\alpha'_{-i} = 0.5$ where $-i$ means all the institutions other than i,

$$\Delta mCoCDaR_{t,\alpha}^{sys|i} = mCoCDaR_{t,\alpha}^{sys|D_t^i=DaR_{t,0.9}^i,D_t^{-i}=DaR_{t,0.5}^{-i}} - mCoCDaR_{t,\alpha}^{sys|D_t^i=DaR_{t,0.5}^i,D_t^{-i}=DaR_{t,0.5}^{-i}}\,.$$

This difference is the incremental/marginal systemic drawdown risk contribution of institution i distress to the financial system, while other institutions are at their normal states.

3 Optimization Problems

3.1 Optimization Problem 1 and 2

Minimizing CVaR (Superquantile) error:

$$\hat{\beta}^\alpha = argminCVaR2\_err(L)$$

3.2 Optimization Problem 3

Minimizing Koenker and Basset error:

$$\hat{\beta}^\alpha = argminKB\_err(L)$$

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