Drawdown Beta and Portfolio Optimization

Rui Ding⋆ and Stan Uryasev‡

⋆,‡Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY, USA, 11794
⋆rui.ding.1@stonybrook.edu, (917) 929-3490
‡stanislav.uryasev@stonybrook.edu, (352) 213-3457

Abstract

This paper introduces a new dynamic portfolio performance risk measure called Expected Regret of Drawdown (ERoD) which is an average of drawdowns exceeding a specified threshold (e.g., 10%). ERoD is similar to Conditional Drawdown-at-Risk (CDaR) which is the average of some percentage of largest drawdowns. CDaR and ERoD portfolio optimization problems are equivalent and result in the same set of optimal portfolios. Necessary optimally conditions for ERoD portfolio optimization lead to Capital Asset Pricing Model (CAPM) equations. ERoD Beta, similar to the Standard Beta, relates returns of the securities and those of a market. ERoD Beta equals to [average losses of a security over times intervals when market is in drawdown exceeding the threshold] divided by [average losses of the market in drawdowns exceeding the threshold]. Therefore, a negative ERoD Beta identifies a security which has positive returns when market is in drawdown. ERoD Beta accounts for only time intervals when the market is in drawdown and conceptually differs from Standard Beta which does not distinguish up and down movements of the market. Moreover, ERoD Beta provides quite different results compared to the Downside Beta based on Lower Semi-deviation. ERoD Beta is conceptually close to CDaR Beta which is based on a percentage of worst case market drawdowns. We have built a website reporting CDaR and ERoD Betas for stocks and SP 500 index as an optimal market portfolio. The case study showed that CDaR and ERoD Betas exhibit persistence over time and can be used in risk management and portfolio construction.

1 Introduction: Drawdown Betas

The Capital Asset Pricing Model (CAPM) Sharpe [19], Sharpe [20] is a fundamental model in portfolio theory and risk management. It is based on a Markowitz mean-variance portfolio optimization problem Markowitz [10]. Tremendous literature is available on CAPM, see for instance, critical review papers Galagedera [6], Rossi [17].

The Standard Beta relates expected return of a security and expected excess return of a market. Beta has been used as a key indicator of asset performance in portfolio management. Variance risk measure used in standard CAPM has a conceptual drawback: it does not distinguish losses and gains of a portfolio. Markowitz [9] considered Semi-Variance based only on negative returns. The associated beta was called Downside Beta. Although, the concept sounds conceptually attractive, Downside Beta and Standard Beta have close values. Therefore, Downside Beta provides little information in addition to Standard Beta.

Various non-symmetric risk measures have been proposed as an alternative to variance. In particular, Conditional Value-at-Risk (CVaR) introduced by Rockafellar and Uryasev [11] for continuous distributions is the conditional expected loss exceeding Value-at-Risk (VaR), and generalized to discrete distributions in Rockafellar and Uryasev [12].

CAPM has been extended to non-symmetric risk measures such as Generalized Deviations by Rockafellar et al. [14]. This paper demonstrated that CAPM equations are necessary optimally conditions for portfolio optimization problems. In particular, Beta was computed for CVaR and Lower Semi-Deviation (square root of Semi-Variance). Review paper by Krokhmal et al. [8] discusses these and other non-symmetric risk measures and provides formulas for Beta.

A considerable drawback of Variance, CVaR, Semi-Deviation and many other risk measures is that they are static characteristics, which do not account for persistent consecutive portfolio losses (may be resulting in a large cumulative loss). Dynamic Drawdown risk measure is actively used in portfolio management as an alternative to static measures. Portfolio managers are trying to build portfolios with low drawdowns. The most popular drawdown characteristic is the Maximum Drawdown. However, the Maximum Drawdown is not the best risk measure from practical perspective: it accounts for only one specific event on a price sample-path. For instance, Goldberg and Mahmoud [7] suggested so called Conditional Expected Drawdown (CED), which is the tail mean of maximum drawdown distribution. Let us consider the market historical sample-path for the recent 15 years.

There were two major drawdowns of SP500 in the recent 15 years: 1) 2008 Financial Crisis; 2) COVID-19 Crisis. CED will notice 2008 Financial Crisis (which is not very relevant at this time) and completely ignores COVID-19 Crisis, which is the most important risk event in the recent years.

Chekhlov et al. [2] proposed Conditional Drawdown-at-Risk (CDaR) which averages a specified percentage of the largest portfolio drawdowns over an investment horizon. CDaR is defined as CVaR of the drawdown observations of the portfolio cumulative returns. CDaR possesses theoretical properties of a deviation measure, see, Chekhlov et al. [3].

CDaR has been used to identify systemic dependencies in the financial market. Ding and Uryasev [4] considered CDaR regression for measurement of systemic risk contributions of financial institutions and for fund style classifications.

Zabarankin et al. [22] developed CAPM relationships based on CDaR. The paper derived necessary optimality conditions for CDaR portfolio optimization. These conditions resulted in CDaR Beta relating cumulative returns of a market (optimal portfolio) and individual securities. CDaR Beta equals:

$$\beta^i_{CDaR} = \frac{\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{CDaR_\alpha(w^M)} \ ,$$

where

- i = index of a security, $i = 1, ..., I$;
- s = index of sample path of returns of securities, $s = 1, ..., S$;
- p_s = probability of a sample path s;
- t = time, $t = 1, ..., T$;
- w^i_{st} = uncompounded cumulative return of asset i at time moment t on sample path s;
- w^M = vector of uncompounded cumulative returns of market portfolio (optimal portfolio) including components w^M_{st}, $t = 1, ..., T$, $s = 1, ..., S$;
- $\tau(s,t)$ = time moment of the most recent maximum of market cumulative return preceding t on scenario s;
- q^*_{st} = indicator which is equal to $\frac{1}{(1-\alpha)T}$ for the largest $(1-\alpha)T$ drawdowns of market portfolio w^M and zero otherwise;
- $CDaR_\alpha(w^M) = \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^M_{s,\tau(s,t)} - w^M_{st})$ = average of the largest $(1-\alpha)\%$ drawdowns of market portfolio w^M (e.g., if $\alpha = 0.9$ then CDaR accounts for 10% largest drawdowns).

This paper introduced a new drawdown based risk measure called Expected Regret of Drawdown (ERoD). By definition, ERoD is an average of drawdowns exceeding a threshold ϵ. The Expected Regret (also termed Low Partial Moment) is defined as the average of losses exceeding a fixed threshold. Therefore, ERoD is the Expected Regret of drawdown observations over considered period. Testuri and Uryasev [21] established the equivalence between Expected Regret and CVaR risk measures. This equivalence also follows from Quantile Quadrangle, see, Rockafellar and Uryasev [15]; CVaR is the Risk and Partial Moment is the Regret in Quantile Quadrangle. We build on this equivalence result and demonstrated the equivalence of CDaR and ERoD portfolio optimization. Similar to CDaR optimization, the ERoD optimization can be reduced to convex and linear programming. Also, necessary conditions of extremum for ERoD optimization can be formulated similar to the necessary conditions for CDaR optimization. Therefore, formula for ERoD Beta can be derived similar to CDaR Beta. Moreover, CDaR Beta and ERoD Beta coincide for some confidence level α in CDaR and some threshold ϵ in ERoD.

We show (see, Theorem 1) that the ERoD Beta equals:

$$\hat{\beta}^i_{ERoD} = \frac{\frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{\tilde{E}_\epsilon(w^M)}$$

where, in addition to the notations used for CDaR Beta,

- $\tilde{E}_\epsilon(w^M) = \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^M_{s,\tau(s,t)} - w^M_{st})$ = threshold adjusted ERoD with threshold ϵ for return w^M ;
- $d^M_{st} = w^M_{s,\tau(s,t)} - w^M_{st}$ = drawdowns of the market portfolio;
- $q^\star_{st} = \mathbb{1}(d^M_{st} \geq \epsilon)$ = indicator function which is equal to 1 for $d^M_{st} \geq \epsilon$ and 0 otherwise.

ERoD Beta indicates good hedges against market drawdowns. Instruments with low and negative ERoD Beta are quite beneficial portfolio construction.

We have done the following calculations for stock data (with at least 15 years history) and SP500 index. We calculated $ERoD_{0+}$ Beta accounting for positive drawdowns of SP500 index. Also, we calculated $CDaR_{0.9}$ Beta accounting for the largest 10% drawdowns of SP500. The resulting ERoD Beta, CDaR Beta, Standard Beta, and other metrics are posted at the Drawdown Beta Website [5]. Appendix A contains a quick description of this website.

To evaluate impact of 2008 financial crisis and stability of considered betas, we compared ERoD, CDaR, and Standard Betas in different historic periods. Section 6 reports correlation across time of considered betas in Dow30, SP100, SP500 indices. We observed that ERoD and CDaR Betas are more sensitive to drawdowns in historical data than the Standard Beta. All Betas are more stable for larger stocks.

Appendix B reports web links to case studies related to drawdown measure: 1) Portfolio Optimization with Drawdown Constraints; 2) CoCDaR-Approach Systemic Risk Contribution Measurement; 3) Style Classification with mCoCDaR Regression.

2 Conditional Drawdown-at-Risk

We call by a sample-path a set of consecutive vectors of returns of instruments. A sample path may be just a table of historical returns of instruments or joint returns simulated with some model. Suppose that $\{r_t\}_{1\le t\le T}$ is a sample path of scalar returns of some instrument. Let us denote:

$\{w_t\}_{1\le t\le T}$ = vector of uncompounded cumulative returns,

$$w_t = \sum_{\nu=1}^{t} r_\nu \ , \quad 1 \le t \le T \ . \tag{1}$$

$\{d_t\}_{1\le t\le T}$ = vector of drawdowns,

$$d_t = \max_{1\le \nu \le t} \{w_\nu\} - w_t \ , \quad 1 \le t \le T \ . \tag{2}$$

In simple words, for every time moment t the drawdown $\{d_t\}$ is the difference between the previous peak and the current cumulative return.

Zabarankin et al. [22] consider a slightly more complicated definition of drawdown with τ-window where all indices in expressions (1), (2) start from $t_k = \max\{t-k, 1\}$. Only the most recent time window with length k is taken into account for calculation of the drawdown. However this modification does not change any formulas and conclusions, therefore for simplicity we use drawdown definition (2).

Conditional Value-at-Risk (CVaR) for a random value X with confidence level α can be defined as follows

$$CVaR_\alpha(X) = \min_C \ \{C + \frac{1}{1-\alpha}\mathbb{E}[(X-C)^+]\} \ ,$$

where $X^+ = \max\{0, X\}$, see Rockafellar and Uryasev [11], Rockafellar and Uryasev [12]. CVaR is the expectation of the α-tail distribution of the random variable X, i.e., it is the average of the largest outcomes with total probability $1-\alpha$.

The Conditional Drawdown-at-Risk (CDaR) for portfolio returns is defined as CVaR of the drawdown observations of the portfolio, see Chekhlov et al. [2], Chekhlov et al. [3]. For a given $\alpha \in [0, 1)$ and time horizon T such that αT is an integer, the α-CDaR is an average over the worst $(1-\alpha)*100\%$ drawdowns occurred in the time horizon. Accordingly, we define the single sample-path $CDaR_\alpha$ as:

$$CDaR_\alpha(w) = \sum_{t=1}^{T} q_t^\star d_t \ , \tag{3}$$

where $q_t^\star = \frac{1}{(1-\alpha)T}$ if d_t is one of the $(1-\alpha)T$ largest portfolio drawdowns, and $q_t^\star = 0$ otherwise. CDaR formula is defined for equally probable observations of drawdowns.

Suppose now that we have S sample-paths of scalar returns $\{r_{st}\}_{1\le t\le T}$ of some instrument and d_{st} is the drawdown at time t on sample-path s. Probability of a sample-path s is denoted by p_s, $s = 1, 2, \ldots, S$. Zabarankin et al. [22] defined $CDaR_\alpha$

$$CDaR_\alpha(w) = \max_{\{q_{st}\}\in Q} \ \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q_{st} d_{st} \tag{4}$$

with risk envelope Q

$$Q = \left\{ \{q_{st}\} \ \middle| \ \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q_{st} = 1, \ 0 \le q_{st} \le \frac{1}{(1-\alpha)T} \right\} \ .$$

The CDaR definition (4) exploits the dual representation of risk through the risk envelope theory, see, for instance, Rockafellar and Uryasev [15]. A coherent risk functional $R(X)$ can be expressed as follows:

$$R(X) = sup_{q \in Q}\, \mathbb{E}[Xq] \ ,$$

where q is a probability measure from a dual risk envelope Q. The CVaR risk envelope is defined by

$$Q_{CVaR}(\alpha) = \{q : q \in [0, \frac{1}{1-\alpha}], \mathbb{E}[q] = 1\} \ .$$

Therefore, CVaR equals

$$CVaR_\alpha(X) = sup_{q \in Q_{CVaR}(\alpha)} E[Xq] \ ,$$

which implies formula (4).

3 Relation Between Expected Regret and Conditional Value-at-Risk

Testuri and Uryasev [21] proved that for CVaR optimization a regret optimization result in the same set of optimal solutions. Specifically, let $f(x, y)$ be a loss function where x is the associated decision vector and y is a random vector.

For each x, we denote the distribution function for the loss $f(x, y)$ by

$$\Psi(x, C) = \mathbb{P}\big\{y \,\big|\, f(x, y) \le C\big\} \ .$$

The α-VaR (α-quantile) of the loss associated with a decision x equals

$$C_\alpha(x) = \min\big\{C \,\big|\, \Psi(x, C) \ge \alpha\big\}.$$

The minimum in the previous equation is attained because $\Psi(x, C)$ is a nondecreasing and right-continuous function in C. We define the Expected Regret of $f(x, y)$ with respect to the threshold C as:

$$G_C(x) = \mathbb{E}[f(x, y) - C]^+ \ .$$

With notation

$$F_\alpha(x, C) = C + \frac{1}{1-\alpha} G_C(x) \ ,$$

CVaR of $f(x, y)$ equals:

$$CVaR_\alpha(x) = \min_C \quad F_\alpha(x, C) \ . \tag{5}$$

The following facts were proved in [11, 12]. As a function of $C \in I\!\!R$, $F_\alpha(x, C)$ is finite and convex (hence continuous), with

$$\begin{aligned} C_\alpha(x) &= \text{lower endpoint of } \operatorname{argmin}_C F_\alpha(x, C), \\ C_\alpha^+(x) &= \text{upper endpoint of } \operatorname{argmin}_C F_\alpha(x, C), \end{aligned}$$

where the argmin refers to the set of C for which the minimum is attained and in this case is a nonempty, closed, bounded interval (perhaps reducing to a single point). In particular, one has

$$C_\alpha(x) \in \operatorname{argmin}_C F_\alpha(x, C), \qquad CVaR_\alpha(x) = F_\alpha(x, C_\alpha(x)) \ .$$

Also, (5) implies

$$\min_{x \in U} \quad CVaR_\alpha(x) = \min_{x \in U, C \in R} \quad F_\alpha(x, C) \ , \tag{6}$$

where U is a feasible set for the vector x. For example, U, could be a linear constraint on the expected return of a portfolio (see, Section 5). Denote:

- $V_\alpha = Argmin_{x \in U, C \in R}\, F_\alpha(x, C)$ = solution set of the right hand side minimization problem (6);
- $U_\alpha^{CVaR} = Argmin_{x \in U}\, CVaR_\alpha(x)$ = solution set of the left hand side minimization problem (6);
- $U_C^{Regret} = Argmin_{x \in U}\, G_C(x)$ = solution set of the minimum regret problem;
- $A_\alpha(x)$ = projection of V_α on the C line, i.e., $A_\alpha = \{C\!:$ there exists x such that $(x, C) \in V_\alpha\}$.

Under condition that the function $G_C(x)$ is continuously differentiable, Testuri and Uryasev [21] proved:

Statement 1. For any $\alpha \in (0, 1)$ and $x^\star \in U_\alpha^{CVaR}$, there exists a pair $(x^\star, C^\star) \in V_\alpha$ such that $x^\star \in U_{C^\star}^{Regret}$. In particular, $(x^\star, C_\alpha(x^\star)) \in V_\alpha$, such that $x^\star \in U_{C_\alpha(x^\star)}^{Regret}$.

Statement 2. For any C and $x^\star \in U_C^{Regret}$, there exists a unique $\alpha \in (0, 1)$ such that $C \in A_\alpha(x^\star)$, $(x^\star, C) \in V_\alpha$, and $x^\star \in U_\alpha^{CVaR}$.

The above statements established the equivalence between CVaR optimization and expected regret optimization. For the special case when $V_\alpha = U_\alpha^{CVaR} \times A_\alpha$, we have $U_\alpha^{CVaR} = U_C^{Regret}$ for any $C \in A_\alpha$.

4 CDaR and ERoD Portfolio Optimization

Let us denote by $w(x)$ the vector of cumulative returns of a portfolio with weights vector x. Also, we denote by $D(w(x))$ the random drawdown value for the portfolio x.

CDaR for portfolio x, by definition, is CVaR of of the random value $D(w(x))$ i.e.,

$$CDaR_\alpha(w(x)) \,=\, CVaR_\alpha(D(w(x)))\ .$$

Expected Regret of Drawdown (ERoD) for portfolio x with threshold ϵ, by definition, is the expected regret of the random value $D(w(x))$ i.e.,

$$ERoD_\epsilon(w(x)) \,=\, \mathbb{E}[(D(w(x)) - \epsilon)^+]\ .$$

For instance, suppose that we want to estimate the average of positive drawdowns of a portfolio (i.e., zero drawdowns are excluded from consideration). We can select a sufficiently small threshold ϵ and evaluate ERoD of a portfolio with this threshold.

We denote

- $x = (x^1, ..., x^I) \;=\;$ vector of weights for n assets in the portfolio;
- $(w^1_{st}, \ldots, w^I_{st}) \;=\;$ vector of uncompounded cumulative returns of portfolio assets at time moment t on scenario s;
- $p_s \;=\;$ probability of the scenario (sample path of returns of securities);
- $w_{st}(x) = \sum_{i=1}^I w^i_{st} x^i \;=\;$ cumulative portfolio return at time moment t on scenario s;
- $w(x) =$ vector of cumulative portfolio returns with components $w_{st}(x)\,,\ s = 1, \ldots, S;\ t = 1, \ldots, T\,;$
- $d_{st}(x) = \max\limits_{1\le\nu\le t}\{w_{s\nu}(x)\} - w_{st}(x) \;=\;$ drawdown of portfolio at time t on sample path s.

ERoD for a portfolio with threshold ϵ is calculated as follows:

$$ERoD_\epsilon(w(x)) \,=\, \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s (d_{st}(x) - \epsilon)^+\ .$$

Following Zabarankin et al. [22], we state CDaR multiple paths minimization over T periods subject to a constraint on the portfolio expected cumulative return at time T:

$$\min_x \quad CDaR_\alpha(w(x)) \quad s.t. \;\; \sum_{s=1}^{S} p_s w_{sT}(x) \ge \delta\ . \tag{7}$$

This problem is similar to Markowitz mean-variance optimization with variance replaced by the α-CDaR. However, an important difference, is that it is a so calledd a Static-Dynamic problem over T periods. The problem is dynamic because there are T time periods; however, it is static in the sense that portfolio x is fixed at the initial time moment $t = 1$ and it is not changed over time. The considered investment strategy is similar to a popular Constant Proportions Strategy.

The above minimization problem (7) is equivalent to the maximization problem below:

$$\max_x \quad \sum_{s=1}^{S} p_s w_{sT}(x) \quad s.t. \;\; CDaR_\alpha(w(x)) \le v\ , \tag{8}$$

in the sense that efficient frontiers of these two problems (7) and (8) coincide.

Further we formulate the ERoD portfolio optimization problem, similar to the CDaR portfolio optimization problem (7):

$$\min_x \quad ERoD_\epsilon(w(x)) \quad s.t. \quad \sum_{s=1}^{S} p_s w_{sT}(x) \ge\ \delta\ , \tag{9}$$

Statement 1 in the previous section 3 implies that for every confidence level α an optimal solution x^* of CDaR minimization problem (7) can be obtained by solving ERoD minimization problem (9) with $\epsilon = C_\alpha(D(w(x^*)))$. Also, Statement 2 in the section 3 implies that for every ϵ an optimal solution x^* of ERoD minimization problem (9) can be obtained by solving the CDaR minimization problem (7) with some confidence level α.

ERoD portfolio minimization problem (9) can be solved very efficiently via convex and linear programming.

5 CAPM: Necessary Optimality Conditions for CDaR and ERoD Portfolio Optimization

Zabarankin et al. [22] provided necessary optimality conditions for the optimization problems (7) and (8) in the form of CAPM equations. In particular, the formula for CDaR Beta was derived similar to the Standard Beta, which relates return of market and individual assets. Baghdadabad et al. [1] present a an attempt to formulate a drawdown-based beta in the CAPM setting, but their derivation does not have a rigorous mathematical justification based portfolio optimization. This paper evaluates correlation of drawdowns in statistical setting.

Further we follow Zabarankin et al. [22] and present Beta for CDaR risk measure. Let $w^M = w(x^\star)$ be the vector of cumulative returns of the optimal portfolio of problem (7) or (8). The necessary optimality conditions for the solution $x^\star$ of both problems (7) and (8) are stated in the form of CAPM:

$$\sum_{s=1}^{S} p_s w^i_{sT} = \beta^i_{CDaR} \sum_{s=1}^{S} p_s w^M_{sT} \ , \tag{10}$$

$$\beta^i_{CDaR} = \frac{\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{CDaR_\alpha(w^M)} \ , \tag{11}$$

where $q^*_{st} = \frac{1}{(1-\alpha)T}$ for the largest $(1-\alpha)T$ drawdowns in the optimal portfolio w^M and zero otherwise, and the time index of the most recent historic maximum in cumulative returns is defined as

$$\tau(s,t) = \max\{\, k \,|\, 1 \le k \le t,\ w^M_{sk} = \max_{1\le \ell \le t} w^M_{s\ell}\ \}. \tag{12}$$

Since there can be multiple historic peaks, we take the most recent one for drawdown calculation. CDaR Beta equation (10) relates expected cumulative returns of market and instruments:

- β^i_{CDaR} = CDaR Beta;
- $\sum_{s=1}^{S} p_s w^M_{sT}$ = cumulative expected return of the market;
- $\sum_{s=1}^{S} p_s w^i_{sT}$ = cumulative expected return of the security i.

On the efficient frontier, CDaR vs. the target return, the optimal solution x^* is the point where the capital asset line makes a tangent cut with the efficient frontier.

According to Statement 1 in Section (3), if $x^\star$ is an optimal solution of (7) then $x^\star$ is an optimal solution of (9) with $\epsilon = VaR_\alpha(D(w(x^\star)))$. Moreover,

$$CDaR_\alpha(w(x^\star)) = CVaR_\alpha(D(w(x^\star))) = \epsilon + \frac{1}{1-\alpha}\mathbb{E}[D(w(x^\star) - \epsilon)^+] = \epsilon + \frac{1}{1-\alpha} ERoD_\epsilon(w(x^\star)) \ . \tag{13}$$

Therefore, CAPM optimally conditions (10), (11) for CDaR optimization (7) and (8) are also the optimally conditions for the ERoD optimization (9).

CAPM optimally conditions (10), (11) were developed for discrete distribution of drawdowns by Zabarankin et al. [22]. However, the equivalence Statements 1,2 in Section (3) are formulated for continuous distributions (see, Testuri and Uryasev [21]). Therefore, further we rigorously prove CAPM optimality conditions for ERoD portfolio optimization problem (9) for discrete distributions.

Theorem 1. *Let $w^M = w(x^\star)$ be the cumulative return vector for an optimal portfolio $x^\star$ of problem (9). The necessary optimality conditions for (9) can be stated in the form of CAPM:*

$$\sum_{s=1}^{S} p_s w^i_{sT} = \beta^i_{ERoD} \sum_{s=1}^{S} p_s w^M_{sT} \ , \tag{14}$$

$$\hat{\beta}^i_{ERoD} = \frac{\frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{\tilde{E}_\epsilon(w^M)} \ , \tag{15}$$

where

- $\tilde{E}_\epsilon(w^M) = \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^M_{s,\tau(s,t)} - w^M_{st})$ = *threshold adjusted ERoD with threshold ϵ for return w^M ;*
- β^i_{ERoD} = *ERoD Beta relating the total expected cumulative return, $\sum_{s=1}^{S} p_s w^M_{sT}$, of optimal portfolio (market) and total expected cumulative return, $\sum_{s=1}^{S} p_s w^i_{sT}$, of security i;*
- $\tau(s,t)$ = *index calculated with (12);*
- $d^M_{st} = w^M_{s,\tau(s,t)} - w^M_{st}$ = *drawdowns of the optimal portfolio;*
- $q^\star_{st} = \mathbb{1}(d^M_{st} \ge \epsilon)$ = *indicator function which is equal to 1 for $d^M_{st} \ge \epsilon$ and 0 otherwise.*

Consequently, for a single path:

$$w_T^i \; = \; \beta_{ERoD}^i \, w_T^M \; ,$$
$$\hat{\beta}_{ERoD}^i \; = \; \frac{\frac{1}{T}\sum_{t=1}^T q_t^\star (w_{\tau(t)}^i - w_t^i)}{\tilde{E}_\epsilon(w^M)} \; ,$$

where

- $\tilde{E}_\epsilon(w^M) = \frac{1}{T}\sum_{t=1}^T q_t^\star (w_{\tau(t)}^M - w_t^M) \; = \;$ *threshold adjusted ERoD with threshold* ϵ *for return* w^M *;*
- $q_t^\star = \mathbb{1}(d_t^M \geq \epsilon) \; = \;$ *indicator for drawdowns* $d_t^M = w_{\tau(t)}^M - w_t^M$ *;*
- $\tau(t) = \max\{\, k \,|\, 1 \leq k \leq t, \; w_k^M = \max_{1\leq \ell \leq t} w_\ell^M \}$ *.*

Proof. Let us denote by $H_\epsilon(w(x)) = \frac{1}{T}\sum_{s=1}^S \sum_{t=1}^T p_s (d_{st}(x) - \epsilon)^+$ the objective function and by $x^\star$ an optimal portfolio vector of problem (9). The cumulative return function $w_{sk}(x)$ linearly depends upon vector x. The objective function $H_\epsilon(w(x))$ is convex in x and cumulative return function in (9) is linear in x. The necessary optimality condition for the convex optimization problem (9) is formulated as follows (see for reference Theorem 3.34 in Ruszczynski [18]):

$$\lambda^\star \, \nabla_x (\sum_{s=1}^S p_s w_{sT}(x^\star) - \delta) \; \in \; \partial_x H_\epsilon(w(x^\star)) \; , \tag{16}$$

where

- $\nabla_x (\sum_{s=1}^S p_s w_{sT}(x^\star) - \delta) \; = \;$ gradient of constraint function at $x = x^\star$;
- $\lambda^\star \; = \;$ Lagrange multiplier such that $\lambda^\star \geq 0$ and $\lambda^\star(\delta - \sum_{s=1}^S p_s w_{sT}(x^\star)) = 0$;
- $\partial_x H_\epsilon(w(x^\star)) \; = \;$ subdifferential of convex in x function $H_\epsilon(w(x))$ at $x = x^\star$.

The gradient of the constraint function, which is liner in x, equals:

$$\nabla_x (\sum_{s=1}^S p_s w_{sT}(x^\star) - \delta) = \sum_{s=1}^S p_s \nabla_x w_{sT}(x) = \sum_{s=1}^S p_s (w_{sT}^1, ..., w_{sT}^n) = (\sum_{s=1}^S p_s w_{sT}^1 \, , ..., \sum_{s=1}^S p_s w_{sT}^n) \; . \tag{17}$$

Objective function $H_\epsilon(w(x)$ is a linear combination of a maximum of linear functions in x. Therefore, according to the standard results in convex analysis,

$$g = (g^1, ..., g^n) \in \partial_x H_\epsilon(w(x^\star)) \; , \quad \text{where} \quad g^i = \frac{1}{T}\sum_{s=1}^S \sum_{t=1}^T p_s q_{st}^\star (w_{s,\tau(s,t)}^i - w_{st}^i) \; . \tag{18}$$

With (17) and (18) we obtain the following system of equations

$$g^i = \frac{1}{T}\sum_{s=1}^S \sum_{t=1}^T p_s q_{st}^\star (w_{s,\tau(s,t)}^i - w_{st}^i) = \lambda \sum_{s=1}^S p_s w_{sT}^i \; , \quad i = 1, \ldots, I \; . \tag{19}$$

Multiplying the left and right hand sides of the previous equation by x^i and summing up terms for $i = 1, \ldots, I$, we have:

$$\frac{1}{T}\sum_{s=1}^S \sum_{t=1}^T p_s q_{st}^\star (w_{s,\tau(s,t)}(x) - w_{st}(x)) = \lambda \sum_{s=1}^S p_s w_{sT}(x) \; ,$$

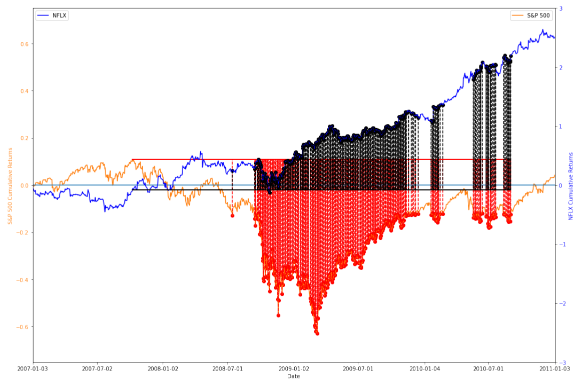
and consequently

$$\lambda = \frac{\frac{1}{T}\sum_{s=1}^S \sum_{t=1}^T p_s q_{st}^\star (w_{s,\tau(s,t)}(x) - w_{st}(x))}{\sum_{s=1}^S p_s w_{sT}(x)} \; . \tag{20}$$

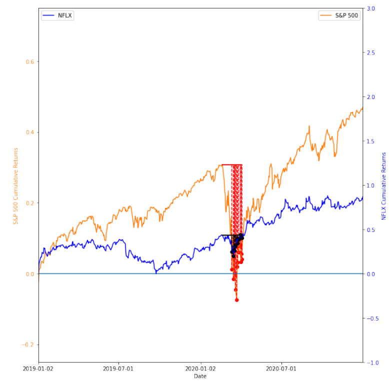
Substituting (20) to (19) gives necessary conditions (14), (15) in CAPM format. □

6 Discussion of CDaR and ERoD Betas

Following standard CAPM assumptions, an optimal portfolio of problems (7), (9) is assumed to be a market portfolio. Therefore, CDaR and ERoD betas can be computed with historical data of a market portfolio and individual assets. These betas show how a security behaved during previous market drawdowns. For instance, let us consider Netflix (NFLX) in the 15-year period from January 2006 to January 2021. Drawdown Beta Website [5] shows that NFLX has a large negative $ERoD_{0+}$ beta of -3.532 with a close to zero positive threshold ϵ. The $ERoD_{0+}$ is based on all daily non-zero drawdowns of SP500 over 15 years. Also, $CDaR_{0.9}$ equals -2.388 for the largest 10% SP500 drawdowns. However, Netlix has a positive Standard Beta value of 0.862 based on monthly returns. The distinct perspective offered by the ERoD and CDaR Betas, can be understood from looking at the largest drawdowns of the SP500 index and comparing with the cumulative returns of NFLX in the same time periods, see Figure 1. The figure shows the top 10% largest drawdowns of SP500 for ERoD of 0.226 and corresponding cumulative returns of NFLX. SP500 had large drawdowns in 2008-2011 during the 2008 financial crisis. At that time, NFLX had strong positive returns. Also, in 2020 during the COVID crisis NFLX had much smaller drawdowns than SP500.

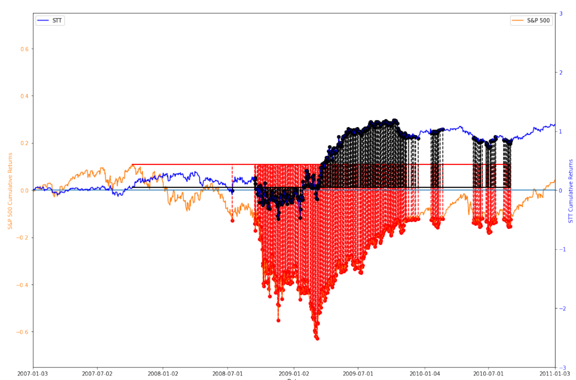


(a) Financial crisis 2008

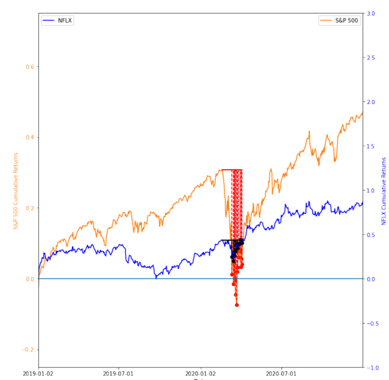


(b) COVID-19 crisis

Figure 1: NFLX returns vs largest 10% SP500 drawdowns (blue curve = NFLX cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = NLFX cumulative returns)

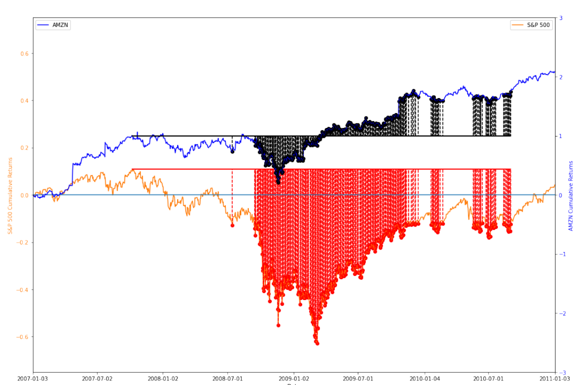


(a) Financial crisis 2008

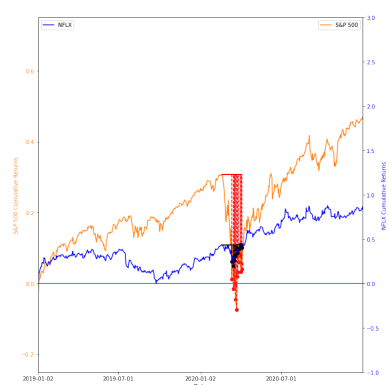


(b) COVID-19 crisis

Figure 2: STT returns vs largest 10% SP500 drawdowns (blue curve = STT cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = STT cumulative returns)



(a) Financial crisis 2008



(b) COVID-19 crisis

Figure 3: AMZN returns vs largest 10% SP500 drawdowns (blue curve = AMZN cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = AMZN cumulative returns)

Figures 2 and 3 show similar graphs for State Street Corporation (STT) and Amazon (AMZN), respectively. In 15-year period, STT reported $ERoD_{0+}$ Beta of -1.15, a $CDaR_{0.9}$ Beta of -1.264, and Standard Beta of 1.404. Likewise, AMZN reported $ERoD_{0+}$ Beta of -1.171, $CDaR_{0.9}$ Beta of -0.255, and Standard Beta of 1.331. We want to emphasize that $CDaR_{0.9}$ and $ERoD_{0+}$ Betas pick up different drawdown characteristics. $CDaR_{0.9}$ is concentrated on unique market conditions, such as financial crisis 2008 and COVID crisis 2020. These unique events most probably will not repeat in future. However, $ERoD_{0+}$ takes into account all drawdowns of SP500. Therefore, the suggested ERoD Beta is an important compliment to CDaR Beta.

Below we report the results of more comprehensive numerical studies that compare different betas in different time periods. The first group of numerical experiments identifies the impact of excluding from the dataset the 2008 financial crisis. We have considered overlapping 15 and 10-year periods:

- Period 1: from 2006/01/01 to 2021/01/01;
- Period 2: from 2011/01/01 to 2021/01/01.

The 15-year dataset (Period 1) includes both 2008 financial crisis and COVID-19 crisis. The 10-year dataset (Period 2) includes only COVID-19 crisis.

Table 1: Betas for DOW30 Stocks: 15-year Period 1 vs 10-year Period 2

	$ERoD_{0+}$ Period 1	$ERoD_{0+}$ Period 2	$CDaR_{0.9}$ Period 1	$CDaR_{0.9}$ Period 2	Standard Period 1	Standard Period 2	Downside Period 1	Downside Period 2
AAPL	-0.606	0.767	-0.062	0.759	0.980	1.107	1.024	1.077
AMGN	-0.094	0.096	-0.201	0.256	0.856	0.906	0.725	0.815
AXP	0.731	1.129	1.037	1.235	1.375	1.134	1.450	1.240
BA	1.321	1.179	1.492	2.056	1.017	1.369	1.254	1.585
CAT	0.798	2.042	1.003	1.698	1.183	1.171	1.106	1.085
CRM	-0.148	0.496	-0.257	0.746	1.428	1.565	1.153	1.206
CSCO	1.036	0.786	0.924	0.604	1.142	1.069	1.001	1.001
CVX	0.132	1.120	0.146	1.059	1.007	1.116	1.103	1.147
DIS	0.221	0.973	0.445	1.245	0.927	0.804	1.031	1.053
GS	0.877	2.332	0.556	2.000	1.334	1.121	1.384	1.301
HD	0.145	0.086	0.203	0.496	1.022	0.852	0.969	1.056
HON	0.803	1.043	1.002	1.271	0.970	1.183	1.050	1.083
IBM	0.022	1.400	0.008	1.035	0.812	0.808	0.794	0.907
INTC	0.333	0.432	0.510	0.432	0.973	1.026	1.006	1.067
JNJ	0.003	0.334	0.076	0.247	0.577	0.796	0.571	0.633
JPM	-0.534	1.580	-0.678	1.647	1.317	1.338	1.447	1.276
KO	-0.122	0.170	0.172	0.311	0.520	0.628	0.593	0.709
MCD	-0.826	-0.599	-0.389	-0.398	0.590	0.678	0.682	0.791
MMM	0.685	1.141	0.647	0.920	0.794	0.745	0.823	0.907
MRK	0.661	0.168	0.903	0.379	0.779	0.451	0.746	0.717
MSFT	-0.074	-0.019	0.256	0.153	1.020	1.242	0.968	1.059
NKE	-0.714	-0.421	-0.167	0.026	1.015	1.295	0.956	0.982
PG	0.144	0.149	0.306	0.126	0.666	0.406	0.583	0.579
TRV	-0.320	0.546	-0.228	0.923	0.907	0.604	1.055	1.004
UNH	0.593	0.117	0.922	0.343	0.720	0.877	0.985	1.042
VZ	0.391	0.067	0.535	0.073	0.758	0.456	0.628	0.536
WBA	0.339	1.012	0.385	0.788	0.816	0.612	0.697	0.791
WMT	-0.570	0.599	-0.581	0.216	0.633	0.328	0.503	0.480

Table 1 provides betas based on 10-year and 15-year historic periods for DOW30 stocks. We compare $ERoD_{0+}$ Beta, $CDaR_{0.9}$ Beta, and Standard Beta (based on monthly returns) and Downside Beta (based on negative monthly returns) for these two periods. Fist, we want to metion that Standard and Downside Betas have similar values. Therefore, Downside Beta brings little new information compared to Standard Beta. The Standard and Downside Betas are quite stable across two time periods. The Standard and Downside Beta is not sensitive to excluding/including from the dataset 2008 financial crisis, i.e., these betas are not tuned to pick up drawdown risk. However, for some stocks, the sign of ERoD Beta and CDaR Beta changes from Period 1 to Period 2. Therefore, drawdown betas "notice" the major risk event (2008 crisis) in the historical data.

To evaluate impact of 2008 crisis on different betas, we calculated the correlation coefficients between betas for 10-year and 15-year historic data. Table 2 shows correlation coefficients for betas for stocks in DOW30, SP100, and SP500 indices. The first row (DOW30) in Table 2 presents correlation coefficients for columns 1 and 2, columns 3 and 4, and columns 5 and 6 of Table 1.

Table 2: Correlation coefficients for ERoD, CDaR and Standard Betas between 15-year Period 1 and 10-year Period 2

	$ERoD_{0+}$-Beta Correlation	$CDaR_{0.9}$-Beta Correlation	Standard-Beta Correlation
DOW30	0.458	0.392	0.786
S&P 100	0.476	0.396	0.751
S&P 500	0.357	0.241	0.631

We observe that the Standard Beta is least sensitive to excluding from the data set the 2008 crisis (correlation is significant, ranging from 0.631 to 0.786). The Standard Beta is not tuned to pick up the drawdown risk. However, ERoD and CDaR Betas are impacted more significantly (correlation coefficient for CDaR Beta ranges from 0.241 to 0.396 and for ERoD Beta from 0.357 to 0.476). CDaR Beta which is concentrated on largest drawdowns is changing quite significantly when we exclude from the dataset the 2008 crisis.

Further, to evaluate stability over time of different betas, we considered two non-overlapping 7.5 years historic periods:

- Period 3: from 2006/01/01 to 2013/07/01;
- Period 4: from 2013/07/01 to 2021/01/01.

We provide Tables 3, 4 similar to Tables 1, 2. Table 3 shows three betas in the time Periods 3 and 4 for DOW30 stocks.

Table 3: Betas for DOW30 Stocks: Period 3 and 4

	$ERoD_{0+}$-Beta Period 3	$ERoD_{0+}$-Beta Period 4	$CDaR_{0.9}$-Beta Period 3	$CDaR_{0.9}$-Beta Period 4	Standard-Beta Period 3	Standard-Beta Period 4
AAPL	-0.986	1.126	0.520	1.402	0.894	1.064
AMGN	-0.218	0.468	-0.071	0.346	0.817	0.954
AXP	0.540	1.606	1.544	1.645	1.601	1.056
BA	1.332	1.277	1.476	2.202	0.866	1.365
CAT	0.558	1.892	1.321	1.522	1.252	1.406
CRM	-1.438	0.154	0.455	0.761	1.533	1.170
CSCO	1.119	0.664	1.030	0.670	1.116	1.245
CVX	-0.129	1.323	0.196	1.209	1.042	1.259
DIS	0.081	0.859	0.684	0.981	0.951	1.025
GS	0.640	1.956	0.946	1.663	1.445	1.325
HD	-0.037	0.253	0.382	0.564	1.003	0.901
HON	0.768	0.967	1.076	1.082	1.026	1.105
IBM	-0.410	1.986	0.272	1.600	0.721	1.002
INTC	0.317	0.405	0.719	0.481	0.776	1.205
JNJ	-0.110	0.519	0.164	0.333	0.491	0.728
JPM	-0.910	1.190	-0.120	1.348	1.436	1.243
KO	-0.227	0.356	0.310	0.517	0.411	0.657
MCD	-0.903	-0.478	-0.219	-0.110	0.674	0.795
MMM	0.585	1.145	0.841	0.746	0.881	1.016
MRK	0.727	0.359	0.985	0.331	0.705	0.725
MSFT	-0.047	-0.197	0.487	0.266	0.877	1.177
NKE	-0.788	-0.378	0.109	0.331	1.023	0.941
PG	0.152	0.107	0.353	0.126	0.576	0.634
TRV	-0.511	0.551	0.031	0.748	0.947	0.926
UNH	0.685	0.180	0.961	0.375	0.683	0.925
VZ	0.414	0.285	0.481	0.082	0.741	0.625
WBA	0.223	0.856	0.545	0.515	0.644	0.820
WMT	-0.916	1.007	-0.431	0.347	0.739	0.550

Table 4, similar to Table 2, shows correlation coefficients for betas between two time periods for DOW30, SP100, and SP500 stocks. We observe, that CDaR Beta has a relatively high correlation for very large companies (stocks in DOW30 and SP100 indices). The CDaR Beta correlation coefficient for DOW30 equals 0.515 while Standard Beta shows correlation coefficient 0.674. The relatively high correlation can be explained by the fact that both time periods have significant market drawdowns.

Table 4: Correlation coefficients of ERoD, CDaR and Standard Betas between Periods 3 and 4

	$ERoD_{0+}$-Beta Correlation	$CDaR_{0.9}$-Beta Correlation	Standard-Beta Correlation
DOW30	0.275	0.515	0.676
S&P 100	0.305	0.449	0.645
S&P 500	0.074	0.293	0.577

Standard Beta is the most stable characteristic because it is based on 100% of data (taking into account both up and down market conditions). ERoD and CDaR Betas account for only time intervals when market goes down. An interesting fact is that stability of all 3 betas is higher for larger stocks in DOW30 and SP100 compared to SP500. For instance, for SP100 the CDaR Beta correlation coefficient equals 0.449 and for SP500 equals 0.293 . Significant positive correlations in Table 4 for DOW30 and SP100 stocks show that CDaR and ERoD Betas can be used for constructing portfolios with low drawdowns.

7 Conclusion

This paper extended the approach developed by Zabarankin et al. [22]. We introduced a new drawdown risk measure called Expected Regret of Drawdown (ERoD). We showed equivalence of CDaR and ERoD portfolio optimization problems, which is based on the results of Testuri and Uryasev [21].

Necessary condition of extremum for the ERoD portfolio optimization problem is formulated in CAPM format. We have derived a new ERoD Beta similar to the Standard and CDaR Betas. The ERoD Beta evaluates portfolio performance during market drawdowns exceeding some threshold ϵ . For small values of ϵ the ERoD Beta takes into account all drawdowns included in the dataset.

We have conducted a case study for DOW30, SP100, and SP500 stocks and compared CDaR and ERoD Betas with the Standard Beta. We have found that CDaR and ERoD Betas are more sensitive to market drawdowns than the Standard Beta. For some stocks CDaR and ERoD Betas are negative, while Standard Beta is positive. Therefore, these stocks had positive returns when market was in drawdown, in spite positive correlation of returns of these stocks and market returns. CDaR and ERoD Betas are quite different characteristics compared to Standard Beta. We want to mentioned also the so called Downside Beta, which is the normalized correlation of stock and market returns over time periods when the market return is negative. The Downside Beta and Standard Beta have very similar values, therefore, Downside Beta does not provide significant additional information compared to the Standard Beta.

The CDaR and ERoD Betas can be used for constructing portfolios with controlled drawdown. Zero CDaR and zero ERoD constraints can be imposed in the portfolio optimization problems, similar to the zero Standard Beta risk management constraint. It is possible to impose simultaneously zero beta constraints for different betas.

Drawdown Beta Website [5] at the server of Quantitative Finance Program at Stony Brook University shows CDaR, ERoD, and Standard Betas as well as other characteristics for SP500 stocks. This website is described in Appendix A. Also, Appendix B contains web links to case studies related to implementation of drawdown portfolio optimization.

Data Availability Statement

The data that support the findings of this study are downloaded from Yahoo Finance `https://finance.yahoo.com/`.

References

[1] M. R. T. Baghdadabad, F. M. Nor, and I. Ibrahim, *Mean-Drawdown Risk Behavior: Drawdown Risk and Capital Asset Pricing*, J. Bus. Econ. Manag., 14 (2013), pp. S447-S469.

[2] A. Chekhlov, S. Uryasev, and M. Zabarankin, *Portfolio Optimization with Drawdown Constraints*, B. Scherer (Ed.) Asset and Liability Management Tools, Risk Books, London, 2003.

[3] A. Chekhlov, S. Uryasev, and M. Zabarankin, *Drawdown Measure in Portfolio Optimization*, Int. J. Theor. Appl. Finance, 8 (2005), pp. 13-58.

[4] R. Ding and S. Uryasev, *CoCDaR and mCoCDaR: New Approach for Measurement of Systemic Risk Contributions*, J. Risk. Financial. Manag., 13 (2020), pp. 270.

[5] Drawdown Beta Website, `http://qfdb.ams.stonybrook.edu/index_SP_10.html` and `http://qfdb.ams.stonybrook.edu/index_SP_15.html`, Quantitative Finance Program at Stony Brook University, 2021.

[6] D. Galagedera, *A Review of Capital Asset Pricing Model*, Managerial Finance, 33 (2007), pp. 821-832.

[7] L. R. Goldberg and O. Mahmoud, *Drawdown: from Practice to Theory and Back Again*, Math. Finan. Econ., 11 (2017), pp. 275–297.

[8] P. Krokhmal, M. Zabarankin, and S. Uryasev, *Modeling and Optimization of Risk*, Surv. Oper. Res. Manag. Sci., 16 (2011), pp. 49-66.

[9] H. M. Markowitz, *Portfolio selection: Efficient diversification of investments*, Yale University Press, 1959.

[10] H. M. Markowitz, *Foundations of Portfolio Theory*, J. Finance, 46 (1991), pp. 469-477.

[11] R. T. Rockafellar and S. Uryasev, *Optimization of conditional value-at-risk*, J. Risk, 2 (2000), pp. 21-42.

[12] R. T. Rockafellar and S. Uryasev, *Conditional Value-at-Risk for general loss distributions*, J. Bank. Finance, 26 (2002), pp. 1443-1471.

[13] R. T. Rockafellar, S. Uryasev, and M. Zabarankin, *Deviation Measures in Risk Analysis and Optimization*, SSRN Electronic Journal (2002), 10.2139/ssrn.365640.

[14] R. T. Rockafellar, S. Uryasev, and M. Zabarankin, *Optimality Conditions in Portfolio Analysis with Generalized Deviation Measures*, Math. Program., 108 (2006), pp. 515-540.

[15] R. T. Rockafellar and S. Uryasev, *The fundamental risk quadrangle in risk management, optimization and statistical estimation*, Surv. Oper. Res. Manag. Sci., 18 (2013), pp. 33-53.

[16] R. T. Rockafellar, O. R. Johannes, and S. I. Miranda, *Superquantile regression with applications to buffered reliability, uncertainty quantification and conditional value-at-risk*, European J. Oper. Res., 234 (2014) ,pp. 140-154.

[17] M. Rossi, *The Capital Asset Pricing Model: A Critical Literature Review*, Glob. Bus. Econ. Rev., 18 (2016), pp. 604.

[18] A. Ruszczynski, *Nonlinear Optimization*, Princeton University Press, 2011.

[19] W. F. Sharpe, *Capital Asset Prices: A Theory of Market Equilibrium under Condition of Risk*, J. Finance, 19 (1964), pp. 425-442.

[20] W. F. Sharpe, *Asset allocation: Management style and performance measurement*, J. Portf. Manag., 18 (1992), pp. 7-19.

[21] C. Testuri and S. Uryasev, *On Relation Between Expected Regret and Conditional Value at Risk*, S. Rachev (Ed.) Handbook of Computational and Numerical Methods in Finance, Birkhauser, 2004, pp. 361-373.

[22] M. Zabarankin Michael, P. Konstantin, and S. Uryasev, *Capital asset pricing model (CAPM) with drawdown measure*, European J. Oper. Res., 234 (2014), pp. 508-517.

A Drawdown Beta Website

Zabarankin et al. [22] and this paper laid down a theoretical foundation for implementation of Drawdown Beta Website [5] at a server of Quantitative Finance Program at Stony Brook University.

The website calculates the following characteristics for SP500 stocks using 10 or 15 years of end-of-the-day stock prices.

- $CDaR_{0.9}-$Beta = CDaR Beta for a stock based on the largest 10% SP500 drawdowns;
- $ERoD_{0+}-$Beta = ERoD Beta for a stock based on all positive (non-zero) SP500 drawdowns;
- Standard-Beta = Beta based co-variance of monthly returns of stock and SP500;
- Max-Drawdown (%) = Maximum drawdown of a stock;
- Annual Return (%) = Effective annual return of stock;
- U-ratio = Ratio of average daily return to average daily drawdown:

$$\frac{(Average\ Daily\ Return)}{(Average\ Drawdown)/(Average\ Length\ of\ Drawdown)}\ .$$

Ticker	$ERoD_{0+}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta	Max-Drawdown (%)	Annual Return (%)	U-ratio
AAPL	0.7669	0.7592	1.1068	43.80	29.21	0.9355
AMGN	0.0955	0.2561	0.9059	24.77	17.64	0.5061
AXP	1.1291	1.2351	1.1336	49.64	12.30	0.6999
BA	1.1794	2.0575	1.3689	77.92	14.87	0.5985
CAT	2.0418	1.6975	1.1711	44.63	9.88	0.6097
CRM	0.4962	0.7463	1.5652	38.81	20.68	0.4809
CSCO	0.7863	0.6039	1.0693	41.95	11.20	0.5052
CVX	1.1199	1.0585	1.1160	55.77	3.15	0.2157
DIS	0.9725	1.2453	0.8036	43.11	18.08	1.1706
GS	2.3317	2.0003	1.1209	49.39	5.90	0.3112
HD	0.0861	0.4961	0.8524	37.99	25.40	0.7290
HON	1.0428	1.2714	1.1826	43.01	17.89	0.5559
IBM	1.4004	1.0346	0.8076	43.72	1.68	0.3187
INTC	0.4317	0.4320	1.0261	34.53	12.29	0.5629
JNJ	0.3338	0.2470	0.7957	27.37	12.78	0.6608
JPM	1.5801	1.6466	1.3378	43.63	14.19	0.5535
KO	0.1702	0.3111	0.6277	36.99	8.88	0.4679
MCD	-0.5986	-0.3984	0.6775	36.90	14.45	0.7634
MMM	1.1409	0.9200	0.7453	51.20	10.19	0.5330
MRK	0.1683	0.3791	0.4505	27.26	12.16	0.5395
MSFT	-0.0186	0.1526	1.2415	28.04	25.82	0.8398
NKE	-0.4211	0.0260	1.2954	39.78	22.49	0.9424
PG	0.1488	0.1259	0.4062	25.46	11.29	0.6172
TRV	0.5457	0.9233	0.6036	46.28	12.42	0.5190
UNH	0.1165	0.3428	0.8771	35.90	26.99	0.7290
V	-0.4126	0.0698	1.3116	36.36	29.52	0.7325
VZ	0.0673	0.0733	0.4561	20.11	9.58	0.5198
WBA	1.0123	0.7880	0.6118	60.42	2.55	0.2194
WMT	0.5993	0.2161	0.3281	36.44	12.85	0.8169

Figure 4: DOW30 Stocks: performance characteristics posted at Drawdown Beta Website [5].

As an example, Figure 4 shows the table for Dow Jones stocks using a 10-year historic period from 2011-01-01 to 2021-01-01. DOW stock is not included in the table because it does not have 10 years history in the database.

B Web Links to Case Sudies Related to Drawdown Risk Measures

This appendix provides links to case studies that are related to the drawdown risk measure and its applications:

- Portfolio Optimization with Drawdown Constraints:

  http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-on-a-single-path/

  http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-on-multiple-paths/

  http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-single-path-vs-multiple-paths/

- CoCDaR-Approach Systemic Risk Contribution Measurement:

  http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-cocdar-approach-systemic-risk-contribution-measurement/

- Style Classification with mCoCDaR Regression:

  http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-style-classification-with-mcocdar-regression/