CASE STUDY: Portfolio Optimization, CVaR vs. ST_DEV (cvar_dev, st_dev)

Background

This case study compares three setups of a single-period portfolio optimization problem when risk is measured by CVaR Deviation, Standard Deviation calculated with the matrix of scenarios, and Standard Deviation calculated with the covariance matrix. The second and the third setups are equivalent representations of the Markowitz (1952) problem trading-off mean and variance of portfolio return. The original Markowitz problem finds a minimum-variance portfolio under restriction on mean return. Here we keep a similar setup but with the CVaR deviation as a replacement to the Standard deviation.

Notations

I = number of instruments (clusters) in the portfolio; $i=\{1,\dots,I\}$ index of instrument in the portfolio;

J = number of scenarios, $j=\{1,\dots,J\}$ index of scenarios;

$\mathbf{x} = (x_1, \dots, x_I)$ = vector of exposures (in fractions of available budget) of instruments $i=1,\dots I$;

$\bar{r}_i$ = expected rate of return of i-th instrument;

r_{ij} = rate of return of the i-th instrument under the risk scenario j ;

$\vec{r} = \left(r_1, \dots, r_I\right)$ = random vector of rates of return of instruments, $i=1,\dots I$;

$\vec{r}_j = \left(r_{1j}, \dots, r_{Ij}\right)$ = vector of rates of return of instruments, $i=1,\dots I$, under scenario j ;

$L(\mathbf{x},\vec{r}_j) = -\sum_{i=1}^{I} r_{ij}\, x_i =$ loss under scenario j ;

Cov = Covariance Matrix built by using the matrix of scenarios;

α = confidence level in CVaR deviation , $CVaR_\alpha^\Delta\left(L(\mathbf{x},\vec{r})\right)$, of the portfolio;

r = lower bound on return.

Optimization Problem 1 (risk is measured by CVaR deviation)

minimizing portfolio CVaR Deviation

$$\min_x \; CVaR_\alpha^\Delta\left(L(\mathbf{x},\vec{r})\right) \tag{CS.1}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.2}$$

constraint on the portfolio rate of return

$$\sum_{i=1}^{I} \bar{r}_i x_i \ge r\,, \tag{CS.3}$$

lower bounds on weights

$$x_i \ge 0, \quad i = 1,\dots,I\,. \tag{CS.4}$$

Optimization Problem 2 (risk is measured by standard deviation calculated by using the covariance matrix) *minimizing risk measured by standard deviation* calculated with the covariance matrix, ***Cov***

$$\min ST\_DEV\left(Cov\right) \tag{CS.5}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.6}$$

constraint on the portfolio rate of return

$$\sum_{i=1}^{I} \bar{r}_i x_i \geq r \ , \tag{CS.7}$$

lower bounds on weights

$$x_i \geq 0, \quad i = 1,\ldots,I \ . \tag{CS.8}$$

Optimization Problem 3 (risk is measured by standard deviation calculated with the matrix of scenarios)

minimizing risk measured by standard deviation calculated with the matrix of scenarios

$$\min ST\_DEV\left(L(\mathrm{x},\vec{r})\right) \tag{CS.9}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.10}$$

constraint on the portfolio rate of return

$$\sum_{i=1}^{I} \bar{r}_i x_i \geq r \ , \tag{CS.11}$$

lower bounds on weights

$$x_i \geq 0, \quad i = 1,\ldots,I \ . \tag{CS.12}$$