CASE STUDY: Portfolio Optimization with Exponential, Logarithmic, and Linear-Quadratic Utilities (exp_eut, log_eut, pm2_pen, avg_g)

Background

Utility functions are quite popular in various financial applications. This case study compares portfolio optimization problems with Exponential, Logarithmic, and Linear-Quadratic utility functions. The rate of return dataset for a portfolio is provided for benchmarking purposes by EpiRisk Research company via Drs. Roger Wets and Michael Tian. The EpiRisk Research relies on letting the manager of a fixed-income portfolio solve a sequence of so-called tacking (optimization) models, described below, to shape the returns' distribution. The shape of the distribution is adjusted by selecting the coefficients of the appraisal (~ utility) function.

Notations

I = number of instruments in the portfolio; $i = \{1,\ldots,I\}$ index of instruments in the portfolio;

J = number of scenarios; $j = \{1,\ldots,J\}$ index of scenarios;

x_i = portion of wealth invested in instrument i, $i = \{1,\ldots,I\}$;

$x = (x_1,\ldots,x_I)$ = vector of decision variables;

l_i = lower bound on the portion of wealth invested in instrument i, $i = \{1,\ldots,I\}$;

u_i = upper bound on the portion of wealth invested in instrument i, $i = \{1,\ldots,I\}$;

p_i^0 = price of instrument i at initial time 0;

$p^0 = (p_1^0,\ldots,p_I^0)$ = price-vector at initial time 0;

p_{ji}^1 = price of instrument i for scenario j at time 1;

$\theta_{ji} = (p_{ji}^1 - p_i^0)/p_i^0$ = rate of return of instrument i for scenario j;

$\tilde{\theta}_{ji} = p_{ji}^1 / p_i^0 = \theta_{ji} + 1$ = return of instrument i for scenario j;

θ_i = random value having J equally probable scenarios of rates of return, $\{\theta_{1i},\ldots,\theta_{Ji}\}$, $i = 1,\ldots,I$;

$\tilde{\theta}_i$ = random value having J equally probable scenarios of returns, $\{\tilde{\theta}_{1i},\ldots,\tilde{\theta}_{Ji}\}$, $i = 1,\ldots,I$;

$\theta = (\theta_1,\ldots,\theta_I)$ = random vector of rates of return;

$\tilde{\theta} = (\tilde{\theta}_1,\ldots,\tilde{\theta}_I)$ = random return vector;

$\theta_j = (\theta_{j1},\ldots,\theta_{jI})$ = vector of rates of return for scenario j;

$\tilde{\theta}_j = (\tilde{\theta}_{j1},\ldots,\tilde{\theta}_{jI})$ = return vector for scenario j;

$R(\theta_j, x) = \sum_{i=1}^{I} \theta_{ji} x_i$ = rate of return of the portfolio for scenario j;

$R(\tilde{\theta}_j, x) = \sum_{i=1}^{I} \tilde{\theta}_{ji} x_i = \sum_{i=1}^{I} (\theta_{ji} + 1) x_i = R(\theta_j, x) + \sum_{i=1}^{I} x_i$ = return of the portfolio for scenario j;

$R(\tilde{\theta}_j, x) = R(\theta_j, x) + 1$, if $\sum_{i=1}^{I} x_i = 1$;

$avg\_g\left(R\left(\theta,x\right)\right) = E\left[R\left(\theta,x\right)\right] =$ Average Gain function;

$pm2\_pen\left(-R\left(\theta,x\right),w\right) = E\left(max\left\{0,-R\left(\theta,x\right)-w\right\}\right)^2$ =Partial Moment Two Penalty function, where w is a threshold value;

$exp\_eut\left(R\left(\theta,x\right)\right) = E\left[-e^{-aR\left(\theta,x\right)}\right]$ = Exponential utility function, where a>0;

$log\_eut\left(R\left(\tilde{\theta},x\right)\right) = E\left[ln\left(R\left(\tilde{\theta},x\right)\right)\right]$ = Logarithmic utility function**.**

$E\left\{F\left(R\left(\theta,x\right)\right)\right\}$ = Linear-Quadratic utility function, where

$F(R)$ = an appraisal (= monitoring) function, a linear-quadratic function modelling the decision maker attitude towards risk.

$F(R)$ is defined as follows:

$$F\left(R\right) = \begin{cases} -qR - \frac{1}{2}\left[u\left(r-s\right) - q\left(r+s\right)\right], & \text{when } R < s \\ \left[\left(u+q\right)R^2 - 2\left(qr+us\right)R - \tilde{k}\right] / \left[2\left(r-s\right)\right], & \text{when } R \in \left[s,r\right]; \\ u\left(R-r\right), & \text{when } R \geq r \end{cases}$$

r = the hurdle rate;

s = the sub-hurdle rate, $s < r$;

u, q = constants, $q < -u < 0$;

$\tilde{k} = \left(ur^2 - qr^2 - 2usr\right)$ = constant;

The function $F(R)$ can be represented as follows:

$$F\left(R\right) = F_1\left(R\right) + F_2\left(R\right) + F_3\left(R\right);$$

$$F_1\left(R\right) = u\left(R-r\right), -\infty < R < \infty;$$

$$F_2\left(R\right) = \begin{cases} 0, & \text{when } R > r \\ \frac{u+q}{2\left(r+s\right)}\left(R-r\right)^2, & \text{when } R \leq r \end{cases};$$

$$F_3\left(R\right) = \begin{cases} 0, & \text{when } R > s \\ -\frac{u+q}{2\left(r+s\right)}\left(R-r\right)^2, & \text{when } R \leq s \end{cases}$$

***Optimization Problem* 1**

Maximizing Exponential Utility

$$max\left[exp\_eut\left(R\left(\theta,x\right)\right)\right] \tag{CS.1}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.2}$$

bounds on positions

$$l_i \le x_i \le u_i,\ i=1,2,\ldots,I. \tag{CS.3}$$

***Optimization Problem* 2**

Maximizing Linear-Quadratic Utility

$$\begin{aligned} &max\ E\left\{F\left(R\left(\theta,x\right)\right)\right\} = \\ &= max\left\{u \cdot avg\_g\left(R\left(\theta,x\right)\right) + \right. \\ &+ \frac{u+q}{2\left(r-s\right)} pm2\_pen\left(-R\left(\theta,x\right),-r\right) - \\ &\left. - \frac{u+q}{2\left(r-s\right)} pm2\_pen\left(-R\left(\theta,x\right),-s\right)\ \right\} \end{aligned} \tag{CS.4}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.5}$$

bounds on positions

$$l_i \le x_i \le u_i,\ i=1,2,\ldots,I. \tag{CS.6}$$

***Optimization Problem* 3**

Maximizing Logarithmic Utility

$$max\left[log\_eut\left(R\left(\tilde{\theta},x\right)\right)\right] = max\left[log\_eut\left(R\left(\theta,x\right)+1\right)\right] \tag{CS.7}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.8}$$

bounds on positions

$$l_i \le x_i \le u_i,\ i=1,2,\ldots,I. \tag{CS.9}$$