Dynamic Portfolio Selection with Linear Control Policies for Coherent Risk Minimization

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Outline

- Introduction
- Optimization Model
- Robust Optimization Techniques
- Numerical Experiments
- Conclusion
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Review: Portfolio Selection

- **Mean-Variance Portfolio Selection** [Markowitz ’52]
  - determines optimal portfolio weights on financial assets to produce low-risk high-return investments.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i,j \in N} \sigma_{ij} w_i w_j \quad \text{\large variance (risk)} \\
\text{subject to} & \quad \sum_{i \in N} \bar{r}_i w_i = \bar{r} \quad \text{\large target return (profitability)} \\
& \quad \sum_{i \in N} w_i = 1 \quad \text{\large portfolio weights} \\
& \quad w_i \geq 0, \quad i \in N \\
\end{align*}
\]

- \( w_i \) : portfolio weight (asset \( i \)) \#Decision Variable
- \( \bar{r}_i \) : average return (asset \( i \))
- \( \sigma_{ij} \) : covariance of returns (assets \( i,j \))

(if \( \sigma_{ij} \) is small, increase \( w_i \) & \( w_j \))
(Single-Period) Portfolio Selection

- Risk (variance) is reduced by exploiting return correlations between assets.
- Time-series (serial/intertemporal) dependence of asset returns is not considered, even though historical asset returns are collected as time-series data.

### Historical asset returns

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 1</td>
<td>+1.2%</td>
<td>+0.8%</td>
<td>−0.3%</td>
<td>−1.8%</td>
<td>+1.4%</td>
</tr>
<tr>
<td>Asset 2</td>
<td>−1.5%</td>
<td>−0.2%</td>
<td>+1.9%</td>
<td>+1.2%</td>
<td>+0.2%</td>
</tr>
<tr>
<td>Asset 3</td>
<td>+2.1%</td>
<td>+1.1%</td>
<td>−2.5%</td>
<td>−3.4%</td>
<td>+1.7%</td>
</tr>
</tbody>
</table>
Empirical Studies

- Empirical studies discovered that past returns contain information about expected returns and volatility.
  - G/ARCH models are commonly used for time-series predictions of volatility. [Engle ‘82, Bollerslev ‘86]
  - Returns of large stocks lead those of smaller stocks. [Lo & MacKinlay ‘90]
  - Stock price tends to continue rising if it is going up (a.k.a. momentum) [Jegadeesh & Titman ‘93 ‘01]
  - Portfolio performance can be improved by time-series predictions of stock returns [DeMigule+ ’14]

- These results suggest that portfolio performance can be improved by fully considering time-series dependence of asset returns.
Stochastic Control Approach

- designs control policies (i.e., decision rules) for dynamically rebalancing a portfolio.
- enables conditional decision-making depending on past outcomes.
- In general, we must solve a large-scale dynamic programming problem or partial differential equations.

[Merton & Samuelson ’90, Infanger ’08, Boyd+ ’14]

Past Outcomes (e.g., asset returns)

<table>
<thead>
<tr>
<th>Control Policy</th>
<th>Portfolio Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>input</td>
<td>output</td>
</tr>
</tbody>
</table>
Optimization of Control Policies

Nonlinear Control Policies
- A computational framework based on the kernel method was proposed. [T. & Gotoh. ‘11 ‘14] etc.
- The kernel method is often employed for estimating nonlinear machine learning models.
- can be applied to only a small number of assets.
- are likely to overfit noisy financial data.

Linear Control Policies
- Decision rules are restricted to the class of linear functions of past outcomes.
- may cause a substantial loss of optimality but can enjoy lower computational complexity.
- find many applications in dynamic decision-making problems. [Calafiore ‘08 ’09] etc.
Coherent Risk Measures [Artzner+ ’99]
• are defined as functionals that satisfy desirable properties for quantifying financial risks.
  ✓ Conditional value-at-risk (CVaR) [Rockafellar & Uryasev ’00 ’02]
• are subject to estimation errors because they depend on a small portion of sampled scenarios.
Robust Optimization

- Robust Optimization
  - focuses on the worst-case realization of uncertain parameters (e.g. asset returns).
  - finds many applications in portfolio selection problems. [Bertsimas+ ’11, Kim+ ’18]
  - The worst-case coherent risk measure was derived for single-period portfolio selection to mitigate the fragility of coherent risk measures. [Gotoh+ ’13]

- Our motivation is to extend the robust optimization techniques [Gotoh+ ’13] to dynamic portfolio selection with linear control policies.
Main Contributions

- We address the dynamic portfolio selection problem of minimizing a coherent risk measure to construct an effective linear control policy.
  - We formulate this problem as a time-series-based optimization model.
  - We analyze the optimization model in the dual form to understand that the intertemporal covariance in asset returns is crucial for better investment performance of linear control policies.
  - We adopt robust optimization techniques for our optimization model in order to cope with uncertainty about asset returns.
- Our main result is that the worst-case coherent risk measure can be decomposed into the empirical risk measure and the penalty terms.
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**Control Policy**

- **Control Policy** (for dynamic portfolio selection)
  - is a function to adjust portfolio weights based on past asset returns.
  - enables conditional decision-making depending on past outcomes.

**Control Policy**

**INPUT**
- asset returns

**OUTPUT**
- portfolio weights

**Historical asset returns**

<table>
<thead>
<tr>
<th>Period</th>
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<tr>
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<td>+1.2%</td>
<td>+0.2%</td>
</tr>
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<td>Asset 3</td>
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Control Policy

- Control Policy
  - is a function to adjust portfolio weights based on past asset returns.
  - enables conditional decision-making depending on past outcomes.

Historical asset returns

<table>
<thead>
<tr>
<th>Period</th>
<th>Asset 1</th>
<th>Asset 2</th>
<th>Asset 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+1.2%</td>
<td>−1.5%</td>
<td>+2.1%</td>
</tr>
<tr>
<td>1</td>
<td>+0.8%</td>
<td>−0.2%</td>
<td>+1.1%</td>
</tr>
<tr>
<td>2</td>
<td>−0.3%</td>
<td>+1.9%</td>
<td>−2.5%</td>
</tr>
<tr>
<td>3</td>
<td>−1.8%</td>
<td>+1.2%</td>
<td>−3.4%</td>
</tr>
<tr>
<td>4</td>
<td>+1.4%</td>
<td>+0.2%</td>
<td>+1.7%</td>
</tr>
</tbody>
</table>

Control Policy

- INPUT: asset returns
- OUTPUT: portfolio weights
Linear Control Policy

\[ y_{jt} = b_j + \sum_{k=K} \sum_{i=I} a_{ijk} (r_{i,t-k} - \bar{r}_i) \]

**OUTPUT**
- portfolio weights

**INPUT**
- past excess asset returns

- **Given Constants**
  - \( r_{jt} \): return (asset \( j \), period \( t \))
  - \( \bar{r}_j \): average return (asset \( j \))

- **Decision Variables**
  - \( y_{jt} \): portfolio weight (asset \( j \), period \( t \))
  - \( b_j \): nominal portfolio weight (asset \( j \))
  - \( a_{ijk} \): linear feedback (assets \( i,j \), lag \( k \))
Optimization Model

\[
\begin{align*}
\text{minimize} & \quad \rho(-\mathbf{r}(\mathbf{y})) \cdot \text{coherent risk measure} \\
\text{subject to} & \quad y_{jt} = b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk} (r_{i,t-k} - \bar{r}_i), \quad (j, t) \in J \times (T \setminus K), \\
& \quad \sum_{j \in J} b_j = 1, \\
& \quad \sum_{j \in J} a_{ijk} = 0, \quad (i, k) \in I \times K, \\
& \quad b_j \geq 0, \quad j \in J, \\
& \quad y_{jt} \geq 0, \quad (j, t) \in J \times (T \setminus K). 
\end{align*}
\]

- can be formulated as a linear optimization problem (e.g. by using the conditional value-at-risk).
- Mean-risk objective is also applicable.
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To cope with uncertainty about asset returns, we apply robust optimization techniques to our model.

**Portfolio Return** (period $t$):

$$r_t(y) := \sum_{j \in J} r_{jt} y_{jt} = \sum_{j \in J} r_{jt} \left( b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk} (r_{i,t-k} - \bar{r}_i) \right)$$

Intertemporal product of asset returns:

$$s_{ijkt} := r_{jt} (r_{i,t-k} - \bar{r}_i)$$

$$r_t(a, b \mid \varepsilon, \delta) := \sum_{j \in J} b_j (r_{jt} - \varepsilon_{jt}) + \sum_{k \in K} \sum_{i \in I} \sum_{j \in J} a_{ijk} (s_{ijkt} - \delta_{ijkt})$$
Worst-case Coherent Risk Measure

- **Portfolio Return with Perturbations** (period $t$):
  \[
  r_t(a, b \mid \varepsilon, \delta) := \sum_{j \in J} b_j (r_{jt} - \varepsilon_{jt}) + \sum_{k \in K} \sum_{i \in I} \sum_{j \in J} a_{ijk} (s_{ijkt} - \delta_{ijkt})
  \]

  **Uncertainty Sets of Perturbations**
  \[
  \mathcal{E} := \{ \varepsilon \in \mathbb{R}^{J \times T} \mid \|\varepsilon_t\| \leq \lambda, \quad t \in T \}
  \]
  \[
  \mathcal{D} := \{ \delta \in \mathbb{R}^{I \times J \times K \times T} \mid \|\delta_{kt}\| \leq \lambda_k, \quad (k, t) \in K \times T \}
  \]

  **Main Theorem:**
  \[
  \max\{ \rho(-r(a, b \mid \varepsilon, \delta)) \mid (\varepsilon, \delta) \in \mathcal{E} \times \mathcal{D} \} = \rho(-r(y)) + \lambda \|b\|^\circ + \sum_{k \in K} \lambda_k \|a_k\|^\circ \quad [\text{Gotoh+ '13}]
  \]

  - **empirical version of coherent risk measure**
  - **penalty terms based on the dual norm**
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Experimental Design

- **Dataset** (downloaded from K.R. French’s website)
  - U.S. stock monthly returns
  - 2001-2010 (120 months) for training
  - 2011-2018 (96 months) for testing

- **Objective** \((\alpha - 1) \times \text{Average Return} + \alpha \times \text{CVaR}\)
  - \(\alpha \in (0, 1)\) : Risk aversion parameter

- **Portfolio Selection Strategies**
  - **EWP**: Equally weighted portfolio
  - **SPP**: Single-period portfolio selection
  - **LC(\(|K|\))**: Linear control policies
  - **LC-W(\(|K|\))**: Linear control policies minimizing the worst-case coherent risk (+ holdout validation)
    - \(|K|\) : the number of input time periods
- LC(5) achieved the highest return when $\alpha = 0.99$.
- SPP produced very low return when $\alpha = 0.01$.
  - All the money was invested in one asset.
LC(2) achieved the highest return when $\alpha \in \{0.75, 0.99\}$.
SPP produced very low return when $\alpha = 0.01$. 
LC-W(5) produced higher returns than LC(5).
  • Our penalty terms successfully improved the out-of-sample investment performance.
Return: 25 Assets (Size & Book-to-Market)

- LC-W(3) produced higher returns than LC(3).
  - Our penalty terms successfully improved the out-of-sample investment performance.
The optimal policy was to invest in high-return assets 1 and 2.

This policy is related to a contrarian investment strategy, where one purchases (sells) assets that produced low (high) returns in the previous period.
The optimal policy was to invest mostly in low-risk assets 1 and 4.

When many industries generate high returns (e.g., during a boom economy), the weight on the high-return asset 1 should be increased.
The optimal policy was to invest in a combination of high-return assets 2 and 3, and low-risk asset 4.

Asset 2 contributes strongly to the portfolio construction.
Optimal Policy: 6 Assets (S & BM, $\alpha = 0.99$)

<table>
<thead>
<tr>
<th>$\bar{r}_j$ (%)</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
<th>$j = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.71</td>
<td>0.94</td>
<td>0.98</td>
<td>0.62</td>
<td>0.62</td>
<td>0.57</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma_j$ (%)</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
<th>$j = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.18</td>
<td>5.32</td>
<td>5.76</td>
<td>4.08</td>
<td>4.17</td>
<td>5.45</td>
<td></td>
</tr>
</tbody>
</table>

(b) $\alpha = 0.99$

<table>
<thead>
<tr>
<th>$b_j$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
<th>$j = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.24</td>
<td>0</td>
<td>0.49</td>
<td>0.26</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_{ijk}$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
<th>$i = 5$</th>
<th>$i = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>($k = 1$)</td>
<td>0</td>
<td>263.8</td>
<td>0</td>
<td>-415.5</td>
<td>151.7</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-834.6</td>
<td>0</td>
<td>1548.7</td>
<td>-714.0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>253.0</td>
<td>0</td>
<td>-350.5</td>
<td>97.5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>-22.7</td>
<td>0</td>
<td>232.1</td>
<td>-209.5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>426.6</td>
<td>0</td>
<td>-931.4</td>
<td>504.9</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>27.9</td>
<td>0</td>
<td>-222.4</td>
<td>194.5</td>
<td>0</td>
</tr>
</tbody>
</table>

- The optimal policy was to invest in a combination of high-return asset 2, and low-risk assets 4 and 5.
- Asset 2 can still have a strong effect on portfolio construction.
Conclusion

- We considered the problem of developing an effective linear control policy for dynamic portfolio selection.
- We firstly formulated the optimization model based on a coherent risk measure.
- We then derived the penalty terms based on the robust optimization techniques to cope with uncertainty about asset returns.
- Numerical results demonstrated that
  - Our control policies outperformed the single-period portfolio selection especially when high returns were strongly preferred.
  - When the number of assets was large, our penalty terms improved the out-of-sample investment performance of the control policies.