CASE STUDY: Stochastic Utility Problem (avg_max_risk, pm_pen_ni,linear)

Background

This case study solves Stochastic Utility (or Expected Utility) Problem which is approximated by sampling stochastic parameters of this problem (Sampling Average Approximation approach). The problem formulation and data are based on dataset which is considered in Nemirovski et al. (2009). The dataset was provided for testing purposes by Prof. George Lan. The problem formulation, as presented in Nemirovski et al. (2009), is as follows

$$\min_{x \epsilon X}\{E[\Phi(\sum_{i=1}^{I}(i/n+\xi_i)x_i)]\}, \qquad \text{(CS.0)}$$

where

$\Phi(t) = \max\{v_1 + s_1 t, \dots, v_M + s_M t\}$ = piecewise linear convex function; v_m, s_m are constants, $m = 1, \dots, M$;

$X = \{x \epsilon R^I \mid x \geq 0, \sum_{i=1}^{I} x_i = 1\}$;

$\xi_i \sim N(0,1)$ = independent normally distributed random values, $i = 1, \dots, I$.

An equivalent formulation to (CS.0) in terms of PSG functions is presented in approximation format with scenarios in (CS.1–CS.3) and in the format using normally distributed random variables in (CS.4–CS.6).

The Case Study presents four pairs of solved problem instances with 500, 1000, 2000, 5000 variables. Each pair includes solution of problem (CS.1–CS.3) and (CS.4–CS.6). All approximations use 10000 scenarios with sampled random coefficients, ξ_i, $i = 1, \dots, I$.

References

- Nemirovski A., Juditsky A., Lan G. and A. Shapiro (2009): Robust stochastic approximation approach to stochastic programming, SIAM J. Optim., Vol. 19, No. 4, 1574–1609.

Notations

I = number of decision variables;

x_i = decision variable, $i = 1, \dots, I$;

$\vec{x} = (x_1, x_2, \dots, x_I)$ = vector of decision variables;

$\Phi(t)$ = piecewise linear convex function,

$\Phi(t) = \max\{v_1 + s_1 t, \dots, v_M + s_M t\}$,

where v_m and s_m are constants, $m = 1, \dots, M$;

$t = \sum_{i=1}^{I}(i/n + \xi_i)x_i$ = random linear function;

$\xi_i \sim N(0,1)$ = independently distributed normal random values, $i = 1, \dots, I$;

$L_m(\vec{x}, \vec{\theta}^m) = v_m + s_m t = v_m + \sum_{i=1}^{I} s_m(i/n + \xi_i)x_i$ = m-th random loss function, $m = 1, \dots, M$;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th loss function; $\theta_0^m = v_m$,

$\theta_i^m = -s_m(i/n + \xi_i)x_i, i = 1, \dots, I$;

$\vec{\theta}_j^m = (\theta_{j0}^m, \theta_{j1}^m, \dots, \theta_{jI}^m)$ = j-th scenario of the random vector $\vec{\theta}^m$, j=1, 2, …, J;

$\theta_{ji}^m = -s_m(i/n + \xi_{ji})$ = value of random coefficient θ_i^m, generated by sample of parameter ξ_i in scenario j;

$L_m(\vec{x}, \vec{\theta}_j^m) = \theta_{j0}^m - \sum_{i=1}^{I} \theta_{ji}^m x_i$ = j-th scenario of m-th random loss function;

$avg\_\max\_risk\left(L_1(\vec{x}, \vec{\theta}^1), L_2(\vec{x}, \vec{\theta}^2), \dots, L_M(\vec{x}, \vec{\theta}^M)\right)$ = Average Max Risk for Loss for random loss

functions $L_m(\vec{x}, \vec{\theta}^m)$, $m = 1, \dots, M$, where, by definition,

$avg\_\max\_risk\left(L_1(\vec{x}, \vec{\theta}^1), L_2(\vec{x}, \vec{\theta}^2), \dots, L_M(\vec{x}, \vec{\theta}^M)\right) = \frac{1}{J}\sum_{j=1}^{J} * \max_{m=1,\dots,M}\{L_m(\vec{x}, \vec{\theta}_j^m)\}$,

i.e., $avg\_\max\_risk\left(L_1(\vec{x}, \vec{\theta}^1), L_2(\vec{x}, \vec{\theta}^2), \dots, L_M(\vec{x}, \vec{\theta}^M)\right)$ = mean value of function $\Phi(t)$;

Optimization Problem 1(approximated with scenarios utility problem)

Minimizing utility function

$$\min_{\vec{x}} \quad avg\_\max\_risk\left(L_1(\vec{x},\vec{\theta}^1), L_2(\vec{x},\vec{\theta}^2), \dots, \; L_M(\vec{x},\vec{\theta}^{\mathrm{M}})\right) \qquad \text{(CS.1)}$$

subject to

Budget constraint

$$\sum_{i=1}^{I} x_i = 1 \qquad \text{(CS.2)}$$

Bounds on variables

$$0 \le x_i, \; i = 1, \dots, I \qquad \text{(CS.3)}$$

Formulating Utility Problem (CS.0) using PSG functions for normal distributions

Since $t = \sum_{i=1}^{I}(i/n + \xi_i)x_i$ is a linear random function and $\xi_i \sim N(0,1)$ are independent normal variables, then t is normally distributed with parameters $t \sim N(\bar{t},\ \sum_{i=1}^{I} x_i^2)$, where $\bar{t} = \sum_{i=1}^{I}(i/n)x_i$.

Let us assume that the order of linear pieces $m = 1, \dots, M$ of function $\Phi(t)$ is such that $s_{m-1} < s_m$ ($s_0 = -\infty$) and for all pieces $\Delta t_m = [t_{m-1}, t_m]$ is such that $t_{m-1} < t_m$ ($t_0 = -\infty, t_M = +\infty$) and $\Phi(t) = v_m + s_m t$ if $t \epsilon \Delta t_m$.

The following statement is valid for any nonsingular random value t.

Let $f_m(t) = \begin{cases} \Phi(t), t\epsilon[t_{m-1}, t_m] \\ 0, \text{ otherwise} \end{cases}$. Then $E[\Phi(t)] = \sum_{\mathrm{m}=1}^{M} E[f_m(t)]$ where, for $m = 2, \dots, M-1$

$$E[f_m(t)] = \int_{t_{m-1}}^{t_m} (v_m + s_m t)p(t)\mathrm{d}t = v_m P_{\Delta t_m} + s_m\big(\mu(t_{m-1}) + t_{m-1}P(t_{m-1}) - \mu(t_m) - t_m P(t_m)\big),$$

where $P(r)$ = probability that $t \ge r$;
$P_{\Delta t_m} = P(t_{m-1}) - P(t_m)$ = probability that $t \epsilon \Delta t_m$;
$\mu(r) = \int_r^{\infty}(t - r)p(t)\mathrm{d}t$ = first order upper partial moment.

For m=1 and M

$$E[f_1(t)] = v_1\big(1 - P(t_1)\big) + s_1\big(\bar{t} - \mu(t_1) - t_1 P(t_1)\big),$$
$$E[f_M(t)] = v_M P(t_{M-1}) + s_M\big(\mu(t_{M-1}) + t_{M-1}P(t_{M-1})\big).$$

Summing up all $E[f_m(t)]$ we obtain

$$E[\Phi(t)] = v_1 + s_1\bar{t} + \sum_{\mathrm{m}=1}^{M-1}(s_{m+1} - s_m)\mu(t_m)\,.$$

PSG function which calculate the first order upper partial moment for linear random function with independent normally distributed coefficients is Partial Moment Penalty for Loss Normal Independent. To calculate $E[\Phi(t)]$ using PSG functions let us define:
$L(\vec{x},\vec{\theta}) = t = \sum_{i=1}^{I}(i/n + \xi_i)x_i$ = Loss Function;
$\vec{\theta} = (\theta_1, \theta_2, \dots, \theta_I)$ = vector of independent random coefficients for Loss Function; $\theta_i \sim N(i/n, 1)$;
$pm\_pen\_ni_{t_m}\left(L(\vec{x},\vec{\theta})\right)$ = Partial Moment Penalty for Loss Normal Independent with threshold t_m;
$avg(L(\vec{x},\vec{\theta}) = \bar{t} = \sum_{i=1}^{I}(i/n)x_i$ = Average Loss function.

Optimization Problem 2 (utility problem with normally distributed random values)

Minimizing utility function

$$\min_x \left[v_1 + s_1 avg\left(L(\vec{x}, \vec{\theta})\right) + \sum_{\mathrm{m}=1}^{M-1} (s_{m+1} - s_m) \cdot pm\_pen\_ni_{t_m}\left(L(\vec{x}, \vec{\theta})\right)\right] \quad \text{(CS.4)}$$

subject to

Budget constraint

$$\sum_{i=1}^{I} x_i = 1 \quad \text{(CS.5)}$$

Bounds on variables

$$0 \le x_i,\ i = 1, \ldots, I \quad \text{(CS.6)}$$