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Buffered-ranking intervals for virtual profit efficiency analysis

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Abstract
The efficiency ranking of a decision making units (DMU) measures its relative position among a group of DMUs over sets of feasible virtual prices that characterize preferences for input and output variables. But the efficiency ranking of a DMU conveys no information about the gap between this DMU and those superior and peer DMUs. So we propose an alternative efficiency measure named buffered-ranking for efficiency analysis. The statement that the efficiency buffered-ranking of a DMU is k implies that its efficiency score reaches the average of the top k efficiency scores of all DMUs. The proposed buffered-ranking is monotone with the conventional ranking, and conveys more information about its relation with superior and peer DMUs. When the efficiency score is based on virtual profit, i.e. the difference between virtual revenue and virtual cost, the calculation of the best buffered-ranking is equivalent to a continuous linear program. We also study the worst buffered-ranking that is opposite to the best buffered-ranking. Experiments demonstrate the advantages of buffered-ranking over conventional ranking.

Keywords Buffered-ranking · Virtual profit efficiency · Linear program · bPOE

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1 Introduction

Both profitability and profit can simultaneously consider the input and output dimensions of the production process and their market relations. Profitability is defined multiplicatively as revenue divided by cost, while profit represents the standard measure of economic performance in business economics, i.e., the difference between revenue and cost. This initial representation of economic performance determines the subsequent characterization of profitability efficiency, defined multiplicatively as the ratio of observed profitability to maximum profitability, and profit efficiency, defined additively as observed profit less maximum profit. Pastor et al. (2022) provides a comprehensive study on how to decompose overall efficiency into technical efficiency and allocative efficiency in terms of both profitability-based and profit-based scores.

Both profitability and profit depend on market (observed) prices of inputs and outputs, while some market prices are not readily available. This is the case in contexts such as higher education, health care, and technology management, among others. But it is still possible to elicit subjective information about how valuable these inputs/outputs are relative to each other. Thus, many models undertake efficiency analysis based on self-appraisal, i.e., each DMU is evaluated by choosing a setting of virtual prices that is most favorable to it. Podinovski (2001) evaluates each DMU with the best virtual profitability efficiency score by choosing a most desirable setting of virtual prices.

Efficiency rankings that show the relative position can provide more instructions for further decision-making (Jablonsky 2018). If the efficiency ranking of a DMU is k, only $k-1$ DMUs achieve higher efficiency scores than this DMU. Based on the principle of self-appraisal, the best ranking of a DMU is obtained by minimizing the number of DMUs with efficiency scores higher than this DMU. Salo and Punkka (2011) study the best ranking that a DMU can attain over feasible virtual input/output prices when the efficiency score is virtual profitability. Section 3.2 extends it to the best ranking when the efficiency score is virtual profit. When a DMU achieves its best virtual efficiency score, it does not necessarily achieve its best ranking.

The best ranking suffers from two concerns. First, the efficiency ranking k conveys neither the magnitude of its inferiority to the $k-1$ superior DMUs, nor the magnitude of its superiority over some inferior DMUs. The statement that a DMU has an efficiency ranking k only implies that $k-1$ DMUs have higher efficiency scores than this DMU. This efficiency ranking of a DMU keeps constant k, even when all $k-1$ superior DMUs improve a lot on their productivity by producing more outputs and consuming fewer inputs.

Second, calculating the best ranking is an NP-hard mixed-integer linear program (MILP). Thus, when the number of DMUs is large, it has a challenging numerical issue. To achieve the best ranking, the threshold hyperplane associated with the optimal virtual price vector should minimize the number of DMUs that lie above it. As shown in Sect. 3.3, this problem is equivalent to a minimization of the cardinality of the upper tail (CUT). Norton et al. (2018, Proposition 4) proved that the computational complexity of minimizing CUT was NP-hard.

Following the above concerns, we propose buffered-rankings for efficiency analysis. Roughly speaking, the statement that the buffered-ranking of a DMU is k implies that

Table 1 The contribution of our paper

	Virtual profitability	Virtual profit
Best efficiency score	Podinovski (2001)	
Best efficiency ranking	Salo and Punkka (2011)	Section 3.2
Best efficiency buffered-ranking		Section 4

the efficiency score of this DMU reaches the average of the top k scores among all DMUs. The number k is a position that can capture the gaps between the DMU under evaluation and those superior and peer DMUs. Furthermore, buffered-ranking is monotone with conventional ranking, which makes buffered-ranking a good alternative to conventional ranking. Buffered-ranking is sensitive to any change in superior DMUs and peer DMUs. The contribution of our paper is summarized in Table 1.

For two reasons, our paper defines buffered-ranking only in difference (profit) form. First, the definition of buffered-ranking relies on the sum of top efficiency scores. The sum of individual virtual profits is the total virtual profit, but it is hard to interpret the sum of individual virtual profitabilities. Second, calculating the best buffered-ranking in difference form is equivalent to a continuous linear program, which can be efficiently solved with off-the-shelf optimization software. But calculating the best buffered-ranking in ratio (profitability) form is still computationally intractable.

Our paper also studies the worst buffered-ranking. The best buffered-ranking measures efficiency, while the worst buffered-ranking measures inefficiency. The best buffered-ranking of a DMU is obtained by choosing a most desirable virtual price vector. In contrast, the worst buffered-ranking of a DMU is obtained by choosing a least desirable virtual price vector.

The rest of the paper is organized as follows. Section 2 is a brief introduction to how to evaluate a DMU with the best virtual profitability score and the best virtual profit score. Section 3 provides a brief introduction to how to evaluate a DMU with the best ranking in terms of virtual profitability, and extends it to the best ranking in terms of virtual profit. Section 4 gives formal definitions of buffered-rankings and computations of the best and worst buffered-rankings in terms of virtual profit. Section 5 gives a probabilistic interpretation of buffered-rankings. Section 6 reports the experimental results of three datasets. This paper is concluded in Sect. 7.

2 Best efficiency scores

2.1 Profitability versus profit

Let $\mathcal{P}$ be the production possibility set, i.e.,

$$\mathcal{P} := \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^N \times \mathbb{R}_+^M : \mathbf{x} \text{ can produce } \mathbf{y}\right\}. \tag{1}$$

$\mathcal{P}$ is a nonempty, closed, and convex set, and satisfies the requirement of no free lunch and the requirement of free disposability of inputs and outputs. $\mathcal{P}$ is unknown in practical efficiency analysis. Commonly it is estimated as an empirical production possibility set in the framework of constant returns to scale (CRS) or variable returns to scale (VRS). Suppose there is a group of J DMUs to be evaluated: $\text{DMU}_1, \ldots,$ DMU_J. Each DMU consumes N types of inputs and produces M types of outputs. DMU_j consumes $\mathbf{x}_j := (x_{1j}, \ldots, x_{Nj})^\intercal$ and produces $\mathbf{y}_j := (y_{1j}, \ldots, y_{Mj})^\intercal$. Let the DMU under assessment be DMU_o where $o \in \mathcal{J} := \{1, \ldots, J\}$. The empirical production possibility sets in the framework of CRS and VRS are

$$\hat{\mathcal{P}}^{\text{CRS}} = \left\{ (\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \sum_{j=1}^{J} \lambda_j \mathbf{x}_j, \mathbf{y} \leq \sum_{j=1}^{J} \lambda_j \mathbf{y}_j, \lambda_j \geq 0, \forall j \right\} \tag{2}$$

$$\hat{\mathcal{P}}^{\text{VRS}} = \left\{ (\mathbf{x}, \mathbf{y}) : \mathbf{x} \geq \sum_{j=1}^{J} \lambda_j \mathbf{x}_j, \mathbf{y} \leq \sum_{j=1}^{J} \lambda_j \mathbf{y}_j, \sum_{j=1}^{J} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\}. \tag{3}$$

To deal with multiple inputs and multiple outputs, we need prices for inputs and outputs. Let $\mathbf{w} = (w_1, \ldots, w_N)^\intercal \geq \mathbf{0}$ be market (observed) prices for inputs and $\mathbf{p} := (p_1, \ldots, p_M)^\intercal \geq \mathbf{0}$ be market (observed) prices for outputs. There are two possible main paradigms for efficiency scores: profitability and profit. The profitability of a production $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ is defined as the ratio of the revenue $\mathbf{p}^\intercal \mathbf{y}$ to the cost $\mathbf{w}^\intercal \mathbf{x}$

$$\text{R}(\mathbf{x}, \mathbf{y}; \mathbf{w}, \mathbf{p}) := \mathbf{p}^\intercal \mathbf{y} / \mathbf{w}^\intercal \mathbf{x}. \tag{4}$$

Please refer to Grifell-Tatjé and Lovell (2015) for the origins of profitability as an indicator of economic performance. The profitability efficiency of $(\mathbf{x}_o, \mathbf{y}_o)$ is the normalized profitability, i.e. the ratio of observed to maximum profitability. The profitability efficiency of production $(\mathbf{x}_o, \mathbf{y}_o) \in \mathcal{P}$ is

$$\text{RE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p}) := \frac{\mathbf{p}^\intercal \mathbf{y}_o / \mathbf{w}^\intercal \mathbf{x}_o}{\max_{(\mathbf{x},\mathbf{y}) \in \mathcal{P}} \{\mathbf{p}^\intercal \mathbf{y} / \mathbf{w}^\intercal \mathbf{x}\}}. \tag{5}$$

A production $(\mathbf{x}_o, \mathbf{y}_o) \in \mathcal{P}$ is profitability efficient if $\text{RE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p}) = 1$.

The profit of a production $(\mathbf{x}, \mathbf{y}) \in \mathcal{P}$ is defined here as the difference between the revenue $\mathbf{p}^\intercal \mathbf{y}$ and the cost $\mathbf{w}^\intercal \mathbf{x}$

$$\text{D}(\mathbf{x}, \mathbf{y}; \mathbf{w}, \mathbf{p}) := \mathbf{p}^\intercal \mathbf{y} - \mathbf{w}^\intercal \mathbf{x}. \tag{6}$$

Petersen (2018) modifies the Nerlovian Profit Indicator (Färe and Grosskopf 2006; Färe et al. 2013), and proposes profit efficiency as the normalized difference between observed profit and maximum profit

$$\text{DE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p}) := \frac{\{\mathbf{p}^\intercal \mathbf{y}_o - \mathbf{w}^\intercal \mathbf{x}_o\} - \max_{(\mathbf{x},\mathbf{y}) \in \mathcal{P}} \{\mathbf{p}^\intercal \mathbf{y} - \mathbf{w}^\intercal \mathbf{x}\}}{\|(\mathbf{w}, \mathbf{p})\|_2}. \tag{7}$$

Due to the normalization, profit efficiency $\mathrm{DE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p})$ is unaffected by units of measurement and has the desirable property of homogeneity of degree 0 in prices $(\mathbf{w}, \mathbf{p})$. $\mathrm{DE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p})$ is the opposite of the Euclidean distance between point $(\mathbf{x}_o, \mathbf{y}_o)$ and the hyperplane $\mathbf{p}^\top \mathbf{y} - \mathbf{w}^\top \mathbf{x} = \max_{(\mathbf{x},\mathbf{y})\in\mathcal{P}} \{\mathbf{p}^\top \mathbf{y} - \mathbf{w}^\top \mathbf{x}\}$. If $\mathrm{DE}(\mathbf{x}_o, \mathbf{y}_o; \mathbf{w}, \mathbf{p}) = 0$, a DMU with production $(\mathbf{x}_o, \mathbf{y}_o)$ is profit efficient, i.e., the achievement of the maximum profit of the production possibility set, under the price vector $(\mathbf{w}, \mathbf{p})$. Different from profitability efficiency (5) which falls in the unit interval $[0, 1]$, profit efficiency (7) falls in an unbounded interval $(-\infty, 0]$.

2.2 Virtual profitability versus virtual profit

For non-profit organizations, such as schools, universities, and hospitals, price information about some inputs and outputs is unavailable. We have to resort to virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$ that describe preferences on inputs and outputs. For instance, if all DMUs believe that one unit of output 1 is at least as valuable as a unit of output 2 but not more valuable than three units of output 2, then the constraints $\mu_2 \le \mu_1 \le 3\mu_2$ must hold (Salo 1995). Let the feasible set of virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$ be

$$\mathcal{S} := \left\{ (\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathbb{R}^{M+N} \middle| \boldsymbol{\nu} \ge \mathbf{0}, \boldsymbol{\mu} \ge \mathbf{0}, \mathbf{A}\boldsymbol{\nu} \ge 0, \mathbf{B}\boldsymbol{\mu} \ge 0 \right\}. \tag{8}$$

where $\mathbf{A}$ and $\mathbf{B}$ are coefficient matrices derived from common preference on how valuable different amounts of inputs and outputs are. If no preference is imposed on inputs and outputs, $\mathbf{A}$ and $\mathbf{B}$ are null matrices. $\mathcal{S}$ is positive-homogeneous, because $(\lambda\boldsymbol{\nu}, \lambda\boldsymbol{\mu}) \in \mathcal{S}$ for any $\lambda > 0$ and any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$.

Consequently, profitability efficiency and profit inefficiency should be rewritten as virtual profitability efficiency and virtual profit efficiency by replacing market (observed) prices $(\mathbf{w}, \mathbf{p})$ with virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$

$$\mathrm{RE}(\mathbf{x}_o, \mathbf{y}_o; \boldsymbol{\nu}, \boldsymbol{\mu}) := \frac{\boldsymbol{\mu}^\top \mathbf{y}_o / \boldsymbol{\nu}^\top \mathbf{x}_o}{\max_{(\mathbf{x},\mathbf{y})\in\mathcal{P}} \{\boldsymbol{\mu}^\top \mathbf{y} / \boldsymbol{\nu}^\top \mathbf{x}\}} \tag{9}$$

$$\mathrm{DE}(\mathbf{x}_o, \mathbf{y}_o; \boldsymbol{\nu}, \boldsymbol{\mu}) := \frac{\{\boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o\} - \max_{(\mathbf{x},\mathbf{y})\in\mathcal{P}} \{\boldsymbol{\mu}^\top \mathbf{y} - \boldsymbol{\nu}^\top \mathbf{x}\}}{\|(\boldsymbol{\nu}, \boldsymbol{\mu})\|_2}. \tag{10}$$

Based on self-appraisal, i.e., each DMU is evaluated by choosing $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$ most favorable to itself. DMU_o can be evaluated by the best virtual profitability efficiency score

$$\mathrm{RE}(\mathbf{x}_o, \mathbf{y}_o) := \sup_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \left\{ \frac{\boldsymbol{\mu}^\top \mathbf{y}_o / \boldsymbol{\nu}^\top \mathbf{x}_o}{\max_{(\mathbf{x},\mathbf{y})\in\mathcal{P}} \{\boldsymbol{\mu}^\top \mathbf{y} / \boldsymbol{\nu}^\top \mathbf{x}\}} \right\}. \tag{11}$$

However, to the best of our knowledge, no model has been proposed to study the best virtual profit efficiency score by

$$\mathrm{DE}(\mathbf{x}_o, \mathbf{y}_o) := \sup_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \frac{\{\boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o\} - \max_{(\mathbf{x},\mathbf{y})\in\mathcal{P}} \{\boldsymbol{\mu}^\top \mathbf{y} - \boldsymbol{\nu}^\top \mathbf{x}\}}{\|(\boldsymbol{\nu}, \boldsymbol{\mu})\|_2}. \tag{12}$$

The best virtual profitability efficiency score has resemble to the technical efficiency measured by the classical data envelopment analysis (Charnes et al. 1978). For example, Podinovski (2001) shows that

$$\sup_{\boldsymbol{\nu}>0,\boldsymbol{\mu}>0}\left\{\frac{\boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}_o/\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}_o}{\max_{j\in\mathcal{J}}\left\{\boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}_j/\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}_j\right\}}\right\}=\sup_{\boldsymbol{\nu}>0,\boldsymbol{\mu}>0}\quad\boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}_o/\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}_o \quad s.t.\quad \boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}_j/\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}_j\le 1,\ j\in\mathcal{J}. \tag{13}$$

Petersen (2018) proposes to evaluate technical efficiency of $(\mathbf{x}_o, \mathbf{y}_o)$ with

$$\sup_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}'}\frac{\{\boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}_o-\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}_o\}-\max_{(\mathbf{x},\mathbf{y})\in\hat{\mathcal{P}}^{\mathrm{VRS}}}\{\boldsymbol{\mu}^{\mathsf{T}}\mathbf{y}-\boldsymbol{\nu}^{\mathsf{T}}\mathbf{x}\}}{\|(\boldsymbol{\nu},\boldsymbol{\mu})\|_2} \tag{14}$$

where the feasibility set $\mathcal{S}'$ is determined by the dual program of the linear program of the directional distance function (Chambers and Pope 1996)

$$\max\left\{\beta\in\mathbb{R}_+ : (\mathbf{x}_o-\beta\boldsymbol{\nu}/\|(\boldsymbol{\nu},\boldsymbol{\mu})\|_2,\, y_o+\beta\boldsymbol{\mu}/\|(\boldsymbol{\nu},\boldsymbol{\mu})\|_2)\in\hat{\mathcal{P}}^{\mathrm{VRS}}\right\}. \tag{15}$$

The definition of technical efficiency depends on the estimation of the production possibility set and the definition of the distance of $(\mathbf{x}_o, \mathbf{y}_o)$ to the production frontier, its feasibility set of virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is determined in an endogenous manner. Following Podinovski (2001) and Salo and Punkka (2011), our paper assumes that the feasibility set of virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is determined in an exogenous manner.

3 Efficiency ranking intervals

One concern about the best efficiency score is that the value of efficiency score itself does not convey any information about its relative position among all DMUs, unless it reaches the maximum (1 for profitability efficiency and 0 for profit efficiency). Another concern about the DEA measure is its sensitivity to which DMUs are included in or excluded from the analysis: for instance, the introduction or removal of a single outlier may shift the efficient frontier drastically and thus disrupt efficiency scores, which may be perplexing to users. This section first introduces the best and worst rankings in terms of virtual profitability that is proposed by Salo and Punkka (2011), then extends to the best and worst rankings in terms of virtual profit.

3.1 Ranking interval in terms of virtual profitability

Salo and Punkka (2011) evaluate the relative efficiency of DMU_o with the ranking interval in terms of virtual profitability

$$\left[\mathrm{rRE}_o^{U*}, \mathrm{rRE}_o^{L*}\right]$$

where

$$\mathrm{rRE}_o^{U*} := \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) \tag{16}$$

$$\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) := 1 + \left|\left\{ j \in \mathcal{J} | \mathrm{RE}_j(\boldsymbol{\nu},\boldsymbol{\mu}) > \mathrm{RE}_o(\boldsymbol{\nu},\boldsymbol{\mu}) \right\}\right| \tag{17}$$

$$\mathrm{rRE}_o^{L*} := \max_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu}) \tag{18}$$

$$\mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu}) := \left|\left\{ j \in \mathcal{J} | \mathrm{RE}_j(\boldsymbol{\nu},\boldsymbol{\mu}) \geq \mathrm{RE}_o(\boldsymbol{\nu},\boldsymbol{\mu}) \right\}\right|. \tag{19}$$

The symbol $|\mathcal{A}|$ denotes the cardinality of a set $\mathcal{A}$. $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu})$ and $\mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$ are the upper ranking and the lower ranking of DMU_o under $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}$ in terms of virtual profitability. If $\mathrm{RE}_o(\boldsymbol{\nu},\boldsymbol{\mu})$ is unique, $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) = \mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$. Or else $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) < \mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$. In these DMUs with the same efficiency score $\mathrm{RE}_o(\boldsymbol{\nu},\boldsymbol{\mu})$, $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu})$ takes the highest ranking, and $\mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$ takes the lowest ranking. Because the mapping $x \mapsto ax$, $a > 0$, is order-preserving, we have

$$\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) = 1 + \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\intercal \mathbf{y}_j / \boldsymbol{\nu}^\intercal \mathbf{x}_j > \boldsymbol{\mu}^\intercal \mathbf{y}_o / \boldsymbol{\nu}^\intercal \mathbf{x}_o \right\}\right| \tag{20}$$

$$\mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu}) = \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\intercal \mathbf{y}_j / \boldsymbol{\nu}^\intercal \mathbf{x}_j \geq \boldsymbol{\mu}^\intercal \mathbf{y}_o / \boldsymbol{\nu}^\intercal \mathbf{x}_o \right\}\right|. \tag{21}$$

rRE_o^{U*} is the best ranking that DMU_o can achieve by choosing a virtual price vector $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}$ most desirable to DMU_o, while rRE_o^{L*} is the worst ranking that DMU_o can achieve by choosing a virtual price vector $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}$ least desirable to DMU_o. Since $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu}) \leq \mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$, $\forall(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}$, we have $\mathrm{rRE}_o^{U*} \leq \mathrm{rRE}_o^{L*}$. Both optimal solutions exist, because $\mathrm{rRE}_o^U(\boldsymbol{\nu},\boldsymbol{\mu})$ and $\mathrm{rRE}_o^L(\boldsymbol{\nu},\boldsymbol{\mu})$ assume values in $\mathcal{J}$.

It is a good idea to evaluate all DMUs with interval efficiencies obtained from both optimistic and pessimistic viewpoints, because this kind of interval efficiency evaluation can avoid the injustice caused by only one strategy. DMU_i dominates DMU_j, if $\mathrm{rRE}_i^{L*} < \mathrm{rRE}_j^{U*}$. This implies that the lower ranking of DMU_i is higher than the upper ranking of DMU_j under any virtual price vector $(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}$.

Salo and Punkka (2011) show that the calculation of rRE_o^{U*} is equivalent to the following MILP

$$\begin{aligned} \mathrm{rRE}_o^{U*} = \min_{\boldsymbol{\nu},\boldsymbol{\mu},\mathbf{z}} \quad & 1 + \sum_{j=1}^{J} z_j \\ s.t.\ & \boldsymbol{\mu}^\intercal \mathbf{y}_j \leq \boldsymbol{\nu}^\intercal \mathbf{x}_j + C z_j, \quad z_j \in \{0,1\}, \quad \forall j \in \mathcal{J} \\ & \boldsymbol{\nu}^\intercal \mathbf{x}_o = 1, \quad \boldsymbol{\mu}^\intercal \mathbf{y}_o = 1 \\ & \mathbf{A}\boldsymbol{\nu} \geq 0, \quad \mathbf{B}\boldsymbol{\mu} \geq 0, \quad \boldsymbol{\nu} \geq 0, \quad \boldsymbol{\mu} \geq 0, \end{aligned} \tag{22}$$

where C is a sufficiently large constant. Similarly, the calculation of rRE_o^{L*} is equivalent to

$$\mathrm{rRE}_o^{L*} = \max_{\boldsymbol{\nu},\boldsymbol{\mu},\mathbf{z}} \quad \sum_{j=1}^{J} z_j \tag{23}$$

$$
\begin{aligned}
s.t. \quad & \boldsymbol{\nu}^\top \mathbf{x}_j \le \boldsymbol{\mu}^\top \mathbf{y}_j + C(1 - z_j), \quad z_j \in \{0, 1\}, \quad \forall j \in \mathcal{J} \\
& \boldsymbol{\nu}^\top \mathbf{x}_o = 1, \quad \boldsymbol{\mu}^\top \mathbf{y}_o = 1 \\
& \mathbf{A}\boldsymbol{\nu} \ge 0, \quad \mathbf{B}\boldsymbol{\mu} \ge 0, \quad \boldsymbol{\nu} \ge 0, \quad \boldsymbol{\mu} \ge 0.
\end{aligned} \tag{24}
$$

3.2 Ranking interval in terms of virtual profit

The relative efficiency of DMU_o also can be evaluated with the ranking interval in terms of virtual profit

$$
\left[\mathrm{rDE}_o^{U*}, \mathrm{rDE}_o^{L*}\right]
$$

where

$$
\mathrm{rDE}_o^{U*} := \min_{(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}\setminus\{\mathbf{0}\}} \mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \tag{25}
$$

$$
\mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) := 1 + \left|\left\{ j \in \mathcal{J} | \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) > \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \right\}\right| \tag{26}
$$

$$
\mathrm{rDE}_o^{L*} := \max_{(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}\setminus\{\mathbf{0}\}} \mathrm{rDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \tag{27}
$$

$$
\mathrm{rDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) := \left|\left\{ j \in \mathcal{J} | \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) \ge \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \right\}\right|. \tag{28}
$$

Because the mapping $x \mapsto ax + b$, $a > 0$, is order-preserving, we have

$$
\mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) := 1 + \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j > \boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o \right\}\right| \tag{29}
$$

$$
\mathrm{rDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) := \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j \ge \boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o \right\}\right|. \tag{30}
$$

Then

$$
\mathrm{rDE}_o^{U*} := 1 + \min_{(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}\setminus\{\mathbf{0}\}} \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j > \boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o \right\}\right| \tag{31}
$$

$$
\mathrm{rDE}_o^{L*} := \max_{(\boldsymbol{\nu},\boldsymbol{\mu}) \in \mathcal{S}\setminus\{\mathbf{0}\}} \left|\left\{ j \in \mathcal{J} | \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j \ge \boldsymbol{\mu}^\top \mathbf{y}_o - \boldsymbol{\nu}^\top \mathbf{x}_o \right\}\right|. \tag{32}
$$

When $(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{0}$, $\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j = 0$, $\forall j \in \mathcal{J}$, which leads to $\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = 1$, $\forall j \in \mathcal{J}$. For this reason, $(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{0}$ is always an optimal solution for rDE_o^{U*}. To exclude the degenerate solution $(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{0}$, we can impose the following constraint

$$
\boldsymbol{\nu}^\top \mathbf{x}_o + \boldsymbol{\mu}^\top \mathbf{y}_o = 1. \tag{33}
$$

Since the objective functions of rDE_o^{U*} are scale-invariant to $(\boldsymbol{\nu}, \boldsymbol{\mu})$, and $\mathcal{S}$ is positively homogeneous, Eq. (33) is a sufficient condition for $(\boldsymbol{\nu}, \boldsymbol{\mu}) \ne \mathbf{0}$. Other constraints can also fulfill the exclusion, e.g., $\boldsymbol{\nu}^\top \mathbf{x}_o + \boldsymbol{\mu}^\top \mathbf{y}_o = \kappa$ with $\kappa > 0$. To calculate rDE_o^{L*}, we should exclude the degenerate solution $\mathbf{0}$ in the same way.

3.3 Two concerns about efficiency rankings

Because the following two concerns apply to both rRE and rDE, for simplicity in this subsection we mainly study two concerns about rDE. The extension to rRE is straightforward. The first concern is that the upper ranking $\mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ conveys no information about the magnitude of its inferiority to $\mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) - 1$ superior DMUs, i.e., about the gaps $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) - \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})$ for $\{j \in \mathcal{J} \mid \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) > \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\}$. The statement $\mathrm{rDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = k$ only implies that among J scores $\{\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}), j = 1, \ldots, J\}$ only $k-1$ scores are higher than $\mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})$, thus neglecting all information about $k-1$ gaps $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) - \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})$. In calculating the best ranking of DMU_o, we just minimize the number of DMUs with efficiency scores higher than DMU_o. Hence rDE_o^{U*} is insensitive to the gap between $\mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})$ and those $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$s in the front rank. Even when all superior DMUs improve their efficiency by producing more outputs or consuming fewer inputs, the best ranking rDE_o^{U*} and the optimal solution keep unchanged.

Let us consider two virtual price vectors

- under $(\boldsymbol{\nu}^A, \boldsymbol{\mu}^A)$, the descending order of J virtual profit efficiency scores is

$$0 > -0.1 > \mathrm{DE}_o(\boldsymbol{\nu}^A, \boldsymbol{\mu}^A) = -1 > -1.1 > -1.2 > -1.3 > -1.4 > \cdots$$

- under $(\boldsymbol{\nu}^B, \boldsymbol{\mu}^B)$, the descending order of J virtual profit efficiency scores is

$$0 > -0.9 > \mathrm{DE}_o(\boldsymbol{\nu}^B, \boldsymbol{\mu}^B) = -1 > -1.1 > -1.2 > -1.3 > -1.4 > \cdots$$

Which virtual price vector is more favorable to DMU_o? Under both virtual price vectors, the upper ranking of DMU_o is 3. But the relative position of DMU_o under $(\boldsymbol{\nu}^B, \boldsymbol{\mu}^B)$ is expected to be higher than that under $(\boldsymbol{\nu}^A, \boldsymbol{\mu}^A)$, because under $(\boldsymbol{\nu}^B, \boldsymbol{\mu}^B)$ the gap between DMU_o and the DMU in the second place is narrower than that under $(\boldsymbol{\nu}^A, \boldsymbol{\mu}^A)$. More examples of failure to convey this kind of information can be found in Sect. 6.

Similarly, consider another two virtual price vectors

- under $(\boldsymbol{\nu}^C, \boldsymbol{\mu}^C)$, the descending order of J normalized profits is

$$0 > -0.9 > \mathrm{DE}_o(\boldsymbol{\nu}^C, \boldsymbol{\mu}^C) = -1 > -1.1 > -1.2 > -1.3 > -1.4 > \cdots$$

- under $(\boldsymbol{\nu}^D, \boldsymbol{\mu}^D)$, the descending order of J normalized profits is

$$0 > -0.9 > \mathrm{DE}_o(\boldsymbol{\nu}^D, \boldsymbol{\mu}^D) = -1 > -1.01 > -1.2 > -1.3 > -1.4 > \cdots$$

Which virtual price vector is more favorable to DMU_o? Under both virtual price vectors, the upper ranking of DMU_o is 3. But the relative position of DMU_o under $(\boldsymbol{\nu}^C, \boldsymbol{\mu}^C)$ should be higher than that under $(\boldsymbol{\nu}^D, \boldsymbol{\mu}^D)$, because under $(\boldsymbol{\nu}^C, \boldsymbol{\mu}^C)$ the gap between DMU_o and the DMU in the fourth place is wider than that under $(\boldsymbol{\nu}^D, \boldsymbol{\mu}^D)$. Under $(\boldsymbol{\nu}^C, \boldsymbol{\mu}^C)$ DMU_o has a better position than under $(\boldsymbol{\nu}^D, \boldsymbol{\mu}^D)$.

The second concern about efficiency ranking is computational burden. The best ranking in ratio form and the best ranking in difference form can be rewritten as

$$\mathrm{rRE}_o^{U*} = 1 + \min_{\substack{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S} \\ \boldsymbol{\nu}^\top\mathbf{x}_o=1,\boldsymbol{\mu}^\top\mathbf{y}_o=1}} \left|\left\{j \in \mathcal{J} | \boldsymbol{\mu}^\top\mathbf{y}_j/\boldsymbol{\nu}^\top\mathbf{x}_j > 1\right\}\right| \tag{34}$$

$$\mathrm{rDE}_o^{U*} = 1 + \min_{\substack{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S} \\ \boldsymbol{\nu}^\top\mathbf{x}_o+\boldsymbol{\mu}^\top\mathbf{y}_o=1}} \left|\left\{j \in \mathcal{J} | \boldsymbol{\mu}^\top(\mathbf{y}_j - \mathbf{y}_o) - \boldsymbol{\nu}^\top(\mathbf{x}_j - \mathbf{x}_o) > 0\right\}\right|. \tag{35}$$

rRE_o^{U*} is obtained by minimizing the CUT of $\boldsymbol{\mu}^\top\mathbf{y}_j/\boldsymbol{\nu}^\top\mathbf{x}_j$ exceeding 1, and rDE_o^{U*} is obtained by minimizing the CUT of $\boldsymbol{\mu}^\top(\mathbf{y}_j - \mathbf{y}_o) - \boldsymbol{\nu}^\top(\mathbf{x}_j - \mathbf{x}_o)$ exceeding 0.

Since $\left\{j \in \mathcal{J} | \mathrm{DE}_j(\boldsymbol{\nu},\boldsymbol{\mu}) \geq \mathrm{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})\right\} \cup \left\{j \in \mathcal{J} | \mathrm{DE}_j(\boldsymbol{\nu},\boldsymbol{\mu}) < \mathrm{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})\right\} = \mathcal{J}$, we have

$$\mathrm{rDE}_o^{L}(\boldsymbol{\nu},\boldsymbol{\mu}) = J - \left|\left\{j \in \mathcal{J} | -\mathrm{DE}_j(\boldsymbol{\nu},\boldsymbol{\mu}) > -\mathrm{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})\right\}\right|. \tag{36}$$

Hence, the two worst rankings rRE_o^{L*} and rDE_o^{L*} can be rewritten as

$$\mathrm{rRE}_o^{L*} = J - \min_{\substack{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S} \\ \boldsymbol{\nu}^\top\mathbf{x}_o=1,\boldsymbol{\mu}^\top\mathbf{y}_o=1}} \left|\left\{j \in \mathcal{J} | -\boldsymbol{\mu}^\top\mathbf{y}_j/\boldsymbol{\nu}^\top\mathbf{x}_j > -1\right\}\right| \tag{37}$$

$$\mathrm{rDE}_o^{L*} = J - \min_{\substack{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S} \\ \boldsymbol{\nu}^\top\mathbf{x}_o+\boldsymbol{\mu}^\top\mathbf{y}_o=1}} \left|\left\{j \in \mathcal{J} | \boldsymbol{\nu}^\top(\mathbf{x}_j - \mathbf{x}_o) - \boldsymbol{\mu}^\top(\mathbf{y}_j - \mathbf{y}_o) > 0\right\}\right|. \tag{38}$$

rRE_o^{L*} involves the minimization of the CUT of $-\boldsymbol{\mu}^\top\mathbf{y}_j/\boldsymbol{\nu}^\top\mathbf{x}_j$ exceeding -1, and rDE_o^{L*} involves the minimization of the CUT of $\boldsymbol{\nu}^\top(\mathbf{x}_j - \mathbf{x}_o) - \boldsymbol{\mu}^\top(\mathbf{y}_j - \mathbf{y}_o)$ exceeding 0.

Norton et al. (2018, Proposition 4) proved that the computational complexity of the CUT minimization problem was NP-hard. Thus, all four best rankings cannot be solved in polynomial time. For a large-scale NP-hard optimization, one has to resort to heuristic algorithms that determine near-optimal solutions. This is achieved by trading optimality, accuracy, or precision for speed.

4 Buffered-ranking intervals

This section proposes to measure DMU_o with buffered-ranking interval in terms of virtual profit

$$[\mathrm{brDE}_o^{U*}, \mathrm{brDE}_o^{L*}]$$

where

$$\mathrm{brDE}_o^{U*} := \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \mathrm{brDE}_o^{U}(\boldsymbol{\nu},\boldsymbol{\mu}) \tag{39}$$

$$\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) := \max\{k \in [1, J] | \mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\} \tag{40}$$

$$\mathrm{UA}_k[\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})] := \frac{1}{k}\left(\sum_{j=1}^{\lfloor k \rfloor} \mathrm{DE}_{[j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) + (k - \lfloor k \rfloor)\mathrm{DE}_{[\lceil k \rceil]}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) \tag{41}$$

$$\mathrm{brDE}_o^{L*} := \max_{(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}} \mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \tag{42}$$

$$\mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) := J + 1 - \max\{k \in [1, J] | \mathrm{LA}_k[\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})] \leq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\} \tag{43}$$

$$\mathrm{LA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) := \frac{1}{k}\left(\sum_{j=1}^{\lfloor k \rfloor} \mathrm{DE}_{[J+1-j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) + (k - \lfloor k \rfloor)\mathrm{DE}_{[J+1-\lceil k \rceil]}(\boldsymbol{\nu}, \boldsymbol{\mu})\right). \tag{44}$$

Here $\mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ and $\mathrm{LA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ are the k-upper average and the k-lower average of vector $\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}) := (\mathrm{DE}_1(\boldsymbol{\nu}, \boldsymbol{\mu}), \ldots, \mathrm{DE}_J(\boldsymbol{\nu}, \boldsymbol{\mu}))^\top$. $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $\mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ are the upper buffered-ranking and the lower buffered-ranking under $(\boldsymbol{\nu}, \boldsymbol{\mu})$. brDE_o^{U*} and brDE_o^{L*} are the best buffered-ranking and the worst buffered-ranking for DMU_o. When k is an integer, the subscript $[k]$ denotes the kth largest. Hence, the descending order of $\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})$ is

$$\mathrm{DE}_{[1]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \mathrm{DE}_{[2]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \mathrm{DE}_{[3]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \cdots \geq \mathrm{DE}_{[J]}(\boldsymbol{\nu}, \boldsymbol{\mu}).$$

When k is not an integer, we need the floor function $\lfloor \cdot \rfloor$ and the ceil function $\lceil \cdot \rceil$. Thus, the subscript $[\lceil k \rceil]$ is the $\lceil k \rceil$th largest, e.g., $[\lceil 6.5 \rceil]$ is the 7th largest. Though the above definitions seem obscure and convoluted, they are intuitive as explained in the following subsections.

It is straightforward to verify that all buffered-rankings keep unchanged when efficiency scores are transformed as

$$a \times \mathrm{DE} + b, \quad a > 0.$$

Thus, buffered-rankings in terms of virtual profit $\boldsymbol{\mu}^\top \mathbf{y} - \boldsymbol{\nu}^\top \mathbf{x}$ is identical to buffered-rankings in terms of virtual profit efficiency score (Eq. 12).

For two reasons, this paper analyzes only buffered-rankings in difference form. First, the upper buffered-ranking relies on the sum of efficiency scores in UA_k. If $\mathrm{DMU}_{[1]}, \ldots, \mathrm{DMU}_{[k]}$ achieve the top k virtual profits among J DMUs under the virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu})$, the sum of the top k virtual profits is $\sum_{j=1}^k \left\{\boldsymbol{\mu}^\top \mathbf{y}_{[j]} - \boldsymbol{\nu}^\top \mathbf{x}_{[j]}\right\}$. It is the total virtual profit of the top k DMUs under the virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu})$, i.e., the difference between the total virtual revenue $\sum_{j=1}^k \boldsymbol{\mu}^\top \mathbf{y}_{[j]}$ and the total virtual cost $\sum_{j=1}^k \boldsymbol{\nu}^\top \mathbf{x}_{[j]}$. The sum of top-$k$ virtual ratios $\sum_{j=1}^k \left\{\boldsymbol{\mu}^\top \mathbf{y}_{[j]} / \boldsymbol{\nu}^\top \mathbf{x}_{[j]}\right\}$ fails to lead to $\left\{\sum_{j=1}^k \boldsymbol{\mu}^\top \mathbf{y}_{[j]}\right\} / \left\{\sum_{j=1}^k \boldsymbol{\nu}^\top \mathbf{x}_{[j]}\right\}$, i.e., the ratio of the total virtual revenue $\sum_{j=1}^k \boldsymbol{\mu}^\top \mathbf{y}_{[j]}$ to the total virtual cost $\sum_{j=1}^k \boldsymbol{\nu}^\top \mathbf{x}_{[j]}$.

Second, as shown in the following, calculating the best (and worst) buffered-ranking in terms of virtual profit can be reformulated into a continuous linear program, which can be efficiently solved with off-the-shelf optimization software. But calculating the best (and worst) buffered-ranking in terms of virtual profitability is computationally intractable.

The requirement $(\boldsymbol{\nu}, \boldsymbol{\mu}) \neq \mathbf{0}$ is necessary for calculating the best (and worst) ranking in terms of virtual profit, but the requirement is negligible for calculating the best (and worst) buffered-ranking in terms of virtual profit. This difference arises from the distinct ways of handling multiple highest or lowest efficiency scores. When $(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{0}$, $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) = 0, \forall j \in \mathcal{J}$, thus $\mathrm{rDE}_j^U = 1$ and $\mathrm{brDE}_o^U = J, \forall j \in \mathcal{J}$. As a result, $(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{0}$ is always a most desirable virtual price vector for the best ranking in terms of virtual profit, while it is always a least desirable virtual price vector for the best buffered-ranking in terms of virtual profit.

4.1 Best buffered-ranking

4.1.1 Upper buffered-ranking

Roughly speaking, $\mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ is the average of the top k components in $\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})$. But this statement only applies to integer values of $k \in \mathcal{J}$. Equation (41) defines $\mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ in a more general manner for $k \in [1, J]$. For any $\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})$, $\mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ is non-increasing and continuous with $k \in [1, J]$.

The meaning of $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ is obvious. Suppose $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ is an integer in $\{1, \ldots, J\}$. When $k < \mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$, the efficiency score of DMU_o fails to exceed the average efficiency score of the top k DMUs. Equation (40) provides a straightforward generalization of the above statement to non-integer k, where the top-k average score is calculated with UA_k. An explicit expression for $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ is

$$\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = \begin{cases} H(\boldsymbol{\nu}, \boldsymbol{\mu}) & \text{if } \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{[1]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \\ k | \mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) = \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) & \text{if } \mathrm{UA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) < \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) < \mathrm{DE}_{[1]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \\ J & \text{if } \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \mathrm{UA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \end{cases} \tag{45}$$

where $H(\boldsymbol{\nu}, \boldsymbol{\mu})$ is the number of DMUs that achieve the highest efficiency score

$$H(\boldsymbol{\nu}, \boldsymbol{\mu}) := \left| \left\{ j \in J | \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{[1]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \right\} \right|.$$

Theorem 1 *For any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$ and $i, j \in \mathcal{J}$, $rDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $brDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ have the following relations*

1. $rDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) < rDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \quad \Rightarrow \quad brDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq brDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$
2. $rDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = rDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \quad \Rightarrow \quad brDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = brDE_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$
3. $rDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq brDE_j^U(\boldsymbol{\nu}, \boldsymbol{\mu})$.

Proof (1) Because $\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) < \mathrm{rDE}_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$, we have $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) > \mathrm{DE}_i(\boldsymbol{\nu}, \boldsymbol{\mu})$. Because $\mathrm{UA}_{\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu})}(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $\mathrm{UA}_{\mathrm{brDE}_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})}(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq$

$\mathrm{DE}_i(\boldsymbol{\nu}, \boldsymbol{\mu})$, it follows $\mathrm{UA}_{\mathrm{brDE}_j^U(\boldsymbol{\nu},\boldsymbol{\mu})}(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_i(\boldsymbol{\nu}, \boldsymbol{\mu})$. Hence,

$$\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \{k \in [1, J] | \mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_i(\boldsymbol{\nu}, \boldsymbol{\mu})\}.$$

From Eq. (40), this inequality implies $\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \mathrm{brDE}_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$.

(2) Since equal upper rankings imply equal efficiency scores, this is straightforward.

(3) For all $i \leq \mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu})$, $\mathrm{DE}_{[i]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$, which implies

$$\mathrm{UA}_{\mathrm{rDE}_j^U(\boldsymbol{\nu},\boldsymbol{\mu})}(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}).$$

Thus,

$$\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \left\{k \in [1, J] | \mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})\right\}.$$

By the definition of $\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu})$, we have $\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu})$. □

Property (1) means the monotonicity: under any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$, a DMU with a higher upper ranking also achieves a higher or equal upper buffered-ranking. Since $\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = J$ for all DMU_j with $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \mathrm{UA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$, a higher upper ranking does not necessarily follow a higher upper buffered-ranking. Provided that $\mathrm{DE}_i(\boldsymbol{\nu}, \boldsymbol{\mu})$ exceeds the overall average, the strict monotonicity holds, i.e., $\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) < \mathrm{rDE}_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ implies $\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) < \mathrm{brDE}_i^U(\boldsymbol{\nu}, \boldsymbol{\mu})$. Please note that the contrary of property (2) is not necessarily correct for the same reason.

Example 1 Suppose there are 12 DMUs to be evaluated under the virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu})$: $\mathrm{DE}_3(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_8(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{11}(\boldsymbol{\nu}, \boldsymbol{\mu}) = 0$, $\mathrm{DE}_{10}(\boldsymbol{\nu}, \boldsymbol{\mu}) = -0.235$, $\mathrm{DE}_2(\boldsymbol{\nu}, \boldsymbol{\mu}) = -0.471$, $\mathrm{DE}_5(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_7(\boldsymbol{\nu}, \boldsymbol{\mu}) = -0.941$, $\mathrm{DE}_6(\boldsymbol{\nu}, \boldsymbol{\mu}) = -0.412$, $\mathrm{DE}_1(\boldsymbol{\nu}, \boldsymbol{\mu}) = -1.647$, $\mathrm{DE}_4(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_9(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{12}(\boldsymbol{\nu}, \boldsymbol{\mu}) = -2.000$. Three ties are deliberately designed: three highest efficiency scores of 0, three lowest efficiency scores of -2, and two equal middle efficiency scores of -0.941. The UA_k curve and the upper buffered-rankings are shown in Fig. 1.

The upper rankings and upper buffered-rankings of 12 DMUs are presented in Table 2. The upper buffered-ranking for each DMU with the highest efficiency score is 3, which is the highest upper buffered-ranking among 12 DMUs. For each DMU with efficiency score on the interval $(\mathrm{UA}_{12}(\boldsymbol{\nu}, \boldsymbol{\mu}), \mathrm{UA}_3(\boldsymbol{\nu}, \boldsymbol{\mu}))$, the upper buffered-ranking is the inverse of UA_k. All DMUs with efficiency scores below the overall average have upper buffered-rankings 12, which is the lowest among 12 DMUs.

When $H(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq 2$, for any DMU_j with the highest efficiency score, the upper ranking differs from the upper buffered-ranking, i.e., $\mathrm{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = 1$ and $\mathrm{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = H(\boldsymbol{\nu}, \boldsymbol{\mu})$. The upper buffered-ranking is more reasonable in handling multiple highest efficiency scores. Consider two extreme virtual price vectors: (a) all DMUs achieve the same efficiency score, (b) only DMU_o achieves the highest efficiency score. Setting (b) should be more desirable to DMU_o. If the upper ranking is chosen as the criterion, setting (a) and setting (b) are indifferent. However, if the

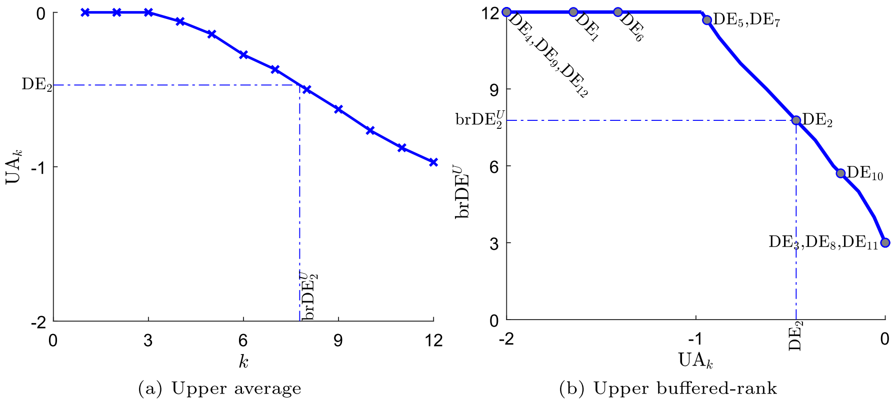


Fig. 1 Upper average and upper buffered-ranking

Table 2 One example of buffered-rankings

	$DE_j(\boldsymbol{\nu},\boldsymbol{\mu})$	$rDE_j^U(\boldsymbol{\nu},\boldsymbol{\mu})$	$brDE_j^U(\boldsymbol{\nu},\boldsymbol{\mu})$	$brDE_j^L(\boldsymbol{\nu},\boldsymbol{\mu})$	$rDE_j^L(\boldsymbol{\nu},\boldsymbol{\mu})$
DMU_1	−1.647	9	12.00	6.81	9
DMU_2	−0.471	5	7.77	1.00	5
DMU_3	0.000	1	3.00	1.00	3
DMU_4	−2.000	10	12.00	10.00	12
DMU_5	−0.941	6	11.69	1.00	7
DMU_6	−1.412	8	12.00	4.89	8
DMU_7	−0.941	6	11.69	1.00	7
DMU_8	0.000	1	3.00	1.00	3
DMU_9	−2.000	10	12.00	10.00	12
DMU_{10}	−0.235	4	5.71	1.00	4
DMU_{11}	0.000	1	3.00	1.00	3
DMU_{12}	−2.000	10	12.00	10.00	12

upper buffered-ranking is chosen as the criterion, setting (b) will be chosen. The upper ranking of DMU_o is 1 in both settings, while the upper buffered-ranking of DMU_o is J under setting (a) and 1 under setting (b).

The upper buffered-ranking of each DMU with efficiency score below the overall average is J. As shown in Fig. 1, upper buffered-rankings of five DMUs (DMU_1, DMU_4, DMU_6, DMU_9, DMU_{12}) are 12. It seems that upper buffered-ranking has less discriminating power than upper ranking. However, technical efficiency measurement is based on self-appraisal, DMU_o will choose the most favorable virtual price vector to improve its upper buffered-ranking.

Generally, the upper average UA_k is not a piecewise linear function of k. Consider $UA_k(\mathbf{E})$ in Example 1. When $k \in [3, 4]$,

$$UA_k(\mathbf{E}) = \frac{3 \times 0 + (k-3) \times (-0.235)}{k} = \frac{(3-k) \times 0.235}{k},$$

which is not linear with k. Therefore, neither the curve in Fig. 1a nor the curve in Fig. 1b is piecewise linear.

4.1.2 Calculating the best buffered-ranking

The best buffered-ranking for DMU_o is the highest upper buffered-ranking of DMU_o by choosing a most desirable virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$. It can be rewritten as

$$\text{brDE}_o^{U*} = \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \max_{\substack{k\in[1,J] \\ \text{UA}_k(\mathbf{E}(\boldsymbol{\nu},\boldsymbol{\mu}))\geq \text{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})}} k. \tag{46}$$

This optimization seems very hard to solve. First, its objective function has the min max form. Second, owing to the order statistic UA_k, the constraint $\text{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \text{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})$ is hard to work with.

Lemma 1 (Norton et al. 2018, Theorem 1) *For a Euclidean vector* $\mathbf{z} := [z_1, \ldots, z_J] \in \mathbb{R}^J$ *and a threshold* $h \in \mathbb{R}$,

$$\bar{\eta}_h(\mathbf{z}) := \begin{cases} \max\{k | UA_k(\mathbf{z}) \geq h\} & \text{if } h \leq \max_j z_j \\ 0 & \text{if } h > \max_j z_j \end{cases} \tag{47}$$

can be represented as

$$\bar{\eta}_h(\mathbf{z}) = \min_{a\geq 0} \sum_{j=1}^{J} [a(z_j - h) + 1]^+. \tag{48}$$

The proof of Lemma 1 is mainly based on the fact

$$\bar{\eta}_h(\mathbf{z}) = \min \left\{ k \middle| \min_{\gamma} \left\{ \gamma + \frac{1}{k} \sum_{j=1}^{J} [z_j - \gamma]^+ \right\} \leq h \right\}. \tag{49}$$

Thanks to the lemma, $\text{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ allows for a representation of minimization

$$\text{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = \min_{a\geq 0} \sum_{j=1}^{J} [a(\text{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) - \text{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})) + 1]^+. \tag{50}$$

Therefore

$$\text{brDE}_o^{U*} = \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}, a\geq 0} \sum_{j=1}^{J} [a(\text{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) - \text{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})) + 1]^+. \tag{51}$$

After substituting $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$ with $\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j$, problem (51) becomes

$$\mathrm{brDE}_o^{U*} = \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}, a\geq 0} \sum_{j=1}^{J} \left[a(\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_o + \boldsymbol{\nu}^\top \mathbf{x}_o) + 1\right]^+ . \tag{52}$$

When $a = 0$, for all $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$, the objective function is equal to J. Since $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq J$ for all $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$, we can neglect the case $a = 0$. Let $a = -1/\gamma$, the above program becomes

$$\mathrm{brDE}_o^{U*} = \inf_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}, \gamma<0} \sum_{j=1}^{J} \left[\frac{\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_o + \boldsymbol{\nu}^\top \mathbf{x}_o - \gamma}{-\gamma}\right]^+ . \tag{53}$$

Since $\mathcal{S}$ and the function $f(\boldsymbol{\nu}, \boldsymbol{\mu}) := \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_o + \boldsymbol{\nu}^\top \mathbf{x}_o$ are positively homogeneous, if $(\boldsymbol{\nu}, \boldsymbol{\mu}, \gamma)$ solves program (53), $(-\boldsymbol{\nu}/\gamma, -\boldsymbol{\mu}/\gamma, -1)$ also solves program (53). Thus, we can rewrite the above optimization as

$$\mathrm{brDE}_o^{U*} = \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \sum_{j=1}^{J} \left[\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_o + \boldsymbol{\nu}^\top \mathbf{x}_o + 1\right]^+ . \tag{54}$$

Therefore, calculating the best buffered-ranking is equivalent to a continuous linear program

$$\begin{aligned} \mathrm{brDE}_o^{U*} = \min_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\beta}} \quad & \sum_{j=1}^{J} \beta_j \\ s.t. \quad & \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_o + \boldsymbol{\nu}^\top \mathbf{x}_o + 1 \leq \beta_j, \beta_j \geq 0, \forall j \in \mathcal{J} \\ & \mathbf{A}\boldsymbol{\nu} \geq 0, \quad \mathbf{B}\boldsymbol{\mu} \geq 0, \quad \boldsymbol{\nu} \geq \mathbf{0}, \quad \boldsymbol{\mu} \geq \mathbf{0}. \end{aligned} \tag{55}$$

4.2 Worst buffered-ranking

4.2.1 Lower buffered-ranking

Since $\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$ is the efficiency score of DMU_j, $-\mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu})$ captures the inefficiency score of DMU_j. For example, if $\boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_j$ is the virtual profit of DMU_j, its opposite is the virtual loss of DMU_j. Due to the fact

$$\mathrm{LA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) = -\mathrm{UA}_k(-\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})), \tag{56}$$

all the following results are self-evident.

Let $L(\boldsymbol{\nu}, \boldsymbol{\mu})$ be the number of DMUs that achieve the lowest efficiency score

$$L(\boldsymbol{\nu}, \boldsymbol{\mu}) := \left|\left\{j \in \mathcal{J} | \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{[J]}(\boldsymbol{\nu}, \boldsymbol{\mu})\right\}\right| .$$

If $L(\boldsymbol{\nu}, \boldsymbol{\mu}) = 1$, LA_k is strictly increasing over the whole domain. Otherwise, LA_k is constant $\mathrm{DE}_{[J]}(\boldsymbol{\nu}, \boldsymbol{\mu})$ on the interval $[1, L(\boldsymbol{\nu}, \boldsymbol{\mu})]$, and strictly increasing on the interval $[L(\boldsymbol{\nu}, \boldsymbol{\mu}), J]$. UA_k and LA_k coincide at $k = J$, i.e.,

$$\mathrm{UA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) = \mathrm{LA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) = \frac{1}{J}\sum_{j=1}^{J} \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}).$$

$\mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ has the following explicit expression

$$\mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) = \begin{cases} J + 1 - L(\boldsymbol{\nu}, \boldsymbol{\mu}) & \text{if } \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) = \mathrm{DE}_{[J]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \\ J + 1 - \{k \in [1, J] | \mathrm{LA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) = \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\} & \text{if } \mathrm{DE}_{[J]}(\boldsymbol{\nu}, \boldsymbol{\mu}) < \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \\ & < \mathrm{LA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \\ 1 & \text{if } \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \mathrm{LA}_J(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})). \end{cases} \tag{57}$$

Theorem 2 shows the relations between $\mathrm{rDE}_j^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $\mathrm{brDE}_j^L(\boldsymbol{\nu}, \boldsymbol{\mu})$. The theorem follows directly from the combination of Theorem 1 and Eq. (56).

Theorem 2 *For any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$ and $i, j \in \mathcal{J}$, $rDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $brDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ have the following relations*

1. $rDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) > rDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \Rightarrow brDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq brDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu})$
2. $rDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) = rDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \Rightarrow brDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) = brDE_i^L(\boldsymbol{\nu}, \boldsymbol{\mu})$
3. $rDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq brDE_j^L(\boldsymbol{\nu}, \boldsymbol{\mu})$.

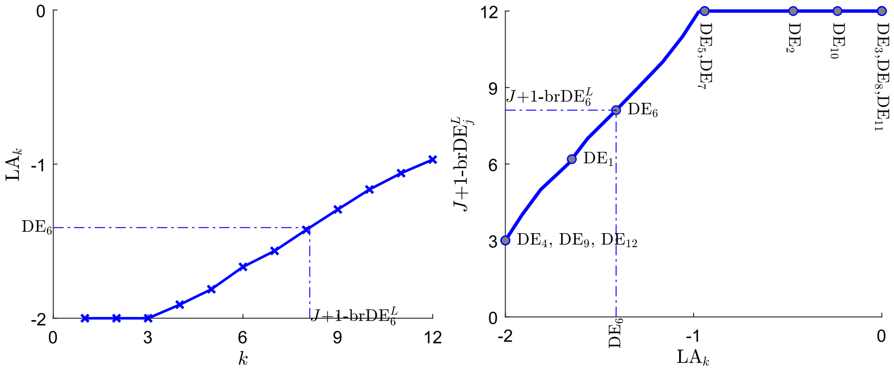
Example 2 The LA_k curve and lower buffered-rankings corresponding to the data in Example 1 are shown in Fig. 2. The lower buffered-ranking for each DMUs that achieves the lowest efficiency score is 10, which is the lowest among 12 DMUs. The lower buffered-ranking for each DMU with efficiency score between $\mathrm{LA}_3(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $\mathrm{LA}_J(\boldsymbol{\nu}, \boldsymbol{\mu}))$ is $J + 1$ minus the inverse of LA_k. Every DMU with efficiency score above the overall average has lower buffered-ranking 1. These DMUs' conventional rankings and buffered-rankings are listed in Table 2, which confirms the relations in Theorems 1 and 2.

4.2.2 Calculating the worst buffered-ranking

The worst buffered-ranking for DMU_o is the lowest lower buffered-ranking of DMU_o by choosing $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$ least desirable to DMU_o. We have

$$\mathrm{brDE}_o^{L*} = \max_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \{J + 1 - \max\{k \in [1, J] | \mathrm{LA}_k[\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})] \leq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\}\} \tag{58}$$

$$= J + 1 - \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \max\{k \in [1, J] | \mathrm{LA}_k[\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})] \leq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\} \tag{59}$$



(a) Lower average (b) Lower buffered-rank

Fig. 2 Lower average and lower buffered-ranking

$$= J + 1 - \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \max_{\substack{k\in[1,J] \\ \text{UA}_k(-\mathbf{E}(\boldsymbol{\nu},\boldsymbol{\mu}))\geq -\text{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})}} k, \tag{60}$$

where (59) $\Rightarrow$ (60) follows from Eq. (56).

$$\min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \max_{\substack{k\in[1,J] \\ \text{UA}_k[\mathbf{E}(\boldsymbol{\nu},\boldsymbol{\mu})]\geq \text{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})}} k$$

is the highest buffered-ranking in terms of efficiency, while

$$\min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \max_{\substack{k\in[1,J] \\ \text{UA}_k(-\mathbf{E}(\boldsymbol{\nu},\boldsymbol{\mu}))\geq -\text{DE}_o(\boldsymbol{\nu},\boldsymbol{\mu})}} k$$

is the highest buffered-ranking in terms of inefficiency. Analog to the best buffered-ranking, the worst buffered-ranking can be rewritten as a continuous linear program

$$\begin{aligned} \text{brDE}_o^{L*} = J + 1 - \min_{\boldsymbol{\nu},\boldsymbol{\mu},\boldsymbol{\beta}} \sum_{j=1}^{J} \beta_j \\ s.t.\ \boldsymbol{\nu}^\top \mathbf{x}_j - \boldsymbol{\mu}^\top \mathbf{y}_j - \boldsymbol{\nu}^\top \mathbf{x}_o + \boldsymbol{\mu}^\top \mathbf{y}_o + 1 \leq \beta_j,\ \beta_j \geq 0, \forall j \in \mathcal{J} \\ \mathbf{A}\boldsymbol{\nu} \geq 0, \quad \mathbf{B}\boldsymbol{\mu} \geq 0, \quad \boldsymbol{\nu} \geq \mathbf{0}, \quad \boldsymbol{\mu} \geq \mathbf{0}. \end{aligned} \tag{61}$$

4.3 The relation between various rankings

This subsection considers the relation between two rankings (rDE$_o^{U*}$ and rDE$_o^{L*}$), and two buffered-rankings (brDE$_o^{U*}$ and brDE$_o^{L*}$). Theorems 1 and 2 show that, for any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$,

$$\text{rDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}) \leq \text{brDE}_j^U(\boldsymbol{\nu}, \boldsymbol{\mu}), \quad \text{rDE}_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq \text{brDE}_j^L(\boldsymbol{\nu}, \boldsymbol{\mu}).$$

Thus we have

$$\text{rDE}_j^{U*} \le \text{brDE}_j^{U*}, \quad \text{rDE}_j^{L*} \ge \text{brDE}_j^{L*}.$$

We also consider two approximate rankings. As shown in the above subsections, the optimization for rDE_o^{U*} is an NP-hard MILP, while the optimization for brDE_o^{U*} is a continuous linear program. So how about approximating the optimization for rDE_o^{U*} with the solution for brDE_o^{U*}? The approximate upper ranking is defined as

$$\widehat{\text{rDE}}_o^U := \text{rDE}_o^U(\boldsymbol{\nu}_o^{U*}, \boldsymbol{\mu}_o^{U*}), \tag{62}$$

where $(\boldsymbol{\nu}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ solves the optimization for brDE_o^{U*}

$$(\boldsymbol{\nu}_o^{U*}, \boldsymbol{\mu}_o^{U*}) := \arg \min_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \text{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}). \tag{63}$$

Similarly, the approximate lower ranking is defined as

$$\widehat{\text{rDE}}_o^L := \text{rDE}_o^L(\boldsymbol{\nu}_o^{L*}, \boldsymbol{\mu}_o^{L*}), \tag{64}$$

where $(\boldsymbol{\nu}_o^{L*}, \boldsymbol{\mu}_o^{L*})$ solves the optimization for brDE_o^{L*}

$$(\boldsymbol{\nu}_o^{L*}, \boldsymbol{\mu}_o^{L*}) := \arg \max_{(\boldsymbol{\nu},\boldsymbol{\mu})\in\mathcal{S}} \text{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}). \tag{65}$$

Theorem 3 shows the relations between four rankings and two buffered-rankings. The quantitative approximating error of $\widehat{\text{rDE}}_o^U$ for rDE_o^{U*} needs further theoretical study.

Theorem 3 *For any $o \in \mathcal{J}$,*

1. $rDE_o^{U*} \le \widehat{rDE}_o^U \le brDE_o^{U*}$
2. $rDE_o^{L*} \ge \widehat{rDE}_o^L \ge brDE_o^{L*}$.

Proof (1) The first inequality follows from the definition of rDE_o^{U*} (Eq. 16), and the second inequality follows from property (3) in Theorem 1.

(2) Analogous to the above. □

The relation $\text{brDE}_o^{U*} \le \text{brDE}_o^{L*}$ does not necessarily hold. A counter-example is $(\mathbf{x}_1, \mathbf{y}_1) = \cdots = (\mathbf{x}_J, \mathbf{y}_J)$. In this counter-example, for any $o \in \mathcal{J}$, $\text{brDE}_o^{U*} = J$ and $\text{brDE}_o^{L*} = 1$. The experimental results in Sect. 6 show that $\text{brDE}_o^{U*} > \text{brDE}_o^{L*}$ rarely happens.

5 Buffered-ranking and bPOE optimization

In this section, we will prove that calculating a buffered-ranking is equivalent to a buffered-probability-of-exceedance (bPOE) minimization (Mafusalov et al. 2018;

Mafusalov and Uryasev 2018). Following the equivalence, one can compute buffered-ranking by bPOE minimization, which is precoded in some professional optimization packages. For example, Portfolio SafeGuard (PSG) contains precoded major classes of nonlinear functions, including Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), probability of exceedance (POE), bPOE, drawdown, partial moment, Omega, and maximum loss. PSG formulates many stochastic optimization problems in analytic formats. The PSG code for the experiments in Sect. 6 is available on the website http://uryasev.ams.stonybrook.edu/index.php/case-study-data-envelopment-analysis-stochastic.

Before presenting the formal definition of bPOE, we introduce two closely related terms: VaR and CVaR. For a random loss ξ, the VaR at confidence level $\alpha \in (0, 1)$ is

$$\mathrm{VaR}_\alpha(\xi) := \inf\{z | F_\xi(z) \geq \alpha\}. \tag{66}$$

CVaR (Rockafellar and Uryasev 2002) of ξ at level α is the conditional expectation exceeding $\mathrm{VaR}_\alpha(\xi)$, i.e., the mean of the α-tail distribution of ξ. If F_ξ is continuous at $\mathrm{VaR}_\alpha(\xi)$,

$$\mathrm{CVaR}_\alpha(\xi) = \mathbb{E}[\xi | \xi \geq \mathrm{VaR}_\alpha(\xi)]. \tag{67}$$

If F_ξ is discontinuous at $\mathrm{VaR}_\alpha(\xi)$, i.e., $\lim_{z \uparrow \mathrm{VaR}_\alpha(\xi)} F_\xi(z) < F_\xi(\mathrm{VaR}_\alpha(\xi))$,

$$\mathrm{CVaR}_\alpha(\xi) = \frac{1 - F_\xi(\mathrm{VaR}_\alpha(\xi))}{1-\alpha}\mathbb{E}[\xi | \xi > \mathrm{VaR}_\alpha(\xi)] + \frac{F_\xi(\mathrm{VaR}_\alpha(\xi)) - \alpha}{1-\alpha}\mathrm{VaR}_\alpha(\xi). \tag{68}$$

For general distributions, $\mathrm{CVaR}_\alpha(\xi)$ can be represented with a minimization

$$\mathrm{CVaR}_\alpha(\xi) = \inf_\gamma \left\{ \gamma + \frac{\mathbb{E}[\xi - \gamma]^+}{1-\alpha} \right\}. \tag{69}$$

CVaR has been widely used in portfolio and risk management (Pekár et al. 2021; Kara et al. 2019). It has also found successful applications in efficiency analysis (Adam and Branda 2021; Branda and Kopa 2014).

bPOE is developed as the inverse of the CVaR. Let $\sup \xi$ denote the essential supremum of ξ, and $z \in \mathbb{R}$ be a fixed threshold. The bPOE of ξ at threshold z equals

$$\mathrm{bPOE}_z(\xi) := \begin{cases} \sup\{1-\alpha | \mathrm{CVaR}_\alpha(\xi) \geq z\} & \text{if } z \leq \sup \xi \\ 0 & \text{if } z > \sup \xi. \end{cases} \tag{70}$$

bPOE minimization has great potential to find successful applications in various areas, e.g., machine learning (Norton et al. 2017; Norton and Uryasev 2019), portfolio management (Shang et al. 2018; Norton et al. 2021), network optimization (Norton et al. 2018), and risk management (Davis and Uryasev 2016; Rockafellar and Uryasev 2020).

Theorem 4 *$brDE_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$ and $brDE_o^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ have the following bPOE representations*

$$brDE_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) = J \times bPOE_0\left(\tilde{E}(\boldsymbol{\nu}, \boldsymbol{\mu}) - DE_o(\boldsymbol{\nu}, \boldsymbol{\mu})\right) \tag{71}$$

$$brDE_o^L(\boldsymbol{\nu}, \boldsymbol{\mu}) = J + 1 - J \times bPOE_0\left(DE_o(\boldsymbol{\nu}, \boldsymbol{\mu}) - \tilde{E}(\boldsymbol{\nu}, \boldsymbol{\mu})\right), \tag{72}$$

where $\tilde{E}(\boldsymbol{\nu}, \boldsymbol{\mu}) = \boldsymbol{\mu}^\top \tilde{\mathbf{y}} - \boldsymbol{\nu}^\top \tilde{\mathbf{x}}$ and $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ follows the uniform discrete distribution with support $\{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_J, \mathbf{y}_J)\}$.

Proof For a given setting $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$, $\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu}) = \boldsymbol{\mu}^\top \tilde{\mathbf{y}} - \boldsymbol{\nu}^\top \tilde{\mathbf{x}}$ follows the uniform discrete distribution with support $\{\boldsymbol{\mu}^\top \mathbf{y}_1 - \boldsymbol{\nu}^\top \mathbf{x}_1, \ldots, \boldsymbol{\mu}^\top \mathbf{y}_J - \boldsymbol{\nu}^\top \mathbf{x}_J\}$. For any $\alpha \in (0, 1)$

$$\mathrm{VaR}_\alpha\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) = \mathrm{DE}_{[\lfloor J(1-\alpha)\rfloor + 1]}(\boldsymbol{\nu}, \boldsymbol{\mu}), \tag{73}$$

which follows

$$\mathbb{E}[\xi | \xi > \mathrm{VaR}_\alpha(\xi)] = \frac{1}{\lfloor J(1-\alpha)\rfloor} \sum_{j=1}^{\lfloor J(1-\alpha)\rfloor} \mathrm{DE}_{[j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) \tag{74}$$

$$F_\xi(\mathrm{VaR}_\alpha(\xi)) = 1 - \frac{\lfloor J(1-\alpha)\rfloor}{J}. \tag{75}$$

According to Eq. (68)

$$\mathrm{CVaR}_\alpha\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) = \frac{1}{J(1-\alpha)} \sum_{j=1}^{\lfloor J(1-\alpha)\rfloor} \mathrm{DE}_{[j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) + \left(1 - \frac{\lfloor J(1-\alpha)\rfloor}{J(1-\alpha)}\right) \mathrm{DE}_{[\lfloor J(1-\alpha)\rfloor + 1]}(\boldsymbol{\nu}, \boldsymbol{\mu}). \tag{76}$$

When $\alpha = 1 - k/J$, i.e., $k = J(1-\alpha)$),

$$\mathrm{CVaR}_{1-k/J}\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) = \frac{1}{k}\left(\sum_{j=1}^{\lfloor k\rfloor} \mathrm{DE}_{[j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) + (k - \lfloor k\rfloor)\mathrm{DE}_{[\lfloor k\rfloor + 1]}(\boldsymbol{\nu}, \boldsymbol{\mu})\right). \tag{77}$$

When k is an integer, $k - \lfloor k\rfloor = 0$. Otherwise $\lfloor k\rfloor + 1 = \lceil k\rceil$. Therefore,

$$\mathrm{CVaR}_{1-k/J}\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) = \frac{1}{k}\left(\sum_{j=1}^{\lfloor k\rfloor} \mathrm{DE}_{[j]}(\boldsymbol{\nu}, \boldsymbol{\mu}) + (k - \lfloor k\rfloor)\mathrm{DE}_{[\lceil k\rceil]}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) \tag{78}$$

which coincides with $\mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu}))$ (41).

$$
\begin{aligned}
&\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu}) && (79)\\
&\quad = \max\{k \in [1, J] | \mathrm{UA}_k(\mathbf{E}(\boldsymbol{\nu}, \boldsymbol{\mu})) \geq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\} && (80)\\
&\quad = \max\left\{k \in [1, J] | \mathrm{CVaR}_{1-k/J}\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu})\right) \geq \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\right\} && (81)\\
&\quad = \max\left\{k \in [1, J] | \mathrm{CVaR}_{1-k/J}\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu}) - \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\right) \geq 0\right\} && (82)\\
&\quad = J \times \mathrm{bPOE}_0\left(\tilde{\mathrm{E}}(\boldsymbol{\nu}, \boldsymbol{\mu}) - \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu})\right). && (83)
\end{aligned}
$$

In this problem, the second case $z > \sup \xi$ in the definition of bPOE (70) can be neglected, since $\max_j \mathrm{DE}_j(\boldsymbol{\nu}, \boldsymbol{\mu}) - \mathrm{DE}_o(\boldsymbol{\nu}, \boldsymbol{\mu}) \geq 0$ for any $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$.

With the bPOE representation of $\mathrm{brDE}_o^U(\boldsymbol{\nu}, \boldsymbol{\mu})$, we can immediately obtain the proof for the bPOE presentation of $\mathrm{brDE}_o^L(\boldsymbol{\nu}, \boldsymbol{\mu})$ by Eq. (58). □

6 Experiments

All experiments are implemented by Matlab with Gurobi and CVX. Gurobi is one of the fastest and most powerful mathematical programming commercial solvers available for MILP problems. CVX (Grant and Boyd 2014) is a Matlab-based modeling system for convex optimization. Our experiments use CVX to solve all continuous linear programs for buffered-rankings, and use Gurobi to solve all MILPs for conventional rankings. In experiments on middle-size and large-size datasets, Gurobi fails to complete calculation of some conventional rankings within an hour. We set the TimeLimit parameter in Gurobi 3600 s. It means that every Gurobi optimization must terminate within one hour, even though optimization conditions are not satisfied. All code and datasets are available on Github.[1] We ran all experiments on a laptop with Intel Core i7-7500 CPU @2.70 G. For simplicity, the paper imposes no common preference on virtual prices $(\boldsymbol{\nu}, \boldsymbol{\mu})$, i.e., $\mathbf{A}$ and $\mathbf{B}$ are null matrices.

6.1 Experiment with PFT1981

The dataset PFT1981[2] is available from Project Follow Through (PTF), which is a large-scale social experiment in public school education. Charnes et al. (1981) considers three outputs: Reading (total reading scores), Math (total math scores), and Coopersmith (total Coopersmith scores), and five inputs: Education (education level of mother), Occupation (occupation index), Parental (parental visit index), Counseling (counseling index), and Teachers (number of teachers). There are 70 DMUs (sites) to be evaluated. Because the dataset is very small, Gurobi can solve all MILPs associated with ranking models within 45 s. All continuous linear programs associated with buffered-ranking models can be solved within 0.1 s. For saving space, this subsection neglects detailed presentation of running time, since for small-size datasets the differ-

[1] https://github.com/YongqiaoWang/buffered-ranking.

[2] Available at https://www.rdocumentation.org/packages/deaR/versions/1.2.5/topics/PFT1981.

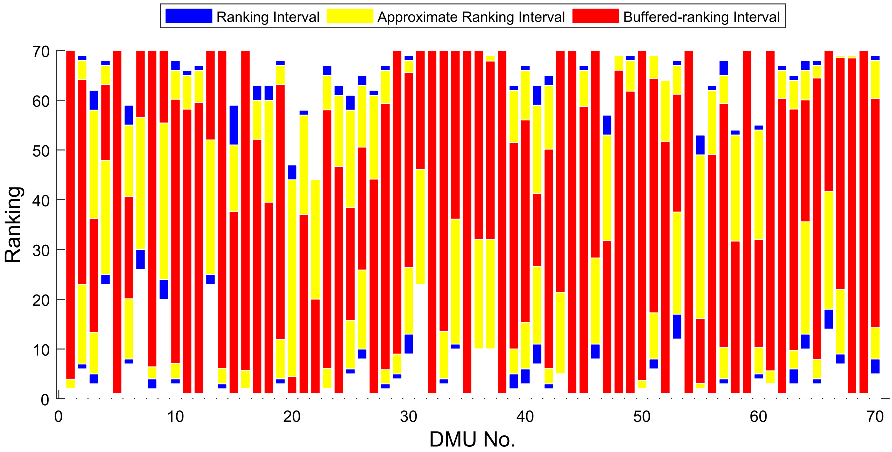


Fig. 3 Difference-based ranking intervals and buffered-ranking intervals for DMUs in the experiment with PFT1981

ence of computational burden between conventional ranking and buffered-ranking is negligible with modern computers.

Four rankings and two buffered-rankings in terms of virtual profit are shown in Fig. 3. 27 DMUs achieve the best ranking of 1 and the best buffered-ranking of 1. 25 DMUs achieve the worst ranking of 70 and the worst buffered-ranking of 70. Among them, 8 DMUs achieve the whole ranking (and buffered-ranking) interval $[1, J]$. In other words, each of these 8 DMUs is the most virtual profit efficient among 70 DMUs when it chooses the most favorable virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$, and is the most virtual profit inefficient among 70 DMUs when it chooses the least favorable virtual price vector $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$.

These rankings and buffered-rankings comply with the inequalities in Theorem 3. Provided that the ranking (and buffered-ranking) interval of a DMU is not equal to the whole interval $[0, J]$, the buffered-ranking interval is narrower than the ranking interval. brDE_o^{U*} is monotonic to rDE_o^{U*}, and brDE_o^{L*} is monotonic to rDE_o^{L*}. For many DMUs, the approximate ranking interval $[\widehat{\text{rDE}_o^U}, \widehat{\text{rDE}_o^L}]$ is very close to the true ranking interval $[\text{rDE}_o^{U*}, \text{rDE}_o^{L*}]$.

To illustrate the first concern of conventional rankings, we show the computational results of three DMUs that have the same best ranking of 3: DMU_3, DMU_{61}, and DMU_{57}, in Table 3. Let $(\boldsymbol{\nu}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ be the optimal virtual price vector for the best ranking of DMU_o. The first panel lists the computational results of $o = 3$, the second panel lists that of $o = 61$, and the third panel lists that of $o = 57$. In each panel, column 1 is DMU id, and column 2 lists the normalized virtual profit efficiency score. After the normalization, the highest virtual profit efficiency score is 0, and the virtual profit efficiency score of DMU_o is -1. Column 3 and column 4 list upper rankings and upper buffered-rankings under the virtual price vector $(\boldsymbol{\nu}_o^{U*}, \boldsymbol{\mu}_o^{U*})$. Three DMUs in comparison are highlighted in bold.

Though DMU_3, DMU_{61}, and DMU_{57} achieve the same best ranking of 3, their gaps with the DMU in the second place are different. After normalization, all their gaps

Table 3 Three DMUs with the same best difference-based rankings of 3 in PFT1981

$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_3				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{61}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{57}			
j	DE_j	rDE_j^U	$brDE_j^U$	j	DE_j	rDE_j^U	$brDE_j^U$	j	DE_j	rDE_j^U	$brDE_j^U$
44	0.000	1	1.00	69	0.000	1	1.00	52	0.000	1	1.00
52	−0.099	2	2.11	5	− 0.590	2	3.44	69	− 0.933	2	8.62
3	**−1.000**	**3**	**16.97**	**61**	**− 1.000**	**3**	**12.27**	**57**	**− 1.000**	**3**	**11.00**
54	− 1.000	3	16.97	52	− 1.000	3	12.27	54	− 1.000	3	11.00
69	− 1.000	3	16.97	49	− 1.000	3	12.27	44	− 1.000	3	11.00
57	− 1.001	6	17.02	62	− 1.000	3	12.27	50	− 1.065	6	14.20
10	− 1.002	7	17.08	17	− 1.031	7	12.98	5	− 1.104	7	16.30
49	− 1.019	8	18.01	63	− 1.043	8	13.24	17	− 1.173	8	19.59
15	− 1.053	9	19.69	32	− 1.116	9	14.96	3	− 1.240	9	23.18
55	− 1.090	10	21.56	65	− 1.287	10	19.77	40	− 1.241	10	23.23
50	− 1.166	11	26.11	48	− 1.355	11	21.86	55	− 1.245	11	23.46
5	− 1.186	12	27.45	14	− 1.426	12	23.69	63	− 1.254	12	23.99
17	− 1.193	13	27.95	38	− 1.566	13	26.80	61	− 1.300	13	26.75
40	− 1.253	14	32.60	29	− 1.655	14	28.33	49	− 1.301	14	26.81
63	− 1.293	15	35.98	22	− 1.692	15	28.98	15	− 1.340	15	29.40
61	− 1.326	16	38.52	15	− 1.714	16	29.34	10	− 1.357	16	30.51
27	− 1.328	17	38.67	55	− 1.816	17	31.08	65	− 1.475	17	38.63
65	− 1.330	18	38.82	40	− 1.817	18	31.10	38	− 1.496	18	39.79
22	− 1.403	19	43.82	60	− 1.821	19	31.17	22	− 1.524	19	41.18
35	− 1.438	20	46.06	51	−1.964	20	33.69	62	− 1.578	20	43.82
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
59	− 5.301	70	70.00	59	− 19.133	70	70.00	59	− 6.647	70	70.00

with the DMU in the first place are 1. The gap between DMU_3 with the corresponding top 2 efficient DMUs (DMU_{44} and DMU_{52}) is $1 + (1 - 0.099) = 1.901$. The gap between DMU_{61} with the corresponding top 2 efficient DMUs (DMU_{69} and DMU_5) is $1 + (1 - 0.590) = 1.410$. The gap between DMU_{57} with the corresponding top 2 efficient DMUs (DMU_{52} and DMU_{69}) is $1 + (1 - 0.933) = 1.067$. Among them, the gap of DMU_3 is the largest, and the gap of DMU_{57} is the smallest. The conventional rankings fail to convey any information about the quantity of this kind of gap. But buffered-ranking has an obvious advantage over conventional ranking, since DMU_3, DMU_{61}, and DMU_{57} have buffered-rankings 16.97, 12.27, and 11.00, respectively. In the example, these buffered-rankings are monotonic to the gaps between the DMU under evaluation and the corresponding superior DMUs.

We also study three DMUs with the same best rankings of 5: DMU_{25}, DMU_{43}, and DMU_{70}. Table 4 is constructed in a way the same as Table 3, except that the normalization makes the DMU under evaluation have virtual profit of -1. The gap between DMU_{25} and the corresponding top 4 DMUs is narrower than the gap between DMU_{43} and the corresponding top 4 DMUs. DMU_{25} has a better relative position than DMU_{43}, but DMU_{25} and DMU_{43} have the same best ranking of 5. The best buffered-ranking of DMU_{25} is higher than that of DMU_{43}, which shows the advantage of DMU_{25} over DMU_{43}. Though DMU_{70} has a much wider gap with the corresponding top 4 DMUs than DMU_{25} and DMU_{43}, DMU_{70} has a much wider gap with DMUs with places from 8 to 20. Finally, DMU_{70} achieves a better buffered-ranking than DMU_{25} and DMU_{43}.

Table 5 shows three DMUs with close best buffered-rankings and different best rankings: DMU_{14}, DMU_{23}, and DMU_{42}. Their best buffered-rankings are 6.07, 6.10, and 6.13, while their best rankings are 3,2,3. Compared with DMU_{42}, DMU_{14} has a wider gap with the corresponding top 2 DMUs, and has a wider gap with 3 DMUs right below. The upper buffered-ranking of DMU_{14} is very close to that of DMU_{42}, since the wide gap between DMU_{14} and the corresponding superior DMUs is compensated by the wide gap between DMU_{14} and its inferior DMUs (DMU_{17},DMU_5, and DMU_{32}). Under the optimal virtual price $(\boldsymbol{\nu}_{23}^{U*}, \boldsymbol{\mu}_{23}^{U*})$, though DMU_{23} has upper ranking of 2, its advantage over 4 DMUs right below is smaller than the advantage of DMU_{14} (DMU_{42}) over 4 DMUs right below.

6.2 Experiment with Pig

The dataset Pig[3] consists of 248 pig producers with six inputs and two outputs. Six inputs are fertilizer, feedstuf, land, labour, machinery, and other capital. Two outputs are crop and pig.

Four rankings and two buffered-rankings in terms of virtual profit are shown in Fig. 4. 74 DMUs have the best ranking of 1 and the best buffered-ranking of 1. And 68 DMUs have the worst ranking of 248 and the worst buffered-ranking of 248. Among them, 26 DMUs have the whole ranking interval $[1, J]$ and the whole buffered-ranking interval $[1, J]$. In other words, each of these 26 DMUs can be the most profit efficient and the most profit inefficient among 248 DMUs by adjusting the virtual price vector

[3] Available at https://rdrr.io/cran/Benchmarking/man/pigdata.html.

Table 4 Three DMUs with the same best difference-based rankings of 5 in the experiment with PFT1981

$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{25}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{43}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{70}			
j	DE_j	rDE_j^U	$brDE_j^U$	j	DE_j	rDE_j^U	$brDE_j^U$	j	DE_j	rDE_j^U	$brDE_j^U$
44	0.000	1	1.00	44	0.000	1	1.00	69	0.000	1	1.00
49	− 0.741	2	5.39	16	− 0.689	2	5.72	62	− 0.063	2	2.22
56	− 0.919	3	14.08	8	− 0.746	3	7.00	58	− 0.345	3	4.88
22	− 0.945	4	16.37	68	− 0.787	4	8.31	56	− 0.395	4	5.28
25	**− 1.000**	**5**	**22.02**	**43**	**− 1.000**	**5**	**23.48**	**70**	**− 1.000**	**5**	**15.44**
23	− 1.000	5	22.02	23	− 1.000	5	23.48	32	− 1.000	5	15.44
21	− 1.000	5	22.02	59	− 1.000	5	23.48	70	− 1.000	5	15.44
47	− 1.000	5	22.02	33	− 1.005	8	24.12	55	− 1.011	8	15.66
55	− 1.000	5	22.02	39	− 1.012	9	25.05	22	− 1.018	9	15.81
62	− 1.002	10	22.22	27	− 1.047	10	30.21	5	− 1.153	10	18.42
52	− 1.018	11	23.91	56	− 1.063	11	32.79	68	− 1.308	11	21.99
48	− 1.058	12	27.78	2	− 1.072	12	34.28	14	− 1.504	12	25.62
60	− 1.075	13	29.53	20	− 1.082	13	36.05	38	− 1.531	13	26.12
58	− 1.094	14	31.70	21	− 1.085	14	36.58	61	− 1.628	14	27.88
15	− 1.098	15	32.16	15	− 1.106	15	40.63	63	− 1.717	15	29.41
69	− 1.107	16	33.12	47	− 1.116	16	42.65	45	− 1.752	16	29.99
27	− 1.116	17	34.14	19	− 1.130	17	45.32	60	− 1.932	17	32.98
16	− 1.148	18	37.90	35	− 1.131	18	45.50	20	− 2.027	18	34.54
18	− 1.167	19	40.33	26	− 1.135	19	46.25	15	− 2.038	19	34.73
70	− 1.167	19	40.33	34	− 1.157	20	50.53	49	− 2.065	20	35.20
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	2.893	70	70.00	50	1.946	70	70.00	59	18.738	70	70.00

Table 5 Three DMUs with the same best difference-based buffered-rankings of about 6 in the experiment with PFT1981

$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{14}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{23}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{42}			
j	DE_j	rDE_j^U	brDE_j^U	j	DE_j	rDE_j^U	brDE_j^U	j	DE_j	rDE_j^U	brDE_j^U
62	0.000	1	1.00	44	0.000	1	1.00	17	0.000	1	1.00
38	− 0.483	2	2.93	**23**	− **0.894**	**2**	**6.09**	24	− 0.643	2	3.44
14	**1.000**	**3**	**6.07**	47	− 1.000	3	8.85	**42**	− **1.000**	**3**	**6.13**
17	− 1.245	4	8.16	16	− 1.120	4	13.72	28	− 1.295	4	11.20
5	− 1.284	5	8.44	52	− 1.124	5	13.96	27	− 1.466	5	13.09
32	− 1.920	6	12.12	33	− 1.196	6	17.87	45	− 1.528	6	13.52
45	− 1.920	6	12.12	49	− 1.196	6	17.87	47	− 1.528	6	13.52
69	− 1.920	6	12.12	19	− 1.248	8	21.15	58	− 1.528	6	13.52
58	− 2.416	9	14.50	25	− 1.263	9	22.10	38	− 1.575	9	13.87
61	− 3.248	10	18.85	21	− 1.273	10	22.83	62	− 1.633	10	14.29
63	− 3.526	11	20.56	41	− 1.326	11	26.93	20	− 1.902	11	16.30
29	− 3.780	12	22.27	26	− 1.355	12	29.64	11	− 2.047	12	17.37
22	− 4.442	13	26.53	22	− 1.373	13	31.52	12	− 2.738	13	22.59
48	− 5.030	14	30.82	68	− 1.384	14	32.85	52	− 3.404	14	27.54
65	− 5.647	15	34.60	56	− 1.390	15	33.50	39	− 3.634	15	29.23
60	− 5.654	16	34.64	27	− 1.450	16	41.30	14	− 3.868	16	31.12
6	− 5.893	17	36.06	60	− 1.473	17	44.48	22	− 4.069	17	32.81
56	− 6.384	18	39.02	18	− 1.500	18	48.64	35	− 4.618	18	37.45
28	− 6.389	19	39.04	55	− 1.502	19	48.91	37	− 4.730	19	38.39
20	− 6.588	20	40.18	59	− 1.513	20	50.44	2	− 4.867	20	39.58
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
59	− 51.294	70	70.00	50	− 2.662	70	70.00	66	47.135	70	70.00

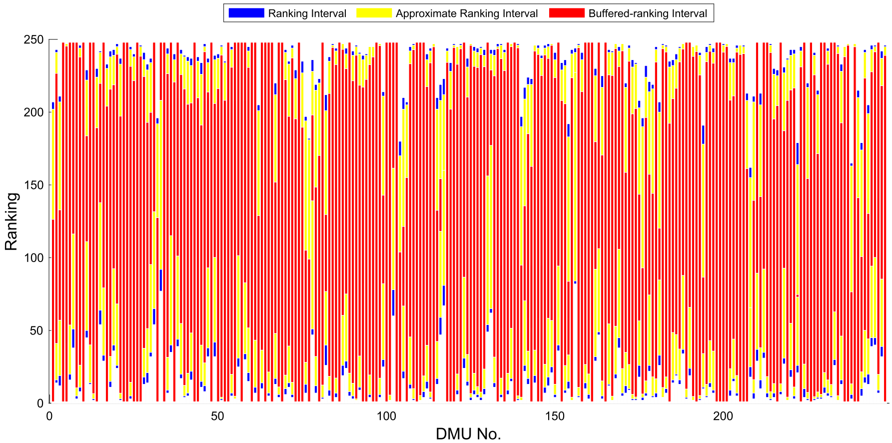


Fig. 4 Difference-based ranking intervals and buffered-ranking intervals for DMUs in Pig

$(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \mathcal{S}$. For all DMUs, the buffered-ranking interval is much narrower than the ranking interval.

To compare the difference between rankings and buffered-rankings, we list the results of three DMUs with the same best rankings of 3: DMU_{164}, DMU_{81}, and DMU_{189}, in Table 6. Among the three DMUs, the gap between DMU_{164} and the corresponding top 2 DMUs is the widest, thus the upper buffered-ranking of DMU_{164} is lower than DMU_{81} and DMU_{189}. Though the gap between DMU_{81} and the corresponding top 2 DMUs is close to the gap between DMU_{189} and the corresponding top 2 DMUs, the gap between DMU_{81} and DMUs with rankings in the range 4–13 is narrower than the gap between DMU_{189} and DMUs with rankings in the range 4–13. Therefore, DMU_{81} has a lower buffered-ranking than DMU_{189}. It confirms that the relative position implied by buffered-ranking can capture information about the quantity of the gap between the DMU under evaluation and those superior and peer DMUs.

Table 7 lists the running time of the six optimizations. Apparently, it is much more time-consuming to calculate four conventional rankings than two buffered-rankings. The average running time for calculating buffered-rankings is 0.05 s, and all buffered-rankings can be obtained within 0.08 s. While the average running time for four rankings is at least 19 s. In some extreme cases, it takes more than one hour to calculate conventional rankings. Especially when its optimal ranking is close to $J/2$, the attempt to locate the global optimum seems rather challenging. The number of failures to complete the optimization within an hour for the best rankings in terms of profit, the worst rankings in terms of profit, the best rankings in terms of profitability, and the worst rankings in terms of profit, are 2, 4, 3, and 2, respectively. In summary, optimizing buffered-rankings has a great computational advantage over optimizing conventional rankings.

Table 6 Three DMUs with the same best difference-based ranking of 3 in the experiment with Pig

$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{164}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{81}				$(\boldsymbol{v}_o^{U*}, \boldsymbol{\mu}_o^{U*})$ for DMU_{189}			
j	DE_j	rDE_j^U	brDE_j^U	j	DE_j	rDE_j^U	brDE_j^U	j	DE_j	rDE_j^U	brDE_j^U
201	0.000	1	1.00	22	0.000	1	1.00	152	0.000	1	1.00
57	− 0.197	2	2.25	93	− 0.676	2	4.09	236	− 0.688	2	4.21
164	**− 1.000**	**3**	**17.47**	**81**	**− 1.000**	**3**	**13.65**	**189**	**− 1.000**	**3**	**11.47**
104	− 1.000	3	17.47	66	− 1.000	3	13.65	74	− 1.000	3	11.47
17	− 1.000	3	17.47	5	− 1.000	3	13.65	120	− 1.000	3	11.47
165	− 1.000	3	17.47	103	− 1.015	6	14.26	1	− 1.000	3	11.47
21	− 1.033	7	19.98	60	− 1.026	7	14.73	104	− 1.140	7	16.00
62	− 1.089	8	25.69	155	− 1.040	8	15.35	207	− 1.182	8	17.48
173	− 1.095	9	26.46	100	− 1.045	9	15.57	145	− 1.185	9	17.59
34	− 1.137	10	33.07	94	− 1.201	10	23.74	53	− 1.202	10	18.25
248	− 1.145	11	34.55	97	− 1.253	11	27.12	146	− 1.397	11	26.84
60	− 1.168	12	39.58	126	− 1.262	12	27.76	96	− 1.439	12	28.97
120	− 1.182	13	42.89	75	− 1.263	13	27.83	133	− 1.451	13	29.61
170	− 1.185	14	43.62	230	− 1.338	14	33.94	26	− 1.480	14	31.24
80	− 1.186	15	43.86	86	− 1.361	15	36.17	231	− 1.515	15	33.30
238	− 1.212	16	50.93	205	− 1.385	16	38.70	122	− 1.563	16	36.18
240	− 1.253	17	65.15	91	− 1.391	17	39.37	219	− 1.630	17	40.11
209	− 1.254	18	65.55	37	− 1.458	18	48.30	27	− 1.649	18	41.19
176	− 1.259	19	67.59	57	− 1.460	19	48.58	108	− 1.659	19	41.78
207	− 1.271	20	72.57	177	− 1.514	20	56.61	226	− 1.693	20	43.73
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
36	− 3.732	248	248.00	110	− 4.644	248	248.00	165	− 37.146	248	248.00

Table 7 Running time (in seconds) of six optimizations in the experiment with Pig

j	Virtual profit				Virtual profitability	
	rDE_j^{U*}	rDE_j^{L*}	brDE_j^{U*}	brDE_j^{L*}	rRE_j^{U*}	rRE_j^{L*}
1	0.09	6.12	0.06	0.08	0.16	0.95
2	0.52	0.22	0.05	0.05	1.16	0.33
3	0.56	31.51	0.05	0.05	0.34	8.61
4	0.09	0.09	0.06	0.06	0.14	0.64
5	0.11	0.11	0.05	0.05	0.11	0.39
6	0.75	0.09	0.06	0.06	0.56	0.11
7	19.40	0.09	0.06	0.08	1.58	0.11
8	0.20	0.09	0.05	0.06	0.20	0.11
9	0.28	0.14	0.06	0.05	0.41	0.61
10	0.09	0.11	0.06	0.06	0.11	0.11
11	11.47	4.75	0.06	0.05	6.97	4.55
12	0.39	0.11	0.05	0.05	0.34	0.11
13	0.09	0.09	0.06	0.05	0.08	0.09
14	0.08	0.80	0.05	0.05	0.08	0.58
15	65.58	0.31	0.05	0.05	4.80	0.19
16	1.14	0.11	0.05	0.06	0.50	0.09
17	0.11	0.34	0.06	0.06	0.11	0.77
18	0.86	0.28	0.06	0.06	0.45	0.55
19	5.45	0.23	0.06	0.05	1.55	0.61
20	0.56	0.13	0.05	0.05	2.33	0.20
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Max	3600.05	3600.55	0.08	0.08	3600.06	1649.53
Min	0.06	0.06	0.03	0.03	0.06	0.06
Average	24.84	45.60	0.05	0.05	20.11	19.36

6.3 Experiment with hospitals

The dataset Hospitals[4] contains anonymous observations for 958 local public hospitals. Hospital outputs are Inpatients (annual numbers of inpatients) and Outpatients (annual numbers of outpatients). Hospital inputs are Labour (total number of employees) and Capital (total number of beds). The dataset was exploited by Besstremyannaya (2011), (2013).

For this large dataset, computational burden is an essential factor for model comparison. Even when the size is 958, all continuous linear programs can be solved within 0.16 s, which implies that the calculation of buffered-rankings is fast and scalable. But the computational time for conventional rankings increases dramatically with the num-

4 Available at https://rdrr.io/cran/rDEA/man/hospitals.html.

Table 8 Running time (in seconds) of six optimizations in the experiment with Hospitals

j	Virtual profit				Virtual profitability	
	rDE_j^{U*}	rDE_j^{L*}	brDE_j^{U*}	brDE_j^{L*}	rRE_j^{U*}	rRE_j^{L*}
1	0.73	1.09	0.09	0.09	0.35	0.33
2	0.21	0.42	0.07	0.08	0.18	0.22
3	0.25	0.31	0.10	0.07	0.13	0.13
4	0.18	0.25	0.07	0.07	0.29	0.29
5	0.20	0.27	0.07	0.06	0.15	0.08
6	96.80	0.48	0.08	0.06	2.97	10.94
7	3600.03	2.81	0.05	0.02	1.60	2.17
8	43.99	277.51	0.03	0.03	0.63	0.68
9	1.60	1.87	0.01	0.02	0.70	0.53
10	0.32	0.11	0.01	0.01	0.10	0.09
11	2538.80	11.15	0.04	0.03	1.28	0.61
12	461.61	0.48	0.03	0.02	0.05	0.05
13	24.08	10.22	0.03	0.02	0.25	0.15
14	16.18	14.50	0.02	0.02	0.15	0.10
15	6.89	57.14	0.02	0.03	1.96	1.82
16	3600.07	1566.51	0.03	0.04	3.62	39.08
17	1458.08	18.73	0.03	0.02	433.24	612.27
18	3.88	0.38	0.01	0.01	0.17	0.03
19	1311.27	2202.33	0.04	0.03	1.50	1.27
20	1122.16	1.50	0.03	0.02	0.11	0.07
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Max	3600.05	3600.55	0.14	0.16	3600.06	3600.23
Min	0.08	0.07	0.01	0.01	0.04	0.03
Average	524.84	615.60	0.05	0.05	220.11	219.36

ber of DMUs. For this large dataset, the computational advantage of buffered-rankings over conventional rankings is significant.

7 Conclusion

This paper proposes buffered-ranking intervals for virtual profit efficiency analysis. The upper buffered-ranking for a DMU is the maximum k that this DMU's efficiency score falls below the average of the corresponding top k efficiency scores. The relative position implied by buffered ranking considers the gap between the DMU under evaluation and superior and peer DMUs. When the efficiency score is in difference form, the optimization for the best buffered-ranking is equivalent to a continuous linear program. As a nice alternative to conventional rankings, buffered-rankings can find wide applications in efficiency analysis.

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Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. All authors declare that they have no conflicts of interest.

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