Drawdown Beta and Portfolio Optimization

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Abstract

This paper introduces a new dynamic portfolio performance risk measure called Expected Regret of Drawdown (ERoD) which is an average of the drawdowns exceeding a specified threshold (e.g., 20%). ERoD is similar to Conditional Drawdown-at-Risk (CDaR) which is the average of some percentage of the largest drawdowns. CDaR and ERoD portfolio optimization problems are equivalent and result in the same set of optimal portfolios. Necessary optimality conditions for ERoD portfolio optimization lead to Capital Asset Pricing Model (CAPM) equations. ERoD Beta, similar to the Standard Beta, relates returns of the securities and those of a market. ERoD Beta is equal to [average losses of a security over time intervals when market is in drawdown exceeding the threshold] divided by [average losses of the market in drawdowns exceeding the threshold]. Therefore, a negative ERoD Beta identifies a security which has positive returns when the market has drawdowns exceeding the threshold. ERoD Beta accounts only for time intervals when the market is in drawdown and conceptually differs from Standard Beta which does not distinguish up and down movements of the market. Moreover, ERoD Beta provides quite different results compared to the Downside Beta based on Lower Semi-deviation. ERoD Beta is conceptually close to CDaR Beta which is based on a percentage of worst case market drawdowns. However, ERoD Beta has some advantage compared to CDaR Beta because the magnitude of the drawdowns is known (e.g., exceeding a 20% threshold), while CDaR Beta is based on a percentage of the largest drawdowns with unknown magnitude. We have built a website reporting CDaR and ERoD Betas for stocks and the SP 500 index as an optimal market portfolio. The case study showed that CDaR and ERoD Betas exhibit persistence over time and can be used in risk management and portfolio construction.

1 Introduction: Drawdown Betas

The Capital Asset Pricing Model (CAPM) (Sharpe [19], Sharpe [20]) is a fundamental model in portfolio theory and risk management. It is based on a Markowitz mean-variance portfolio optimization problem (Markowitz [10]). Tremendous literature is available on CAPM, see for instance, critical review papers Galagedera [6], and Rossi [17].

The Standard Beta relates expected return of a security and expected excess return of a market. Beta has been used as a key indicator of asset performance in portfolio management. The variance risk measure used in the standard CAPM has a conceptual drawback: it does not distinguish losses and gains of a portfolio. Markowitz [9] considered Semi-Variance based only on negative returns. The associated beta was called Downside Beta. Although, the idea sounds conceptually attractive, Downside Beta and Standard Beta have close values. Therefore, Downside Beta provides little information in addition to Standard Beta.

Various non-symmetric risk measures have been proposed as an alternative to variance. In particular, Conditional Value-at-Risk (CVaR) introduced by Rockafellar and Uryasev [11] for continuous distributions is the conditional expected loss exceeding Value-at-Risk (VaR), and generalized to discrete distributions in Rockafellar and Uryasev [12].

CAPM has been extended to non-symmetric risk measures such as Generalized Deviations by Rockafellar et al. [14]. This paper demonstrated that CAPM equations are necessary optimality conditions for portfolio optimization problems. In particular, Beta was computed for CVaR and Lower Semi-Deviation (the square root of Semi-Variance). The review paper by Krokhmal et al. [8] discusses these and other non-symmetric risk measures and provides formulas for Betas.

A considerable drawback of Variance, CVaR, Semi-Deviation and many other risk measures is that they are static characteristics, which do not account for persistent consecutive portfolio losses (may be resulting in a large cumulative loss). The dynamic drawdown risk measure is actively used in portfolio management as an alternative to static measures. Portfolio managers try to build portfolios with low drawdowns. The most popular drawdown characteristic is the Maximum Drawdown. However, the Maximum Drawdown is not the best risk measure from a practical perspective: it accounts for only one specific event on a price sample-path. For instance, Goldberg and Mahmoud [7] suggested the so-called Conditional Expected Drawdown (CED), which is the tail mean of the maximum drawdown distribution. Let us consider the market historical sample-path for the most recent 15 years. There were two major drawdowns of SP500 in the most recent 15 years: 1) the 2008 Financial Crisis; 2) the COVID-19 Crisis. CED will notice the 2008 Financial Crisis (which is not very relevant at this time) and completely ignore the COVID-19 Crisis, which is the most important risk event in recent years.

Chekhlov et al. [2] proposed Conditional Drawdown-at-Risk (CDaR) which averages a specified percentage of the largest portfolio drawdowns over an investment horizon. CDaR is defined as CVaR of the drawdown observations of the portfolio cumulative returns. CDaR possesses the theoretical properties of a deviation measure, see, Chekhlov et al. [3].

CDaR has been used to identify systemic dependencies in the financial market. Ding and Uryasev [4] considered CDaR regression for measurement of systemic risk contributions of financial institutions and for funds style classification.

Zabarankin et al. [22] developed CAPM relationships based on CDaR. The paper derived necessary optimality conditions for CDaR portfolio optimization. These conditions resulted in CDaR Beta relating cumulative returns of a market (optimal portfolio) and individual securities. CDaR Beta equals:

$$\beta^i_{CDaR} = \frac{\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{CDaR_\alpha(w^M)} \ ,$$

where

- i = index of a security, $i = 1, ..., I$;
- s = index of a sample path of returns of securities, $s = 1, ..., S$;
- p_s = probability of a sample path s;
- t = time, $t = 1, ..., T$;
- w^i_{st} = uncompounded cumulative return of asset i at time moment t on sample path s;
- w^M = vector of uncompounded cumulative returns of the market portfolio (optimal portfolio) including components w^M_{st}, $t = 1, ..., T$, $s = 1, ..., S$;
- $\tau(s, t)$ = time moment of the most recent maximum of market cumulative return preceding t on scenario s;
- q^*_{st} = indicator which is equal to $\frac{1}{(1-\alpha)T}$ for the largest $(1-\alpha)T$ drawdowns of market portfolio w^M and zero otherwise;
- $CDaR_\alpha(w^M) = \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^M_{s,\tau(s,t)} - w^M_{st})$ = average of the largest $(1-\alpha)\%$ drawdowns of market portfolio w^M (e.g., if $\alpha = 0.9$, then CDaR accounts for 10% largest drawdowns).

This paper introduces a new drawdown based risk measure called Expected Regret of Drawdown (ERoD). By definition, ERoD is an average of drawdowns exceeding a threshold ϵ. The Expected Regret (also termed Low Partial Moment) is defined as the average of losses exceeding a fixed threshold. Therefore, ERoD is the Expected Regret of drawdown observations over the considered period. Testuri and Uryasev [21] established the equivalence between Expected Regret and CVaR risk measures. This equivalence also follows from the Quantile Quadrangle, see, Rockafellar and Uryasev [15]; CVaR is the Risk and Partial Moment is the Regret in the Quantile Quadrangle. We build on this equivalence result and demonstrate the equivalence of CDaR and ERoD portfolio optimization. Similar to CDaR optimization, the ERoD optimization can be reduced to convex and linear programming. Also, necessary conditions of extremum for ERoD optimization can be formulated similar to the necessary conditions for CDaR optimization. Therefore, a formula for ERoD Beta can be derived similar to CDaR Beta. Moreover, CDaR Beta and ERoD Beta coincide for some confidence level α in CDaR and some threshold ϵ in ERoD.

We show (see, Theorem 1) that the ERoD Beta equals:

$$\hat{\beta}^i_{ERoD} = \frac{\frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st}(w^i_{s,\tau(s,t)} - w^i_{st})}{\tilde{E}_\epsilon(w^M)} \ .$$

ERoD Beta indicates good hedges against market drawdowns. Instruments with low and negative ERoD Beta are quite beneficial in portfolio construction. ERoD Beta has a conceptual advantage compared to CDaR Beta. ERoD beta is based on drawdowns with a known threshold. Therefore, we can control the magnitude of evaluated drawdowns. While CDaR Beta is based on some percentage of largest drawdowns with unknown magnitude.

We have done the following calculations for stock data (with at least 15 years history) and the SP500 index. We calculated $ERoD_{0+}$ Beta accounting for the positive drawdowns of the SP500 index. Also, we calculated

$CDaR_{0.9}$ Beta accounting for the largest 10% of drawdowns of the SP500. The resulting ERoD Beta, CDaR Beta, Standard Beta, and other metrics are posted at the Drawdown Beta Website [5]. Appendix A contains a quick description of this website. We have also compared different thresholds for ERoD and CDaR Betas.

To evaluate the impact of the 2008 financial crisis and stability of the considered betas, we compared ERoD, CDaR, and Standard Betas in different historic periods. Section 6 reports correlation and rank correlation coefficients across time of the considered betas in the Dow30, SP100, and SP500 indices. We observed that the ERoD and CDaR Betas are more sensitive to drawdowns in historical data than the Standard Beta. Betas are more stable for larger stocks.

Appendix B reports web links to case studies related to drawdown measure: 1) Portfolio Optimization with Drawdown Constraints; 2) CoCDaR-Approach Systemic Risk Contribution Measurement; 3) Style Classification with mCoCDaR Regression.

2 Conditional Drawdown-at-Risk

We call a set of consecutive vectors of returns of instruments a sample-path. A sample path may be just a table of historical returns of instruments or joint returns simulated with some model. Suppose that $\{r_t\}_{1\le t\le T}$ is a sample path of scalar returns of some instrument. Let us denote:

$\{w_t\}_{1\le t\le T}$ = vector of uncompounded cumulative returns,

$$w_t = \sum_{\nu=1}^{t} r_\nu \ , \quad 1 \le t \le T \ . \tag{1}$$

$\{d_t\}_{1\le t\le T}$ = vector of drawdowns,

$$d_t = \max_{1\le\nu\le t} \{w_\nu\} - w_t \ , \quad 1 \le t \le T \ . \tag{2}$$

In simple terms, for every time moment t the drawdown $\{d_t\}$ is the difference between the previous peak and the current cumulative return.

Zabarankin et al. [22] consider a slightly more complicated definition of drawdown with a τ-window where all indices in expressions (1),(2) start from $t_k = \max\{t-k, 1\}$. Only the most recent time window with length k is taken into account for calculation of the drawdown. However this modification does not change any formulas and conclusions, therefore for simplicity we use drawdown definition (2).

Conditional Value-at-Risk (CVaR) for a random value X with confidence level α can be defined as follows

$$CVaR_\alpha(X) = \min_C \ \{C + \frac{1}{1-\alpha}\mathbb{E}[(X-C)^+]\} \ ,$$

where $X^+ = \max\{0, X\}$, see Rockafellar and Uryasev [11], Rockafellar and Uryasev [12]. CVaR is the expectation of the α-tail distribution of the random variable X, i.e., it is the average of the largest outcomes with total probability $1-\alpha$.

The Conditional Drawdown-at-Risk (CDaR) for portfolio returns is defined as CVaR of the drawdown observations of the portfolio, see Chekhlov et al. [2], Chekhlov et al. [3]. For a given $\alpha \in [0, 1)$ and time horizon T such that αT is an integer, the α-CDaR is an average over the worst $(1-\alpha) * 100\%$ drawdowns occurred in the time horizon. Accordingly, we define the single sample-path $CDaR_\alpha$ as:

$$CDaR_\alpha(w) = \sum_{t=1}^{T} q_t^\star d_t \ , \tag{3}$$

where $q_t^\star = \frac{1}{(1-\alpha)T}$ if d_t is one of the $(1-\alpha)T$ largest portfolio drawdowns, and $q_t^\star = 0$ otherwise. This CDaR formula is defined for equally probable observations of drawdowns.

Suppose now that we have S sample-paths of scalar returns $\{r_{st}\}_{1\le t\le T}$ of some instrument and d_{st} is the drawdown at time t on sample-path s. Probability of a sample-path s is denoted by p_s, $s = 1, 2, \ldots, S$. Zabarankin et al. [22] defined $CDaR_\alpha$,

$$CDaR_\alpha(w) = \max_{\{q_{st}\}\in Q} \ \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q_{st} d_{st} \tag{4}$$

with risk envelope Q,

$$Q = \left\{ \{q_{st}\} \;\middle|\; \sum_{s=1}^{S}\sum_{t=1}^{T} p_s q_{st} = 1,\ 0 \le q_{st} \le \frac{1}{(1-\alpha)T} \right\} \ .$$

The CDaR definition (4) exploits the dual representation of risk through the risk envelope theory, see, for instance, Rockafellar and Uryasev [15]. A coherent risk functional $R(X)$ can be expressed as follows:

$$R(X) = sup_{q \in Q}\, \mathbb{E}[Xq] \ ,$$

where q is a probability measure from a dual risk envelope Q. The CVaR risk envelope is defined by

$$Q_{CVaR}(\alpha) = \{q : q \in [0, \frac{1}{1-\alpha}],\, \mathbb{E}[q] = 1\} \ .$$

Therefore, CVaR equals

$$CVaR_\alpha(X) = sup_{q \in Q_{CVaR}(\alpha)} E[Xq] \ ,$$

which implies formula (4).

3 Relation Between Expected Regret and Conditional Value-at-Risk

Testuri and Uryasev [21] proved that for CVaR optimization a regret optimization results in the same set of optimal solutions. Specifically, let $f(x, y)$ be a loss function where x is the associated decision vector and y is a random vector.

For each x, we denote the distribution function for the loss $f(x, y)$ by

$$\Psi(x, C) = \mathbb{P}\big\{y \,\big|\, f(x, y) \le C\big\} \ .$$

The α-VaR (α-quantile) of the loss associated with a decision x equals

$$C_\alpha(x) = \min\big\{C \,\big|\, \Psi(x, C) \ge \alpha\big\}.$$

The minimum in the previous equation is attained because $\Psi(x, C)$ is a nondecreasing and right-continuous function in C. We define the Expected Regret of $f(x, y)$ with respect to the threshold C as:

$$G_C(x) = \mathbb{E}[f(x, y) - C]^+ \ .$$

With notation

$$F_\alpha(x, C) = C + \frac{1}{1-\alpha} G_C(x) \ ,$$

CVaR of $f(x, y)$ equals:

$$CVaR_\alpha(x) = \min_C \quad F_\alpha(x, C) \ . \qquad (5)$$

The following facts were proved in [11, 12]. As a function of $C \in I\!R$, $F_\alpha(x, C)$ is finite and convex (hence continuous), with

$$\begin{aligned} C_\alpha(x) \ &= \text{lower endpoint of } \operatorname{argmin}_C F_\alpha(x, C), \\ C_\alpha^+(x) \ &= \text{upper endpoint of } \operatorname{argmin}_C F_\alpha(x, C), \end{aligned}$$

where the argmin refers to the set of C for which the minimum is attained and in this case is a nonempty, closed, bounded interval (perhaps reducing to a single point). In particular, one has

$$C_\alpha(x) \in \operatorname{argmin}_C F_\alpha(x, C), \qquad CVaR_\alpha(x) = F_\alpha(x, C_\alpha(x)) \ .$$

Also, (5) implies

$$\min_{x \in U} \quad CVaR_\alpha(x) = \min_{x \in U, C \in R} \quad F_\alpha(x, C) \ , \qquad (6)$$

where U is a feasible set for the vector x. For example, U could be a linear constraint on the expected return of a portfolio (see, Section 5). Denote:

- $V_\alpha = Argmin_{x \in U, C \in R}\, F_\alpha(x, C)$ = solution set of the right hand side minimization problem (6);
- $U_\alpha^{CVaR} = Argmin_{x \in U}\, CVaR_\alpha(x)$ = solution set of the left hand side minimization problem (6);
- $U_C^{Regret} = Argmin_{x \in U}\, G_C(x)$ = solution set of the minimum regret problem;
- $A_\alpha(x)$ = projection of V_α on the C line, i.e., $A_\alpha = \{C$: there exists x such that $(x, C) \in V_\alpha\}$.

Under the condition that the function $G_C(x)$ is continuously differentiable, Testuri and Uryasev [21] proved:

Statement 1. For any $\alpha \in (0, 1)$ and $x^\star \in U_\alpha^{CVaR}$, there exists a pair $(x^\star, C^\star) \in V_\alpha$ such that $x^\star \in U_{C^\star}^{Regret}$. In particular, $(x^\star, C_\alpha(x^\star)) \in V_\alpha$, such that $x^\star \in U_{C_\alpha(x^\star)}^{Regret}$.

Statement 2. For any C and $x^\star \in U_C^{Regret}$, there exists a unique $\alpha \in (0, 1)$ such that $C \in A_\alpha(x^\star)$, $(x^\star, C) \in V_\alpha$, and $x^\star \in U_\alpha^{CVaR}$.

The above statements establish the equivalence between CVaR optimization and expected regret optimization. For the special case when $V_\alpha = U_\alpha^{CVaR} \times A_\alpha$, we have $U_\alpha^{CVaR} = U_C^{Regret}$ for any $C \in A_\alpha$.

4 CDaR and ERoD Portfolio Optimization

Let us denote by $w(x)$ the vector of cumulative returns of a portfolio with weights vector x. Also, we denote by $D(w(x))$ the random drawdown value for the portfolio x.

CDaR for portfolio x, by definition, is CVaR of of the random value $D(w(x))$ i.e.,

$$CDaR_\alpha(w(x)) \;=\; CVaR_\alpha(D(w(x)))\;.$$

Expected Regret of Drawdown (ERoD) for portfolio x with threshold ϵ, by definition, is the expected regret of the random value $D(w(x))$ i.e.,

$$ERoD_\epsilon(w(x)) \;=\; \mathbb{E}[(D(w(x)) - \epsilon)^+]\;.$$

For instance, suppose that we want to estimate the average of the positive drawdowns of a portfolio (i.e., zero drawdowns are excluded from consideration). We can select a sufficiently small threshold ϵ and evaluate ERoD of a portfolio with this threshold.

We denote

- $x = (x^1, ..., x^I) \;=\;$ vector of weights for n assets in the portfolio;
- $(w^1_{st}, \ldots, w^I_{st}) \;=\;$ vector of uncompounded cumulative returns of portfolio assets at time moment t on scenario s;
- $p_s \;=\;$ probability of the scenario (sample path of returns of securities);
- $w_{st}(x) = \sum_{i=1}^{I} w^i_{st} x^i \;=\;$ cumulative portfolio return at time moment t on scenario s;
- $w(x) =$ vector of cumulative portfolio returns with components $w_{st}(x)\,,\ s = 1,\ldots,S;\ t = 1,\ldots,T\,;$
- $d_{st}(x) = \max\limits_{1\le\nu\le t}\{w_{s\nu}(x)\} - w_{st}(x) \;=\;$ drawdown of portfolio at time t on sample path s.

ERoD for a portfolio with threshold ϵ is calculated as follows:

$$ERoD_\epsilon(w(x)) \;=\; \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s (d_{st}(x) - \epsilon)^+\;.$$

Following Zabarankin et al. [22], we state CDaR multiple paths minimization over T periods subject to a constraint on the portfolio expected cumulative return at time T:

$$\min_x \quad CDaR_\alpha(w(x)) \quad s.t.\ \sum_{s=1}^{S} p_s w_{sT}(x) \ge \delta\;. \tag{7}$$

This problem is similar to Markowitz mean-variance optimization with variance replaced by the α-CDaR. However, an important difference, is that it is a so-called Static-Dynamic problem over T periods. The problem is dynamic because there are T time periods; however, it is static in the sense that portfolio x is fixed at the initial time moment $t = 1$ and it is not changed over time. The considered investment strategy is similar to a popular Constant Proportions Strategy.

The above minimization problem (7) is equivalent to the maximization problem below:

$$\max_x \quad \sum_{s=1}^{S} p_s w_{sT}(x) \quad s.t.\quad CDaR_\alpha(w(x)) \le v\;, \tag{8}$$

in the sense that the efficient frontiers of these two problems (7) and (8) coincide.

Further we formulate the ERoD portfolio optimization problem, similar to the CDaR portfolio optimization problem (7):

$$\min_x \quad ERoD_\epsilon(w(x)) \quad s.t.\quad \sum_{s=1}^{S} p_s w_{sT}(x) \ge\ \delta\;. \tag{9}$$

Statement 1 in the previous section 3 implies that for every confidence level α an optimal solution x^* of CDaR minimization problem (7) can be obtained by solving ERoD minimization problem (9) with $\epsilon = C_\alpha(D(w(x^*)))$. Also, Statement 2 in the section 3 implies that for every ϵ an optimal solution x^* of ERoD minimization problem (9) can be obtained by solving the CDaR minimization problem (7) with some confidence level α.

ERoD portfolio minimization problem (9) can be solved very efficiently via convex and linear programming.

5 CAPM: Necessary Optimality Conditions for CDaR and ERoD Portfolio Optimization

Zabarankin et al. [22] provided necessary optimality conditions for optimization problems (7) and (8) in the form of CAPM equations. In particular, the formula for CDaR Beta was derived similar to the Standard Beta, which relates return of the market and individual assets. Baghdadabad et al. [1] present an attempt to formulate a drawdown-based beta in the CAPM setting, but their derivation does not have a rigorous mathematical justification based on portfolio optimization. This paper evaluates the correlation of drawdowns in a statistical setting.

Further, we follow Zabarankin et al. [22] and present the Beta for the CDaR risk measure. Let $w^M = w(x^\star)$ be the vector of cumulative returns of the optimal portfolio of problem (7) or (8). The necessary optimality conditions for the solution $x^\star$ of both problems (7) and (8) are stated in the form of CAPM:

$$\sum_{s=1}^{S} p_s w^i_{sT} = \beta^i_{CDaR} \sum_{s=1}^{S} p_s w^M_{sT} \ , \tag{10}$$

$$\beta^i_{CDaR} = \frac{\sum_{s=1}^{S} \sum_{t=1}^{T} p_s q^\star_{st} (w^i_{s,\tau(s,t)} - w^i_{st})}{CDaR_\alpha(w^M)} \ , \tag{11}$$

where the time index of the most recent historic maximum in the cumulative returns is defined as

$$\tau(s,t) = \max\{\, k \,|\, 1 \le k \le t,\ w^M_{sk} = \max_{1\le \ell \le t} w^M_{s\ell} \,\}. \tag{12}$$

Since there can be multiple historic peaks, we take the most recent one for drawdown calculation. The CDaR Beta equation (10) relates expected cumulative returns of the market and instruments:

- $\sum_{s=1}^{S} p_s w^M_{sT}$ = cumulative expected return of the market;
- $\sum_{s=1}^{S} p_s w^i_{sT}$ = cumulative expected return of the security i.

On the efficient frontier with the CDaR risk measure against the target return, the optimal solution x^* is the point where the capital asset line makes a tangent cut with the efficient frontier.

According to Statement 1 in section 3, if $x^\star$ is an optimal solution of (7) then $x^\star$ is an optimal solution of (9) with $\epsilon = VaR_\alpha(D(w(x^\star)))$. Moreover,

$$CDaR_\alpha(w(x^\star)) = CVaR_\alpha(D(w(x^\star))) = \epsilon + \frac{1}{1-\alpha}\mathbb{E}[D(w(x^\star) - \epsilon)^+] = \epsilon + \frac{1}{1-\alpha} ERoD_\epsilon(w(x^\star)) \ . \tag{13}$$

Therefore, CAPM optimality conditions (10), (11) for CDaR optimization (7) and (8) are also the optimality conditions for the ERoD optimization (9).

CAPM optimality conditions (10), (11) were developed for discrete distributions of drawdowns by Zabarankin et al. [22]. However, the equivalence Statements 1,2 in section 3 are formulated for continuous distributions (see, Testuri and Uryasev [21]). Therefore, further we rigorously prove the CAPM optimality conditions for ERoD portfolio optimization problem (9) for discrete distributions.

Theorem 1. *Let $w^M = w(x^\star)$ be the cumulative return vector for an optimal portfolio $x^\star$ of problem (9). The necessary optimality conditions for (9) can be stated in the form of CAPM:*

$$\sum_{s=1}^{S} p_s w^i_{sT} = \beta^i_{ERoD} \sum_{s=1}^{S} p_s w^M_{sT} \ , \tag{14}$$

$$\hat{\beta}^i_{ERoD} = \frac{\frac{1}{T}\sum_{s=1}^{S} \sum_{t=1}^{T} p_s q^\star_{st} (w^i_{s,\tau(s,t)} - w^i_{st})}{\tilde{E}_\epsilon(w^M)} \ , \tag{15}$$

where

- $\tilde{E}_\epsilon(w^M) = \frac{1}{T}\sum_{s=1}^{S} \sum_{t=1}^{T} p_s q^\star_{st} (w^M_{s,\tau(s,t)} - w^M_{st})$ = *ERoD with threshold ϵ for return w^M ;*
- β^i_{ERoD} = *ERoD Beta relating the total expected cumulative return, $\sum_{s=1}^{S} p_s w^M_{sT}$, of the optimal portfolio (market) and total expected cumulative return, $\sum_{s=1}^{S} p_s w^i_{sT}$, of security i;*
- $\tau(s,t)$ = *index calculated with (12);*
- $d^M_{st} = w^M_{s,\tau(s,t)} - w^M_{st}$ = *drawdowns of the optimal portfolio;*
- $q^\star_{st} = \mathbb{1}(d^M_{st} \ge \epsilon)$ = *indicator function which is equal to 1 for $d^M_{st} \ge \epsilon$, and 0 otherwise.*

Consequently, for a single path:

$$w^i_T \;=\; \beta^i_{ERoD}\, w^M_T \ ,$$

$$\hat{\beta}^i_{ERoD} \;=\; \frac{\frac{1}{T}\sum_{t=1}^{T} q^\star_t (w^i_{\tau(t)} - w^i_t)}{\tilde{E}_\epsilon(w^M)} \ ,$$

where

- $\tilde{E}_\epsilon(w^M) = \frac{1}{T}\sum_{t=1}^{T} q^\star_t (w^M_{\tau(t)} - w^M_t) \;=\;$ *ERoD with threshold* ϵ *for return* w^M *;*
- $q^\star_t = \mathbb{1}(d^M_t \geq \epsilon) \;=\;$ *indicator for drawdowns* $d^M_t = w^M_{\tau(t)} - w^M_t$ *;*
- $\tau(t) = \max\{\, k \,|\, 1 \leq k \leq t,\ w^M_k = \max_{1\leq \ell \leq t} w^M_\ell \}$.

Proof. Let us denote by $H_\epsilon(w(x)) = \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s (d_{st}(x) - \epsilon)^+$ the objective function and by $x^\star$ an optimal portfolio vector of problem (9). The cumulative return function $w_{sk}(x)$ linearly depends upon vector x. The objective function $H_\epsilon(w(x))$ is convex in x and the cumulative return function in (9) is linear in x. The necessary optimality condition for the convex optimization problem (9) is formulated as follows (see, for reference, Theorem 3.34 in Ruszczynski [18]):

$$\lambda^\star\, \nabla_x (\sum_{s=1}^{S} p_s w_{sT}(x^\star) - \delta) \;\in\; \partial_x H_\epsilon(w(x^\star)) \ , \tag{16}$$

where

- $\nabla_x (\sum_{s=1}^{S} p_s w_{sT}(x^\star) - \delta) \;=\;$ gradient of the constraint function at $x = x^\star$;
- $\lambda^\star \;=\;$ Lagrange multiplier such that $\lambda^\star \geq 0$ and $\lambda^\star(\delta - \sum_{s=1}^{S} p_s w_{sT}(x^\star)) = 0$;
- $\partial_x H_\epsilon(w(x^\star)) \;=\;$ subdifferential of convex in x function $H_\epsilon(w(x))$ at $x = x^\star$.

The gradient of the constraint function, which is liner in x, equals:

$$\nabla_x (\sum_{s=1}^{S} p_s w_{sT}(x^\star) - \delta) = \sum_{s=1}^{S} p_s \nabla_x w_{sT}(x) = \sum_{s=1}^{S} p_s (w^1_{sT}, ..., w^n_{sT}) = (\sum_{s=1}^{S} p_s w^1_{sT} \ , ..., \ \sum_{s=1}^{S} p_s w^n_{sT}) \ . \tag{17}$$

Objective function $H_\epsilon(w(x))$ is a linear combination of a maximum of linear functions in x. Therefore, according to the standard results in convex analysis,

$$g = (g^1, ..., g^n) \in \partial_x H_\epsilon(w(x^\star)) \ , \quad \text{where} \quad g^i = \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st} (w^i_{s,\tau(s,t)} - w^i_{st}) \ . \tag{18}$$

With (17) and (18) we obtain the following system of equations

$$g^i = \frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st} (w^i_{s,\tau(s,t)} - w^i_{st}) = \lambda \sum_{s=1}^{S} p_s w^i_{sT} \ , \quad i = 1, \dots, I \ . \tag{19}$$

Multiplying the left and right hand sides of the previous equation by x^i and summing up terms for $i = 1, \dots, I$, we have:

$$\frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st} (w_{s,\tau(s,t)}(x) - w_{st}(x)) = \lambda \sum_{s=1}^{S} p_s w_{sT}(x) \ .$$

Consequently,

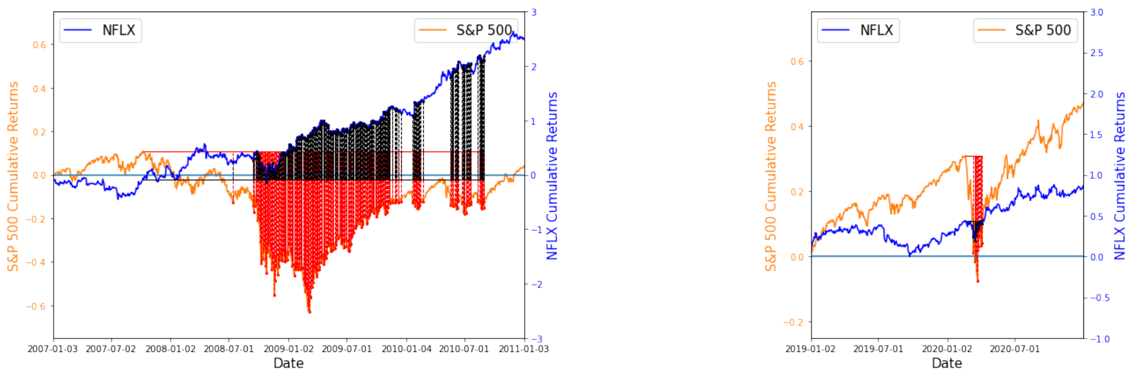
$$\lambda = \frac{\frac{1}{T}\sum_{s=1}^{S}\sum_{t=1}^{T} p_s q^\star_{st} (w_{s,\tau(s,t)}(x) - w_{st}(x))}{\sum_{s=1}^{S} p_s w_{sT}(x)} \ . \tag{20}$$

Substituting (20) to (19) gives necessary conditions (14), (15) in CAPM format. □

6 Discussion of CDaR and ERoD Betas

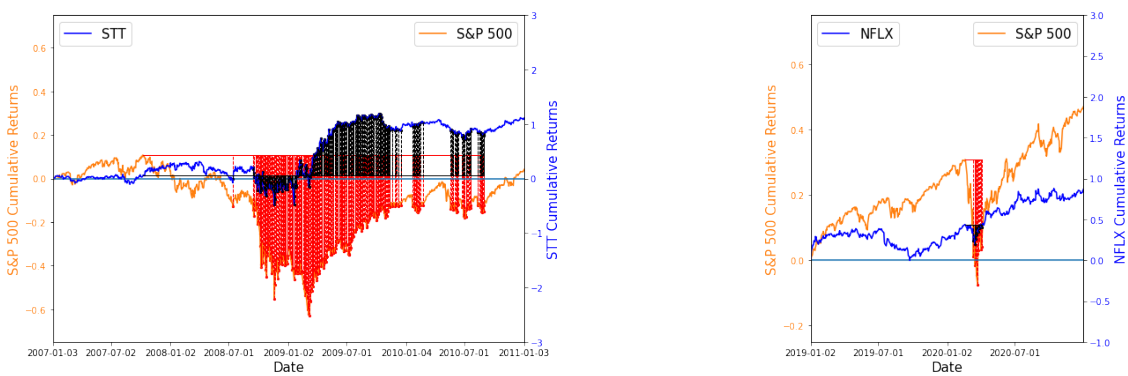
Similar to the standard CAPM, an optimal portfolio of problems (7), (9) is assumed to be a market portfolio. Therefore, CDaR and ERoD Betas can be computed with historical data of the market portfolio and individual assets. These drawdown betas show how securities behaved during previous market drawdowns. We developed the Drawdown Beta Website [5] for SP500 stocks, where the SP500 index is considered as an optimal market portfolio. Drawdown Betas are calculated with daily data. Data are updated every 3 months starting from July 2020, so it is possible to observe how stable Drawdown Betas are over time. In particular, $ERoD_{0+}$ is based on all daily non-zero drawdowns of SP500 over 10 or 15 years. The threshold 0+ denotes some small value (we set $\epsilon = 10^{-6}$). Also, the drawdown website shows $CDaR_{0.9}$ Beta based on the 10% largest drawdowns. Take into account, that the magnitude of drawdowns is not known in $CDaR_{0.9}$, we only know that it is based on largest 10% of drawdowns. We want to mention that $CDaR_0$ with confidence level $\alpha = 0$ is not equal to $ERoD_{0+}$ because $CDaR_0$ accounts for all drawdowns, including zero drawdowns. In principle, we can count the percentage of zero drawdowns for some dataset and select confidence level α such that only nonzero drawdowns are present in $CDaR_\alpha$. However, for another dataset, the percentage of nonzero drawdowns is different, and the α corresponding to $ERoD_{0+}$ depends upon the dataset. ERoD Beta allows for a direct control of the magnitude of the considered drawdowns by setting up a lower bound for these drawdowns.

For instance, let us consider Netflix (ticker NFLX) in the 15-year period from January 2006 to January 2021. The drawdown website shows that NFLX has a large negative $ERoD_{0+}$ Beta of -3.532 and $CDaR_{0.9}$ Beta of -2.388. However, NFLX has a positive Standard Beta value of 0.862 based on monthly returns; so the Standard Beta does not show that Netflix performed very well when SP500 was in drawdown. The distinct perspective offered by the ERoD and CDaR Betas, can be understood by looking at the largest drawdowns of the SP500 index and comparing with the cumulative returns of NFLX in the same time periods, see Figure 1. The figure shows the top 10% largest drawdowns of SP500 and corresponding cumulative returns of NFLX. SP500 had large drawdowns in 2008-2011 during the 2008 financial crisis. At that time, NFLX had strong positive returns. Also, in 2020 during the COVID crisis NFLX had much smaller drawdowns than SP500.



(a) Financial crisis 2008 (b) COVID-19 crisis

Figure 1: NFLX returns vs largest 10% SP500 drawdowns (blue curve = NFLX cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = NFLX cumulative returns)

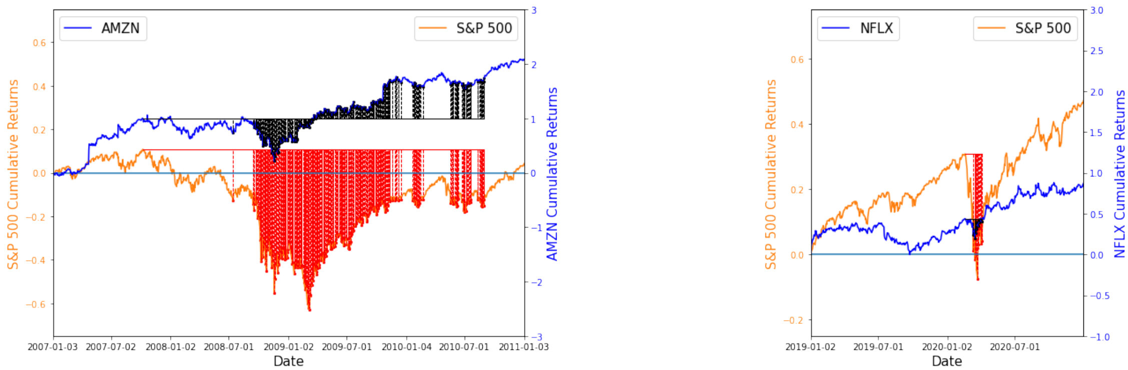


(a) Financial crisis 2008 (b) COVID-19 crisis

Figure 2: STT returns vs largest 10% SP500 drawdowns (blue curve = STT cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = STT cumulative returns)

Figures 2 and 3 show similar graphs for State Street Corporation (STT) and Amazon (AMZN), respectively. In a 15-year period, STT reported $ERoD_{0+}$ Beta of -1.15, $CDaR_{0.9}$ Beta of -1.264, and Standard Beta of 1.404. Likewise, AMZN reported $ERoD_{0+}$ Beta of -1.171, $CDaR_{0.9}$ Beta of -0.255, and Standard Beta of 1.331. We want to emphasize that $CDaR_{0.9}$ and $ERoD_{0+}$ Betas pick up different drawdown characteristics. $CDaR_{0.9}$ is concentrated on unique market conditions, such as financial crisis 2008 and COVID crisis 2020. These unique events most probably will not repeat in future. However, $ERoD_{0+}$ takes into account all nonzero drawdowns of SP500. Therefore, the suggested ERoD Beta is an important alternative to CDaR Beta.

Below we compare values of betas across different time periods. The first group of numerical experiments



(a) Financial crisis 2008

(b) COVID-19 crisis

Figure 3: AMZN returns vs largest 10% SP500 drawdowns (blue curve = AMZN cumulative return, orange curve = SP500 cumulative return, red vertical lines = SP500 drawdowns, black vertical lines = AMZN cumulative returns)

identifies the impact of excluding from the dataset the 2008 financial crisis. We have considered overlapping 15 and 10-year periods:

- Period 1: from 2006/01/01 to 2021/01/01;
- Period 2: from 2011/01/01 to 2021/01/01.

The 15-year dataset (Period 1) includes both the 2008 financial crisis and the COVID-19 crisis. The 10-year dataset (Period 2) includes only the COVID-19 crisis.

Table 1 provides betas based on Periods 1 and 2 for DOW30 stocks. This table compares $ERoD_{0+}$ Beta, $ERoD_{0.1}$ Beta, $CDaR_{0.5}$ Beta, $CDaR_{0.9}$ Beta, Standard (based on monthly returns) Beta and Downside (accounting for only negative monthly returns) Beta. Fist, we want to mention that Standard and Downside Betas have similar values. Therefore, the Downside Beta brings little new information compared to the Standard Beta. The Standard and Downside Betas are quite stable across the two time periods. Standard and Downside Betas are not sensitive to excluding/including from the dataset the 2008 Financial Crisis, i.e., these Betas are not tuned to pick up drawdown risk. However, for some stocks, the sign of ERoD and CDaR Betas changes from Period 1 to Period 2. Therefore, the Drawdown Betas "notice" the main risk event (2008 Crisis) in the historical data.

To evaluate the impact of the 2008 Crisis on different betas, we calculated the correlation coefficients between betas for Periods 1 and 2. Table 2 shows correlation coefficients for betas for stocks in DOW30, SP100, and SP500 indices. The first row (DOW30) in Table 2 presents correlation coefficients for columns 1 and 2, columns 3 and 4, columns 5 and 6, columns 7 and 8, and columns 9 and 10 of Table 1. The second and third rows of Table 2 present correlation coefficients for SP100 and SP500, respectively.

Table 3 reports the Spearman rank correlation coefficients for betas between Periods 1 and 2 for DOW30, SP100, and SP500 indices, similar to Table 2.

Table 2: Correlation coefficients for ERoD, CDaR and Standard Betas between Periods 1 and 2

	$ERoD_{0+}$-Beta	$ERoD_{0.1}$-Beta	$CDaR_{0.5}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta
DOW30	0.458	0.369	0.414	0.392	0.786
S&P 100	0.476	0.379	0.436	0.396	0.751
S&P 500	0.357	0.312	0.368	0.241	0.631

Table 3: Spearman rank correlation coefficients of ERoD, CDaR and Standard Betas between Periods 1 and 2

	$ERoD_{0+}$-Beta	$ERoD_{0.1}$-Beta	$CDaR_{0.5}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta
DOW30	0.476	0.310	0.416	0.384	0.807
S&P 100	0.470	0.367	0.439	0.354	0.796
S&P 500	0.387	0.339	0.403	0.258	0.606

Table 1: Betas for DOW30 Stocks: 15-year Period 1 vs 10-year Period 2

	$ERoD_{0+}$ Period 1	$ERoD_{0+}$ Period 2	$ERoD_{0.1}$ Period 1	$ERoD_{0.1}$ Period 2	$CDaR_{0.5}$ Period 1	$CDaR_{0.5}$ Period 2	$CDaR_{0.9}$ Period 1	$CDaR_{0.9}$ Period 2	Standard Period 1	Standard Period 2	Downside Period 1	Downside Period 2
AAPL	-0.606	0.767	-0.575	0.629	-0.581	0.858	-0.062	0.759	0.980	1.107	1.024	1.077
AMGN	-0.094	0.096	-0.154	0.216	-0.086	0.210	-0.201	0.256	0.856	0.906	0.725	0.815
AXP	0.731	1.129	0.773	1.220	0.713	1.088	1.037	1.235	1.375	1.134	1.450	1.240
BA	1.321	1.179	1.465	2.183	1.358	1.255	1.492	2.056	1.017	1.369	1.254	1.585
CAT	0.798	2.042	0.693	1.589	0.767	1.998	1.003	1.698	1.183	1.171	1.106	1.085
CRM	-1.148	0.496	-1.038	0.796	-1.102	0.511	-0.257	0.746	1.428	1.565	1.153	1.206
CSCO	1.036	0.786	0.956	0.613	1.012	0.759	0.924	0.604	1.142	1.069	1.001	1.001
CVX	0.132	1.120	0.061	1.001	0.115	1.114	0.146	1.059	1.007	1.116	1.103	1.147
DIS	0.221	0.973	0.276	1.341	0.253	1.016	0.445	1.245	0.927	0.804	1.031	1.053
GS	0.877	2.332	0.575	1.952	0.812	2.244	0.556	2.000	1.334	1.121	1.384	1.301
HD	0.015	0.086	0.094	0.617	0.073	0.226	0.203	0.496	1.022	0.852	0.969	1.056
HON	0.803	1.043	0.872	1.334	0.834	1.094	1.002	1.271	0.970	1.183	1.050	1.083
IBM	0.022	1.400	-0.182	0.946	-0.039	1.328	0.008	1.035	0.812	0.808	0.794	0.907
INTC	0.333	0.432	0.357	0.478	0.315	0.426	0.510	0.432	0.973	1.026	1.006	1.067
JNJ	0.003	0.334	-0.034	0.220	0.006	0.379	0.076	0.247	0.577	0.796	0.571	0.633
JPM	-0.534	1.580	-0.788	1.671	-0.549	1.627	-0.678	1.647	1.317	1.338	1.447	1.276
KO	-0.122	0.170	-0.012	0.351	-0.081	0.222	0.172	0.311	0.520	0.628	0.593	0.709
MCD	-0.826	-0.599	-0.686	-0.257	-0.756	-0.490	-0.389	-0.398	0.590	0.678	0.682	0.791
MMM	0.685	1.141	0.572	0.863	0.667	1.147	0.647	0.920	0.794	0.745	0.823	0.907
MRK	0.661	0.168	0.843	0.330	0.713	0.251	0.903	0.379	0.779	0.451	0.746	0.717
MSFT	-0.074	-0.019	0.033	0.209	-0.021	0.079	0.256	0.153	1.020	1.242	0.968	1.059
NKE	-0.714	-0.421	-0.509	0.208	-0.640	-0.321	-0.167	0.026	1.015	1.295	0.956	0.982
PG	0.144	0.149	0.205	0.114	0.175	0.198	0.306	0.126	0.666	0.406	0.583	0.579
TRV	-0.320	0.546	-0.309	1.042	-0.298	0.593	-0.228	0.923	0.907	0.604	1.055	1.004
UNH	0.593	0.117	0.854	0.391	0.666	0.182	0.922	0.343	0.720	0.877	0.985	1.042
VZ	0.391	0.067	0.547	0.082	0.433	0.102	0.535	0.073	0.758	0.456	0.628	0.536
WBA	0.339	1.012	0.331	0.784	0.359	1.005	0.385	0.788	0.816	0.612	0.697	0.791
WMT	-0.570	0.599	-0.791	0.126	-0.631	0.596	-0.581	0.216	0.633	0.328	0.503	0.480

Table 4: Betas for DOW30 Stocks: Period 3 and 4

	$ERoD_{0+}$-Beta Period 3	$ERoD_{0+}$-Beta Period 4	$ERoD_{0.1}$-Beta Period 3	$ERoD_{0.1}$-Beta Period 4	$CDaR_{0.5}$-Beta Period 3	$CDaR_{0.5}$-Beta Period 4	$CDaR_{0.9}$-Beta Period 3	$CDaR_{0.9}$-Beta Period 4	Standard-Beta Period 3	Standard-Beta Period 4
AAPL	-0.986	1.126	-0.720	1.302	-0.796	1.207	0.520	1.402	0.894	1.064
AMGN	-0.218	0.468	-0.183	0.217	-0.163	0.523	-0.071	0.346	0.817	0.954
AXP	0.540	1.606	0.701	1.714	0.659	1.547	1.544	1.645	1.601	1.056
BA	1.332	1.277	1.386	2.493	1.376	1.342	1.476	2.202	0.866	1.365
CAT	0.558	1.892	0.644	1.327	0.626	1.868	1.321	1.522	1.252	1.406
CRM	-1.438	0.154	-1.187	0.889	-1.268	0.207	0.455	0.761	1.533	1.170
CSCO	1.119	0.664	0.977	0.681	1.015	0.641	1.030	0.670	1.116	1.245
CVX	-0.129	1.323	-0.021	1.121	-0.038	1.303	0.196	1.209	1.042	1.259
DIS	0.081	0.859	0.214	1.075	0.197	0.893	0.684	0.981	0.951	1.025
GS	0.640	1.956	0.490	1.670	0.564	1.896	0.946	1.663	1.445	1.325
HD	-0.037	0.253	0.051	0.645	0.040	0.373	0.382	0.564	1.003	0.901
HON	0.768	0.967	0.853	1.105	0.830	1.015	1.076	1.082	1.026	1.105
IBM	-0.410	1.986	-0.313	1.512	-0.337	1.901	0.272	1.600	0.721	1.002
INTC	0.317	0.405	0.344	0.520	0.323	0.386	0.719	0.481	0.776	1.205
JNJ	-0.110	0.519	-0.058	0.276	-0.071	0.565	0.164	0.333	0.491	0.728
JPM	-0.910	1.190	-0.960	1.439	-0.893	1.257	-0.120	1.348	1.436	1.243
KO	-0.227	0.356	-0.060	0.608	-0.108	0.400	0.310	0.517	0.411	0.657
MCD	-0.903	-0.478	-0.746	0.083	-0.787	-0.366	-0.219	-0.110	0.674	0.795
MMM	0.585	1.145	0.570	0.603	0.588	1.154	0.841	0.746	0.881	1.016
MRK	0.727	0.359	0.894	0.190	0.854	0.394	0.985	0.331	0.705	0.725
MSFT	-0.047	-0.197	0.009	0.349	-0.014	-0.090	0.487	0.266	0.877	1.177
NKE	-0.788	-0.378	-0.595	0.592	-0.660	-0.269	0.109	0.331	1.023	0.941
PG	0.152	0.107	0.215	0.072	0.201	0.159	0.353	0.126	0.576	0.634
TRV	-0.511	0.551	-0.399	0.845	-0.415	0.593	0.031	0.748	0.947	0.926
UNH	0.685	0.180	0.886	0.431	0.842	0.245	0.961	0.375	0.683	0.925
VZ	0.414	0.285	0.588	0.027	0.542	0.289	0.481	0.082	0.741	0.625
WBA	0.223	0.856	0.318	0.500	0.320	0.848	0.545	0.515	0.644	0.820
WMT	-0.916	1.007	-0.861	0.114	-0.884	0.976	-0.431	0.347	0.739	0.550

To evaluate stability over time of the different betas, we considered two non-overlapping 7.5 years historic periods:

- Period 3 (containing Financial Crisis 2008): from 2006/01/01 to 2013/07/01;
- Period 4 (containing Covid-19 Crisis): from 2013/07/01 to 2021/01/01.

Further, we provide Table 4 similar to Table 1. Table 4 shows different betas in Periods 3 and 4 for DOW30 stocks. Table 5, similar to Table 2, shows correlation coefficients for betas between two time periods for DOW30, SP100, and SP500 stocks. Standard Beta is the most stable characteristic because it is based on 100% of the data (taking into account both up and down market conditions). ERoD and CDaR Betas account only for time intervals when market goes down. We observe that Drawdown Betas have higher correlation for very large companies in DOW30 and SP100, compared to SP500 companies. $CDaR_{0.9}$ Beta for DOW30 equals 0.515, while Standard Beta correlation coefficient equals 0.676. The relatively high correlation for $CDaR_{0.9}$ can be explained by the fact that time Periods 3 and 4 have significant market drawdowns. Also, for DOW30 and SP100 companies, $ERoD_{0+}$ Beta, $ERoD_{0.1}$ Beta, and $CDaR_{0.5}$ Beta have significant positive correlation ranging from 0.206 to 0.305. Significant positive correlations in Table 5 for DOW30 and SP100 companies show that CDaR and ERoD Betas can be used for constructing portfolios with low drawdowns.

In addition, Table 6 reports the Spearman rank correlation coefficients for betas between Periods 3 and 4 for the three indices.

Table 5: Correlation coefficients of ERoD, CDaR and Standard Betas between Periods 3 and 4

	$ERoD_{0+}$-Beta	$ERoD_{0.1}$-Beta	$CDaR_{0.5}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta
DOW30	0.275	0.206	0.248	0.515	0.676
S&P 100	0.305	0.273	0.267	0.449	0.645
S&P 500	0.074	0.111	0.084	0.293	0.577

Table 6: Spearman rank correlation coefficients of ERoD, CDaR and Standard Betas between Periods 3 and 4

	$ERoD_{0+}$-Beta	$ERoD_{0.1}$-Beta	$CDaR_{0.5}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta
DOW30	0.209	0.098	0.199	0.411	0.724
S&P 100	0.169	0.157	0.166	0.364	0.706
S&P 500	0.071	0.118	0.092	0.311	0.591

7 Conclusion

This paper extended the approach developed by Zabarankin et al. [22]. We introduced a new drawdown risk measure called Expected Regret of Drawdown (ERoD). We showed the equivalence of CDaR and ERoD portfolio optimization problems, which is based on the results of Testuri and Uryasev [21].

Necessary conditions of extremum for the ERoD portfolio optimization problem are formulated in CAPM format. We have derived a new ERoD Beta similar to the Standard and CDaR Betas. The ERoD Beta evaluates portfolio performance during market drawdowns exceeding some threshold ϵ . For small positive values of ϵ the ERoD Beta takes into account all nonzero drawdowns included in the dataset.

We have conducted a case study for DOW30, SP100, and SP500 stocks and compared CDaR and ERoD Betas with the Standard Beta. We have found that CDaR and ERoD Betas are more sensitive to market drawdowns than the Standard Beta. For some stocks CDaR and ERoD Betas are negative, while Standard Beta is always positive. Therefore, these stocks had positive returns when the market was in drawdown, in spite of positive Standard Beta. CDaR and ERoD Betas have quite different characteristics compared to Standard Beta. We want to mention also the so-called Downside Beta, which is the normalized correlation of stock and market returns over time periods when the market return is negative. The Downside Beta and Standard Beta have similar positive values, therefore, Downside Beta does not provide significant additional information compared to the Standard Beta.

The CDaR and ERoD Betas can be used for constructing portfolios with controlled drawdown. Zero CDaR Beta and zero ERoD Beta constraints can be imposed in the portfolio optimization problems, similar to the zero Standard Beta risk management constraint. It is possible to impose simultaneously zero beta constraints for different betas.

The Drawdown Beta Website [5] at the server of the Quantitative Finance Program at Stony Brook University shows CDaR, ERoD, and Standard Betas as well as other characteristics for SP500 stocks. This website is described in Appendix A. Also, Appendix B contains web links to case studies related to implementation of drawdown portfolio optimization.

Data Availability Statement

Data for the case study were downloaded from Yahoo Finance `https://finance.yahoo.com/`.

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A Drawdown Beta Website

Zabarankin et al. [22] and this paper laid down a theoretical foundation for implementation of the Drawdown Beta Website [5] at a server of the Quantitative Finance Program at Stony Brook University.

The website calculates the following characteristics for SP500 stocks using 10 or 15 years of end-of-the-day stock prices.

- $CDaR_{0.9}$Beta = CDaR Beta for a stock based on the largest 10% SP500 drawdowns;
- $ERoD_{0+}$Beta = ERoD Beta for a stock based on all positive (non-zero) SP500 drawdowns;
- Standard Beta = Beta based on co-variance of monthly returns of stock and SP500;
- Max-Drawdown (%) = Maximum drawdown of a stock;
- Annual Return (%) = Effective annual return of stock;
- U-ratio = Ratio of average daily return to average daily drawdown:

$$\frac{(Average\ Daily\ Return)}{(Average\ Drawdown)/(Average\ Length\ of\ Drawdown)}\ .$$

Ticker	$ERoD_{0+}$-Beta	$CDaR_{0.9}$-Beta	Standard-Beta	Max-Drawdown (%)	Annual Return (%)	U-ratio
AAPL	0.7669	0.7592	1.1068	43.80	29.21	0.9355
AMGN	0.0955	0.2561	0.9059	24.77	17.64	0.5061
AXP	1.1291	1.2351	1.1336	49.64	12.30	0.6999
BA	1.1794	2.0575	1.3689	77.92	14.87	0.5985
CAT	2.0418	1.6975	1.1711	44.63	9.88	0.6097
CRM	0.4962	0.7463	1.5652	38.81	20.68	0.4809
CSCO	0.7863	0.6039	1.0693	41.95	11.20	0.5052
CVX	1.1199	1.0585	1.1160	55.77	3.15	0.2157
DIS	0.9725	1.2453	0.8036	43.11	18.08	1.1706
GS	2.3317	2.0003	1.1209	49.39	5.90	0.3112
HD	0.0861	0.4961	0.8524	37.99	25.40	0.7290
HON	1.0428	1.2714	1.1826	43.01	17.89	0.5559
IBM	1.4004	1.0346	0.8076	43.72	1.68	0.3187
INTC	0.4317	0.4320	1.0261	34.53	12.29	0.5629
JNJ	0.3338	0.2470	0.7957	27.37	12.78	0.6608
JPM	1.5801	1.6466	1.3378	43.63	14.19	0.5535
KO	0.1702	0.3111	0.6277	36.99	8.88	0.4679
MCD	-0.5986	-0.3984	0.6775	36.90	14.45	0.7634
MMM	1.1409	0.9200	0.7453	51.20	10.19	0.5330
MRK	0.1683	0.3791	0.4505	27.26	12.16	0.5395
MSFT	-0.0186	0.1526	1.2415	28.04	25.82	0.8398
NKE	-0.4211	0.0260	1.2954	39.78	22.49	0.9424
PG	0.1488	0.1259	0.4062	25.46	11.29	0.6172
TRV	0.5457	0.9233	0.6036	46.28	12.42	0.5190
UNH	0.1165	0.3428	0.8771	35.90	26.99	0.7290
V	-0.4126	0.0698	1.3116	36.36	29.52	0.7325
VZ	0.0673	0.0733	0.4561	20.11	9.58	0.5198
WBA	1.0123	0.7880	0.6118	60.42	2.55	0.2194
WMT	0.5993	0.2161	0.3281	36.44	12.85	0.8169

Figure 4: DOW30 Stocks: performance characteristics posted at Drawdown Beta Website [5].

As an example, Figure 4 shows the table for Dow Jones stocks using a 10-year historic period from 2011-01-01 to 2021-01-01. DOW stock is not included in the table because it does not have 10 years of history in the database.

B Web Links to Case Sudies Related to Drawdown Risk Measure

This appendix provides links to case studies that are related to the drawdown risk measure and its applications:

- Portfolio Optimization with Drawdown Constraints:

  `http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-on-a-single-path/`

  `http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-on-multiple-paths/` `http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/portfolio-optimization-with-drawdown-constraints-single-path-vs-multiple-paths/`

- CoCDaR-Approach Systemic Risk Contribution Measurement:

  `http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-cocdar-approach-systemic-risk-contribution-measurement/`

- Style Classification with mCoCDaR Regression:

  `http://uryasev.ams.stonybrook.edu/index.php/research/testproblems/financial_engineering/case-study-style-classification-with-mcocdar-regression/`